

# CAPITAL AND THE MONETARY BUSINESS CYCLE THEORY 

ESSAYS ON THE AUSTRIAN THEORY OF<br>CAPITAL, INTEREST, AND BUSINESS CYCLE

Doctoral Dissertation

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Prohlašuji na svou čest, že jsem disertační práci "Capital and the Monetary Business Cycle Theory: Essays on the Austrian Theory of Capital, Interest, and Business Cycle" vypracoval samostatně a s použitím uvedené literatury.


#### Abstract

This dissertation explores four big topics in the Austrian economic theory. Chapter 1 elucidates the Austrian theory of capital. It introduces basic tools that are further used in the analysis of the business cycle. It also clarifies some misunderstandings in this theory. Chapter 2 investigates the evolution of the interest rate over the business cycle that is predicted by the Austrian theory of economic fluctuations. Chapter 3 examines the pure time preference theory. It shows with the help of a simple neoclassical graphical and mathematical apparatus that there is a fundamental flaw in this theory. It suggests that the notions of want and good must be explicitly separated, and it concludes that the time preference as well as the subjective exchange ratio between present goods and future goods may take on any value. Chapter 4 explores the business cycle dynamics in the economy with permanently rising natural output. Simple monetary policy rule that was designed to eliminate economic fluctuations is discussed in detail.


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## PREFACE

This study is the outcome of research I have been doing for the last decade. The main focus of this research was the business cycle theory and the monetary theory, and the monetary theory of the business cycle. At the beginning of this investigation I was impressed by very old ideas of Friedrich August von Hayek. His insights were both intellectually demanding and attractive, but they seem to be also confirmed by the performance of real world economies. This feeling was then supported by the financial crisis and the great recession that followed. It seemed that these two events perfectly fitted the Hayekian framework according to which sharp recessions may occur even after the period of price stability and decent economic growth.

It is this theory that may uncover why seemingly smooth economic development is interrupted by a sudden deterioration in economic conditions even when there were no previous signals or symptoms that the economy is on the unsustainable path. According to Hayek, the roots should be searched in the banking and the monetary system in which the link between the act of saving and the act of investment are separated into two independent operations. Although money is one of the greatest inventions of the human race, it also performs the role of a loose joint in the markets that may lead to serious imbalances between the demand for goods and the supply of goods not only within the given period of time but also between various periods of time.
It emerged that this process cannot be well understood without deep investigation of the intertemporal markets and motivations of the acting man in these markets. This research also revealed that the discrepancies in the intertemporal markets may be exposed after a considerable period of time when it is too late to respond in such a way to restore conditions of the intertemporal equilibrium that ruled before the monetary disturbance.

The field of this research turned out to be very broad. The proper understanding of the intertemporal exchange required a thorough study of the theory of interest. The theory of capital also had to be examined to comprehend the complex production process in modern economies. Furthermore, because the major source of economic fluctuations was identified in the monetary system, a long-term investigation must be done not only of the authors writing in the Austrian tradition but also of modern theorists that uncovered fascinating phenomena that were hidden to the eyes of the glorious scholars of the pre-war era.
The present study consists of four separate chapters that have the form of independent articles or books. However, one major line of reasoning might be found in all of them. Since the topic examined is rather complex, ideas in one chapter are often found in other chapters as well. Moreover, since Chapter 2 was prepared as a separate article, its beginning partly coincides with the investigation in Chapter 1.

Chapter 1 is designed as an attempt to clarify basic concepts within the Austrian theory of capital. It forms a basis for further investigations in the following chapters. Since much confusion can be found in the works that explore this difficult theory, Chapter 1 tries to elucidate some of the major topics that must be accurately understood otherwise the demanding ideas of the Austrian theory are lost.
Chapter 2 is focused on the specific dynamics of the interest rate during the business cycle as viewed by the Austrian theory. The theory of capital and the theory of money are integrated in this section, and an attempt is made to uncover that aggressive critique that the Hayekian business cycle theory faced at the top of the Keynesian revolution is partly unfair, partly wrong, and partly beside the point. This chapter also suggests that proper understanding of
economic fluctuations requires a deep investigation of the "real" part of the economy since the impulse to economic boom may originate in changes in technology.

This task is performed in Chapter 3, which is constructed as a critique of the pure time preference theory of Ludwig von Mises and Murray Rothbard. Especially Rothbard blamed the central bank for inflationary policy that leads to the observed boom-bust pattern in aggregate income. However, Chapter 2 uncovered that real forces, such as technological shocks, may play a much greater role, so Chapter 3 is designed to investigate these real forces. The centre of this chapter is, however, an acting man that must decide about the optimal allocation of his income over time. This chapter shows that the a priori time preference that people may all have in-built in their minds and in their behaviour may not be the only determinant of the rate of interest - the price that coordinates intertemporal allocation of resources.

The last chapter, Chapter 4, builds on the investigations of the previous sections. It explores the business cycle that may occur in a growing economy. A direct attack is performed against the general belief about the benefits of price level stabilization. However, this chapter shows that the alternative monetary policy of the MV-rule may improve performance of the economy only if certain conditions are met.

## Chapter 1

## The Austrian Theory of Capital: An Attempt at Clarification

## 1. INTRODUCTION

Chapter 1 of the dissertation introduces the Austrian theory of capital. This specific theory will be the core of business cycle investigations presented in Chapter 2. Thus, in the Austrian model, the theory of the business cycle and the theory of economic growth are tightly interconnected. This alliance might be observed also in modern real business cycle theory and in recent New Keynesian literature. However, the Austrian theory is very specific, as this chapter demonstrates.

In section 2 of this chapter, the Austrian model is introduced. Key ideas of the founder of this theory - Eugen von Böhm-Bawerk - are presented in detail. The emphasis is put on the structure of production and especially on the idea of roundaboutness. Several misunderstandings that emerged in the literature are clarified, especially as regards the higher productivity of roundabout methods.
Section 3 presents a simple modelling tool developed by Hayek (1935), which is known as the Hayek triangle. This tool is used to further clarify the notion of capital in the Austrian theory. The core of this section is the discussion with Frank Knight, the Chicago neoclassical economist and great supporter of Clark's idea that capital represents a homogenous mass that is automatically maintained. An attempt is made to clarify that capital has its own structure that cannot be ignored if important insights in economic theory are not to be lost. A direct outcome of this idea is that the maintenance of capital is not automatic.

The decrease in the time preference is discussed in section 4. A thorough analysis of the restructuring of the production processes is carried out. This section clarifies that the capital theory is virtually dynamic, and processes in the transition period from one structure of capital to another may not be ignored. The reallocation of the labour force during this process is examined as well, and it is concluded that the price system plays a major role in the intertemporal allocation and re-allocation of the factors of production. This section ends with a discussion about the eternal growth provoked by a decrease in impatience in the society. The last part concludes.

## 2. THE AUSTRIAN MODEL

The Austrian business cycle theory stands and falls with the Austrian theory of capital, outlined by Menger (2007), thoroughly developed by Böhm-Bawerk (1890; 1891), and refined by Hayek (1941) and Lachmann (1956). Although it is not the main objective of this dissertation to go into the deep intricacies of the capital theory, for a clear understanding of the theory of the business cycle, some basic tools developed within the Austrian capital theory are necessary to be introduced and thoroughly explained.
The first problem that is immediately encountered in the investigation of this field is that the Austrian authors defined capital differently. Böhm-Bawerk (1890:6) in his magnum opus defined capital as "a complex of produced means of acquisition". Hayek (1941:9) partly diverted from this concept and stressed the fact that capital is mainly represented by nonpermanent resources either produced by man or given by nature. Rothbard (2004:497) narrowed this definition and included only non-permanent produced means of production, whereas for Lachmann (1956:11) capital was represented by the stock of all material
resources, including land. ${ }^{1}$
Although many approaches to the definition of capital can be found, all Austrian authors stress the fact that in the realm of capital theory, the crucial role is played by time. Capital goods are just intermediary products on the (time-consuming) way to the final consumption goods (Böhm-Bawerk 1890:22) since almost no consumption good is created directly by labour and land. Instead of using labour and natural powers to make the goods prepared for immediate consumption directly, man undergoes an indirect way, a long roundabout journey, by creating intermediate products - capital goods - as was shown not only by BöhmBawerk (1891), but by many authors before him, as he himself demonstrated in the critique of the theory of his predecessors (Böhm-Bawerk 1890). ${ }^{2}$
In this connection, it can be said that capital goods are productive since they have the power to create consumption goods - the only goods that can satisfy (by definition) human wants. However, although the formation of capital goods precedes in time the creation of consumption goods, the value of the former is derived from the value of the latter. Only due to the fact that consumption goods may satisfy human wants does capital acquire its value. Once the capital good loses its power to create useful consumption goods, it will immediately become worthless. This may happen either if the consumption good loses its value to the consumer, or if the capital good ceases to be used in the process of its (or in any other consumption good's) production.
This simple logic was used not only by Menger (2007:150) or Böhm-Bawerk (1890; 1891) in demolishing old cost-of-production theories of value of classical economists, but it was also heavily stressed by Irving Fisher (1930) in developing his own theory of interest. However, although the capital goods derive their value from consumption goods, Böhm-Bawerk (1890; 1891) devoted (after criticising other authors and their theories in very great detail) almost one thousand pages to identifying and explaining the fact that one can always observe a difference between the value of factors of production (or capital goods) and the final value of consumption goods they eventually produce. This positive premium or agio, as Böhm-Bawerk called it, is always present in the economy, and it is never eliminated by the competition of entrepreneurs.
Two elements hitherto mentioned are necessary to explain this agio. The first lies in the fact that consumption goods are manufactured indirectly by capital goods and the latter derives its value from the former. The second one is grounded in an indisputable observation that this indirect, or roundabout, process proceeds in time. In other words, present capital goods will and should gradually mature into consumption goods some time in the future; in essence, they are future consumption goods.

Böhm-Bawerk's solution to this apparent puzzle is then condensed in a simple statement:
PRESENT goods are, as a rule, worth more than future goods of like kind and number. This proposition is the kernel and centre of the interest theory which I have to present. (BöhmBawerk 1891:248)

[^0]If people value present goods more than future goods, it is natural that present capital goods (i.e. future consumption goods) should be valued less than present consumption goods, and the competition would never eliminate this value difference. As can be seen in the quotation above, this statement also plays a crucial role in the explanation of the interest phenomenon. For Böhm-Bawerk (1891:299), it represents the fundamental ground for the natural interest on capital - interest exists because people value present goods more than future goods. In other words, people are prepared to exchange more future goods to obtain a lower amount of present goods, or alternatively, they forego present goods only for the compensation of a larger amount of future goods.

This approach may be readily used for the explanation of the interest on loans. In the loan contract, people usually exchange present money for a higher amount of future money, where the separation of principal and interest is made just for commercial and practical convenience (Böhm-Bawerk 1891:296). Hence, the interest is a widespread phenomenon, whose importance should not be underestimated. It is immediately visible on the loan market, however - and this will be crucial further in the text - it is a pervasive phenomenon in the production process, as it is expressed as a value difference between present consumption goods and present factors of production, or alternatively as a value difference between present consumption goods and future consumption goods.
This interest, stemming from the different valuation of present goods and future goods, and by Mises (1996:524) called the originary interest, will be also in the centre of our explanation of the business cycle. However, let us first present how Böhm-Bawerk elucidated its existence. Surprisingly, we will also see that all three grounds expounded in the following paragraphs are more or less built in the modern economic growth theories, where the interest rate itself, though, is usually of secondary importance.

The first cause of the value advantage of present goods lies in the fact that people are usually better provided for in the future than in the present (Böhm-Bawerk 1891:249ff). This statement holds on average in a growing economy and on the individual basis for the majority of people over their lives (before they retire). However, it seems to be much less tenable in a stationary economy. Nevertheless, if people are better equipped with consumption goods in the future rather than in the present, then, according to the theory of marginal utility developed by Menger (2007) and further developed by Böhm-Bawerk (1891) himself, the additional present consumption good has a higher marginal utility than the future consumption good. In essence, the last unit of the present good may satisfy more needed want than the last unit of the future good owing to the fact that a man has a better provision in the future. As a result, the marginal present good is valued more than the future good of the same type and quality.

Böhm-Bawerk (1891:250-252) further developed this idea. Apparently, it may happen in reality that present is better provided for than future. However, in this case the present goods can be stored and moved to the future. Obviously, this can never be done with future goods, which brings about additional advantage to the present goods. Here we encounter the phenomenon of the irreversibility of time (Hayek 1941:345). Moreover, between the moment the present goods are stored and the time at which the future goods will become available, a new, initially unanticipated, want may emerge, which can be satisfied only with the present good. However, the last argument, as well as the first one, holds only for the goods that are easily storable. The goods that will spoil rapidly over time are less prone to acquire such an advantage.

The second cause of the agio between present goods and future goods rests in the inner
inclination of a man to systematically underestimate his future wants (Böhm-Bawerk 1891:253). Frank Fetter (1928:239) stressed that the act of valuation either about present goods or about future goods always takes place in the present. Future valuation of future goods is not possible. ${ }^{3}$ Böhm-Bawerk was aware of this phenomenon and stated that man in the present underestimates future wants for the three following reasons.
The first lies in the incomplete imagination about future wants a man has built in his mind. The second one is connected with a defect of will that makes him prefer present satisfaction even at the expense of future uneasiness or unpleasantness. The third relies on the uncertainty of future life since one never knows whether the gratification from future goods will ever arrive (Böhm-Bawerk 1891:254-255). ${ }^{4}$

According to Böhm-Bawerk, this undervaluation operates regardless of the relative provision of goods between present and future. Compared to the first reason, this second ground should hold also in the stationary economy. Hence, it might form a stronger basis for the existence of the interest phenomenon. Nevertheless, this explanation was heavily criticised even by the economists within the Austrian school. Mises (1996:486), for example, rejected this reasoning by the objection that the general property of human action cannot be based on psychological grounds. He then qualified the original Böhm-Bawerk approach by saying that: "Satisfaction of a want in the nearer future is, other things being equal, preferred to that in the farther distant future" (Mises 1996:483). ${ }^{5}$

For Mises, the preference for the satisfaction of a want in the present rather than in the future is a general prerequisite of human action. It holds everywhere and every time. After all, the satisfaction of some basic wants and needs in present is a necessary condition for the survival to future (Fetter 1928:241). However, according to Mises (1996:487), the fundamental feature of the human action stated above is effective not only in the extreme situation of preservation of human life, but it is present in every choice of the acting man. ${ }^{6}$

So far, we have discussed two grounds for the existence of the positive premium on present goods against future goods suggested by Böhm-Bawerk. However, it is not the objective of this section to defend, extend or develop his theory. The main reason for the preceding discussion was to establish a simple and tractable model that will be helpful in further investigations. The first two causes expounded that interest is a value phenomenon and should be treated along these lines. The first two causes create a phenomenon that Frank Fetter (1928:236), instead of agio, called the time preference that is put on present goods as against future goods and that Fisher (1930) called impatience. ${ }^{7}$ These two reasons will be crucial to form an upward sloping saving curve on the loanable funds market.

The third reason put forward by Böhm-Bawerk, albeit also connected with human valuation in the end, introduced a technical or productivity element to his theory. According to Böhm-

[^1]Bawerk (1891:260), present goods are technically superior to future goods. This seemingly strong statement is based on several observations. The first, and the most fundamental one, is that if the factors of production are employed in time-consuming roundabout methods, instead of used directly in the production of final consumption goods, they are, as a rule, more productive in the sense that they provide higher output of consumption goods.

However, this fact alone can surely not give present goods any technical superiority. By locking present consumption goods for some time in the stock, the future output of consumption goods will not increase by a wave of a magic wand. The more proper reasoning rests in the fact that if the man possesses some amount of present consumption goods, he has an advantage compared to having the same amount of future consumption goods. By having present goods, he can release factors of production from processes that provide consumption goods directly or in a very short time. He may use them instead in the roundabout processes that take a longer time, but that will provide a higher output of consumption goods after completion. And if man prefers a larger amount of goods to a lower amount of goods, the given amount of present goods must be valued more than the given amount of future consumption goods simply due to the fact that by the mechanism just described the given amount of present goods may provide higher output of future goods.
This roundabout process of production lies at the centre of the Austrian theory of capital. During this process, capital goods of various forms are created by investing labour, land (or natural resources in general), and other capital goods. However, this creation of intermediate products (capital goods) is not made for its own purpose. A man undergoes this timeconsuming process only because he believes that in the end, the roundabout methods will provide him with a higher output of consumption goods.

It is quite surprising that Böhm-Bawerk never gave a reason why the roundabout processes should have a higher technical productivity than shorter processes that provide consumption goods directly by applying labour and land. Regarding this point, Böhm-Bawerk just stated:

That roundabout methods lead to greater results than direct methods is one of the most important and fundamental propositions in the whole theory of production. It must be emphatically stated that the only basis of this proposition is the experience of practical life. Economic theory does not and cannot show a priori that it must be so; but the unanimous experience of all the technique of production says that it is so. And this is sufficient; all the more that the facts of experience which tell us this are commonplace and familiar to everybody. But why is it so? The economist might quite well decline to answer this question. For the fact that a greater product is obtained by methods of production that begin far back is essentially a purely technical fact, and to explain questions of technique does not fall within the economist's sphere. (Böhm-Bawerk 1891:20)

It was Hayek (1941) who listed various reasons for a higher technical productivity of roundabout methods, although some traces of the most relevant explanation can be also found in Böhm-Bawerk (1891:82) and Menger (2007:73,154). The roundabout processes usually employ resources that were not used in shorter processes or that were even impossible to be used in the processes that provide consumption goods directly (Hayek 1941:60). They were free (natural) resources that could be employed only if they cooperate with other factors of production in the methods that take much more time.
The consideration of time is again crucial in this case. A man did not possess enough time to employ unused natural resources when he devoted labour and land for direct production of consumption goods. Even proverbial Robinson Crusoe, who undertakes a roundabout process by creating a boat and a net, uses materials that were not used when he was catching fish by
his bare hands. By engaging in roundabout processes, previously unused, free, hence noneconomic goods may become scarce and therefore economic. However, they would never acquire the economic character if the other factors of production were not employed in the roundabout methods of production (Hayek 1941:63).

The simple example with Robinson can be extended to any degree. By additional lengthening of the process of production, new resources that were left idle can be utilised to expand future output. As a result, the roundabout methods are physically more productive simply by the fact that more natural resources and more natural powers, which were impossible to use in the shorter methods due to the lack of time, are utilized. As Böhm-Bawerk (1891:82) put it: " $[\mathrm{N}]$ ew allies are obtained from the immense stores of natural powers, and their activity is enlisted in the work of production."

Hayek (1941:63) attributed the second reason for a higher productivity of roundabout methods to what he called the vertical division of labour. ${ }^{8}$ In the longer methods, the production process can be separated into many sequential activities, and owing to this specialization, the final product is larger (or serves better, or satisfies more needed wants) than in shorter processes in which the division of labour cannot be extended to any appreciable degree. This reasoning therefore builds on the famous argument of Adam Smith (2001).

Hence, all arguments usually raised for the benefit of the horizontal division of labour, where people are specialized in the production of various final consumption goods, can be utilised to support the argument of vertical division of labour in which activities are devoted to a sequential production that is spread over time. ${ }^{9}$
The argument that roundabout methods of production are more productive is sometimes interwoven with some confusion. Since this piece of the mosaic of the Austrian capital theory will be widely used in the business cycle theory, it is necessary to be clarified in some detail. First of all, the argument relies on the generally accepted assumption of efficiency. If numerous methods of equal length capable of producing the same consumption good exist, only the one providing the largest output will be chosen (Hayek 1941:73). ${ }^{10}$ As a result, if the higher output of consumption goods is to be obtained, factors of production must be employed in longer processes. However, it is not guaranteed that every lengthening of the production process will bring about a larger output of consumption goods. The requirement of efficiency must hold again. Thus, only the process with the highest output will be chosen within the range of processes with the given (and now longer) "period of production". At this point, the Austrian authors usually recall the Böhm-Bawerk (1891:82) statement about "wisely chosen" roundabout methods.

It may also happen that a roundabout method (and the word "roundabout" is of particular importance here) gives output in a shorter period of time than a direct method. ${ }^{11}$ Of course, the direct method is then out of consideration as it is not the most efficient among the methods of the particular length. Some goods might be even impossible to produce in shorter methods

[^2](Böhm-Bawerk 1891:20). Hence, in the rest of our analysis, we assume that only the most efficient methods for the given length of the production process will be used. ${ }^{12}$

Nevertheless, even if we select the range of methods that provide the highest output for the given period of time, other confusion can easily trap one's sound reasoning. Although it takes more time for a longer process than for a shorter process to release final consumption goods, the fact that the longer process produces more consumption goods implies that the given amount of consumption goods is produced faster in the former rather than in the latter, otherwise it would be more efficient to produce the given output sequentially in the shorter process (Hayek 1941:77).
This idea is illustrated in Figure No. 1. The upper part of this diagram represents a very short process in which one unit of labour is employed for one period and provides one unit of the final consumption good. This process is repeated every period onwards, so five units of output are produced in five periods and ten units of output in ten periods. The bottom part of the figure depicts a long roundabout process that provides consumption goods in five periods, and in each period one unit of labour is employed as well as in the first process. However, after five periods, this roundabout process will produce ten units of output. As we can see, even though the same amount of labour has been employed, the output is higher due to the fact that the process was longer - the input was invested for a longer period until the final output matured. As a result, the longer process implies a higher output per one unit of input.


Figure No. 1, The productivity of short and long processes

[^3]To illustrate the discussion from the previous paragraphs in a similar picture, we may imagine a third process that is as long as the second one (i.e. five periods), but which produces just four units of output in the end. This process will never be chosen as it is inferior to the second one. Moreover, it is also dominated by the first process since it provides only four units of output after five periods, whereas the first one, after being repeated five times, gives one unit more.
As can be seen, great caution must be taken in identifying either a long or a short method. The first process in Figure No. 1 is short in the sense that it provides final consumption goods in a shorter period of time. The second one is longer for the opposite reasons; one must wait five periods until the consumption goods mature. Yet, it is more productive as it provides not only more goods than the first process (this statement holds also for the third, inferior, process), but more goods for a comparable period of time. In addition, the given amount of final consumption goods (10 units in our case) is produced more rapidly by the roundabout method (in five periods) rather than by the direct method (in 10 periods).
From the three methods just listed, only the first and the second one are efficient. In the given period of time, for which it must be waited for the final consumption good (one period and five periods), no other method can be found that will provide a higher output of final consumption goods. The third process will not be chosen, because it is inefficient.
We can repeat the statement that if the higher output is to be acquired, man must lengthen the process of production. In other words, factors of production must be tied up for a longer period of time. However, for this theory to have any definite meaning, it must be assumed that knowledge in the economy over the relevant period is given and stationary (Hayek 1941:72). Nevertheless, this does not imply that all possible technical knowledge or all inventions are also utilised in the running processes. It only means that people know how the given output can be produced within the shorter process and also that the longer process (that uses different methods, creating a different set of capital goods, and utilising different and maybe better inventions) will eventually provide a higher output of consumption goods. However, this higher output will mature after a longer period of time. Hence, it may easily happen that consumers prefer shorter processes even though they provide a lower amount of consumption goods. In such a case, there may exist latent inventions, methods, and knowledge that would lead to a higher output in the longer methods but that are not used since consumers are not patient enough. ${ }^{13}$
Latent methods and inventions that can be utilised only in longer processes represent the third reason for a higher productivity of roundabout methods. We may call this knowledge endogenous as it is present in the economy. This should be distinguished from the knowledge or invention that is suddenly devised and which shows how the given amount (or even a higher amount) of consumption goods could be produced in a shorter period of time. This second type may be called the exogenous knowledge since it comes from the outside and increases the previous level of knowledge or technologies in the economy. In our simple picture, it can be represented by a new process that provides ten units of output just in four periods instead of five periods, or that the new invention enables the five-period process to produce 14 units. The technical improvement of various processes will be discussed in more

[^4]detail in Chapter 2. ${ }^{14}$
The question is, however, why the methods of production are not continuously lengthened when they provide a higher output of consumption goods. The answer lies in two fundamental elements that limit the time extension of the production process. The first one relies on the notoriously known and utilised argument of diminishing marginal productivity. However, in connection with the Austrian theory of capital, this key economic principle is rather difficult to comprehend. ${ }^{15}$ Although it is an integral part of the work of Böhm-Bawerk (1891:84), Hayek (1941:179ff) admitted that it was rather difficult to find a suitable definition. Nevertheless, he offered the following statement: " $[\mathrm{R}]$ ate of increase of the product due to the extension of the investment period."
It is generally believed that the marginal product (i.e. the marginal increase in output of consumption goods) due to the additional extension of the investment period should be decreasing from some point. The basis for this assumption can be found in the three causes of higher productivity of roundabout processes mentioned above.

First, further time extension of the production process can gradually exhaust previously unused natural resources and natural forces. It is conceivable that their technical contribution to a further increase in output may continuously diminish. Secondly, the vertical division of labour and the resulting specialization can also lose its productive potential, which leads to decreasing marginal productivity of the further extension of the production process. And finally, the storehouse of knowledge and inventions gradually used up in longer processes can also exhaust its productive power. Hence, all three reasons may at some point start to exhibit diminishing marginal productivity. ${ }^{16}$
This rather long exposition of the third reason for the existence of agio between present goods and future goods should give us a basis to construct the investment curve on the loanable funds market. Hence, it was separated from the first two causes that may serve to plot the saving curve. The assumption that the roundabout methods are technically more productive, and the fact that the output from the time extension of production increases at a decreasing rate (it exhibits diminishing marginal productivity) leads to the conclusion that the investment curve is decreasing.

However, the diminishing marginal productivity is not sufficient to limit the never-ending time extension of the production process. It is conceivable that even the smallest increase in the future output may persuade the entrepreneur to extend the roundaboutness of the production process. Yet, as will be shown below, the essential brake is performed by the first two Böhm-Bawerkian reasons for the agio discussed above.

For a lucid explanation of the brake for further lengthening, it may be easier to start with the second cause. The fact that people underestimate their future wants, or as Mises (1996:484) exposed, that people prefer the given satisfaction in the present rather than in the future, limits the time extension of production. It puts a stop to the amount of factors of production that a man is prepared to use in longer processes. A definite point must emerge at which the increase in future output at the expense of present output of consumption goods will not be accepted

[^5]because the present satisfaction will be felt more urgently compared to the potential increase in future output. At this point, the lengthening will halt.

The operation of the first cause runs as follows. Since the lengthening of the production process requires that the factors of production must be diverted from providing goods in a short period of time on behalf of a more remote future, the output of present goods decreases and the output of future goods goes up. As a result, the relative provision of goods in the future improves at the expense of the present, which increases the marginal utility of present goods and reduces the marginal utility of future goods.
As a result, a man will be still more and more reluctant to forego present goods, even though the output of future goods will increase. We may say that the sacrificed satisfaction from present will not be compensated by the satisfaction given by a higher amount of future goods. Hence, the operation of the first cause also brings the lengthening to a standstill. It is obvious that both causes operate in conformity, especially if the second cause depends on the average flow of income. ${ }^{17}$ A more technical treatment of this problem will be given in Chapter 3.

The operation of the three reasons just analysed leads to a positive premium put on present goods against future goods; it leads to the emergence of interest in the economy. And this very existence of interest limits the never-ending lengthening of the production process since its decreasing marginal productivity - still lower and lower increments of future output cannot keep pace with the interest. ${ }^{18}$

## 3. THE HAYEK TRIANGLE

A useful tool widely employed to describe the main tenets of the Austrian capital theory was developed by Hayek (1935a), and further refined by Rothbard (2004), Garrison (2001) or de Soto (2006), hence the name - the Hayek triangle. It must be remembered that this tool is a crude simplification of the true processes in real economy, though it may provide us with some general insights. ${ }^{19}$


Figure No. 2, The Hayekian triangle

Figure No. 2 shows one possible production process. The fact that the production proceeds in

[^6]time is represented by the horizontal line. The vertical leg of the triangle measures the nominal value of final consumption goods and portrays the ultimate objective of the entire production process. By cutting the triangle at any point of time and erecting a vertical line, one can get the value of unfinished goods emerging on the way to final consumption goods.

The proceeds from selling the given intermediate product can be split into three main components. First, the product of the previous stage of production must be bought, a product that is in a lower degree of processing. Secondly, the original means of production (i.e. labour and land) must be remunerated for their services of refining the product for a further stage. And finally, since the process proceeds in time, the interest must accrue during the given stage because, as time elapses, the future goods gradually mature to a stage closer to present goods.

If the factors of production are added proportionally, the hypotenuse can be roughly represented by an exponential curve. However, this model is so stylized that even the linear line would display the most important properties of the Austrian approach. In our analysis onwards, the linear hypotenuse will be used as a reasonable approximation. A property more important than the exact shape of the hypotenuse is the fact that the more remote the stage from the final consumption goods, the lower the value of the intermediate product.

Furthermore, the beginning of the triangle is not very important either, since it is impossible to find where and when the particular process of the creation of the consumption goods started. ${ }^{20}$ Hence, it would be more sensible to depict the triangle with an open beginning. This approach is also supported by the fact that the history of the production process is never important, all processes are in essence forward-looking. The entrepreneurs never look backwards when they buy intermediate products and other capital goods. They never ask about the past of the particular capital good (Mises 1996). The only important thing is to optimally combine factors of production and choose the optimal length of the production process to maximize profits.

What is of particular importance is the slope of the triangle. It displays the value difference between two stages of production. As has been already demonstrated, the future goods have, as a rule, lower value than present goods. Hence, the further the stage is from the final consumption stage, the "more future" goods the particular stage represents, and the lower its value is. Hence, this omnipresent value difference results in the fact that the value of intermediate products gradually increases as the process approaches final consumption goods. Furthermore, the condition of no-arbitrage requires that the value difference (in percentage terms) is the same for every stage in the production process, reflecting also the time period necessary to manufacture the given intermediate product.

Another important dimension of the Hayekian triangle is the length of the horizontal line representing the duration of the roundaboutness of the production process. As was stated above, the approach is always forward-looking. Moreover, if we utilise the knowledge about the diminishing marginal productivity of the roundabout processes and the limit for further lengthening imposed by the first two causes for the agio, it is perfectly clear that the lower the agio between present and future goods, the more roundabout the given process is. And it is exactly this agio that creates the phenomenon of interest in the economy.
To put it in other words, if people are more patient (if their time preference is low), they are prepared to wait for the consumable output for a longer period of time, and the production process can be longer. Such a process, depicted in Figure No. 3, has a lower slope of the hypotenuse since the lower impatience narrows the agio between present goods and future goods, and it decreases the rate of interest in the economy. As a result, this process is more

[^7]roundabout.


Figure No. 3, Production process for a low interest rate

Conversely, if the valuation of present goods far exceeds the valuation of future goods (if people have rather high time preference), the slope of the triangle should go up, and the total roundaboutness of the production process will diminish since the interest rate in this economy will be high. This process is sketched in Figure No. 4.


Figure No. 4, Production process for a high interest rate


Figure No. 5, Decreasing marginal productivity of the roundabout process

Nonetheless, these two figures may be quite misleading. At first glance, it seems that the second process will end up with a higher amount of final consumption goods. Yet, as was discussed at some length before, the exact opposite is true. The triangles are constructed in nominal terms, so the size of the vertical leg does not reflect the true overall output of real consumption goods. The picture more in line with the Austrian theory is depicted in Figure No. 5. Here, the final output, represented by the vertical leg, is reported in real terms rather than in nominal terms and, as can be seen, the more roundabout processes lead up to a higher
output. Again, the triangle mirrors the decreasing marginal productivity of the additional lengthening of the production process, otherwise the economy would be unstable.

It may be instructive to look inside the Hayek triangle again. Our analysis indicated that the production process might be rather fragile. Every stage must be perfectly connected to the previous one. Once one phase is disturbed or stopped, the entire process collapses, and consumption goods are never released.


Figure No. 6, Elimination of one stage in the production process

Nevertheless, the price system and the profit-seeking behaviour should moderate the catastrophic scenario if one stage of the process is eliminated. In Figure No. 6, we can see the elimination of one stage inside the production process. However, the stage just before the eliminated one in this particular production process should experience a dramatic fall in demand, leading to a corresponding drop in prices of its products. On the other hand, the stage just after the eliminated one suffers from an immense shortage of inputs, which will push prices of the intermediate products of the eliminated stage sky-high. Hence, the price difference between the stages skyrockets, creating above-normal profits to any entrepreneur considering the entry to the stage that dropped out. As a result, this suspension of the production at one stage should be replaced very soon by entrepreneurs looking for the profit. In the end, the smooth process of production should be resumed. Once the undisturbed process is re-established again, the value difference between two subsequent stages will be the same for the entire production process; hence, one single interest rate will again rule in the economy.

The Hayek triangle introduced above is a special type of a process known as the continuous input-point output model. The final consumption goods are gradually made by adding factors of production, and all capital goods have only the form of intermediate products - the goods in process. As will be clarified later on, according to Hayek (1941), such a model is especially appropriate to describe the essence of the capital theory.
Moreover, it was also assumed that the only material input at every stage comes from the preceding stage, and no other resources are used apart from land and labour. Yet, such an approach is far remote from reality. Thus, let us suppose that some stage uses also products from other processes. It is obvious that many of these products had to undergo a timeconsuming journey as well. Hence, as can be seen in Figure No. 7, the image of the economy starts to be much more complicated. Formally, the time needed to produce the intermediate product that is finally used in the major process must be also added, which makes the exact expression of its length even more difficult.


Figure No. 7, Input being made in some other roundabout process

A material input from other processes can be inserted into any stage, so the overall picture of the structure of production will become infinitely complex and almost impossible to comprehend. Moreover, the price system must also guarantee a reasonable synchronization in the sense that the required input for the major process must be prepared at the proper time. It is also obvious that the longer the period that is needed to produce the given input (t2 in the picture), the higher ability and accuracy on the part of entrepreneurs is required to keep the smooth course of the production process. Price signals probably play a vital and fundamental role in this synchronization.
Furthermore, the foregoing analysis left aside the durable capital goods and durable consumption goods. The economic essence of a durable consumption good was first illustrated by Jevons (1957:231) who connected an additional triangle to the first one. Such a process is a type of a continuous input-continuous output schema. His approach is shown in Figure No. 8 in which the decreasing shape of the second triangle reflects the flow of services of this durable consumption good that gradually dies out over time.


Figure No. 8, Continuous input-continuous output schema

Suppose that the agio between present goods and future goods - the interest rate - declines. This will also make the second triangle flatter. Two fundamental reasons can be found for such a change in the shape. First, the flow of services will be discounted at a lower rate. And secondly, lower impatience in the production process usually leads to the production of consumption goods with higher durability. This fact was documented, for example, by Fisher (1930). Figure No. 9 represents (in real terms) two continuous input-continuous output processes, where the flatter one is consistent with a lower interest rate.


Figure No. 9, Continuous input-continuous output schema (in real terms) for lower time preferences

The inclusion of a durable capital good, which can be used at any stage of the production process, will bring about another complication to this analysis. First of all, the durable capital good is not entirely used up during one process, since it provides services continuously over time. And secondly, the durable capital good itself is a product of a time consuming process, whose durability usually also depends on the time spent on its creation.
However, with regard to the basic properties of the production process in the real world, even the simple Hayekian triangle of continuous input-point output can provide us with important insights. First of all, even this simple picture clarifies that capital cannot be considered as a homogenous amorphous aggregate. ${ }^{21}$ It does not make much sense to add together the value of all products at a different stage of processing. So any attempt to make a direct correspondence between the value of capital and its size is utterly futile. Compared with labour and land, the quantity of capital changes if the market data (i.e. prices of materials or even the interest rate) change. For the given set of prices and the interest rate, one production process may represent a higher amount of capital, whereas for another set of prices and the interest rate the calculation of the aggregate value may indicate that some other is of higher quantity (Hayek 1935b:242). ${ }^{22}$
Hence, the Austrian authors consider capital as a heterogeneous phenomenon, whose total value is of little importance. The economic science should explore its components and the relationships between them rather than the aggregate quantity of capital (Hayek 1941:6, Lachmann 1956:2). Only a thorough analysis of the relationships between different capital goods, their potential complementarity and consistency for further production processes, may provide us with the necessary insight about the functioning of modern and complex economies. It will be seen that such an approach will be crucial in analysing the business cycle phenomenon. Contemplating capital as an aggregate and amorphous mass will rather blur the true running processes. In this connection, Hayek offered the following definition of the supply of capital:
The datum usually called the " supply of capital" can thus be adequately described only in terms of the totality of all the alternative income streams between which the existence of a certain stock of nonpermanent resources (together with the expected flow of input) enables us to choose. (Hayek 1941:147)

[^8]It was also Hayek who stressed another feature of capital, deliberately disregarded or even rejected by other economists, most prominently by Frank Knight. For Hayek, capital is a set of non-permanent resources that are necessary to be maintained, restored or replaced in order to keep the flow of income (or the output of consumption goods) at some given permanent level (Hayek 1941:88; 1936a:201). Although the capital goods are able to provide the permanent flow of income, they are not permanent resources per se. Only after the proper maintaining of capital goods, is the producing power of capital secured.
On the other hand, according to Knight (1934:264) the replacement of capital goods is just a technical detail that is not worthy of any economic analysis. However, even the simple picture of the continuous input-point output process in Figure No. 6 shows that such an approach is highly problematic. If any stage of the production process is not maintained, the smooth flow of consumption goods is greatly endangered. Furthermore, it is not an easy task for the entrepreneur to decide where the factors of production should be allocated and in what amounts. It cannot be taken for granted that the necessary investment will be always made in necessary amounts. Hence, the maintenance of capital is not a simple technical detail without any reference to human action. Without a planning entrepreneur and a working price system, the replacement of capital goods is never guaranteed.
Furthermore, Hayek (1935b) demonstrated in discussion with Knight that even the concept of maintaining capital intact is problematic without the reference to the flow of income secured by the given set of capital goods and preferred by the acting man. According to Hayek, if data change, sometimes the reduction of capital, sometimes its increase (in value terms) is essential to keep the flow of income from capital at the previous level. In other words, a mere maintenance of capital in value terms may not be sufficient to maintain its productive power. Moreover, as the capital goods are usually replaced by different units, it seems to be highly problematic to distinguish what part of the act of investment represents a mere replacement of the worn out capital stock and what is associated with a net increase in the entire capital stock.
For Knight, only the net increase of the entire capital stock (in the whole society) should be at the focus of the economic science. The mere replacement of depreciated capital goods is just a simple technical datum. For Hayek, on the contrary, the decision of an acting entrepreneur about the new investment to his capital stock is inseparable form the decision of a "mere" replacement. As a result, the theoretical separation of the replacement of capital goods and their net increase is of little importance. It is rather a symptom of unsound economic reasoning. Both acts require a calculating entrepreneur who bases his decision on relevant prices and the interest rate. As a result, the "mere" replacement of capital goods is never a simple technical datum, because it is not automatically guaranteed from the flow of returns to capital. A thorough economic calculation on the part of the entrepreneur is essential as in other problems studied by the economic science.

Let us focus on the third controversy between Hayek and Knight that will be important for our further analysis and that can be easily represented by a simple tool of the Hayekian triangle. So far, we have been analyzing just one production process of some given length and hence productivity. In a stationary economy, or what Mises (1996) called the evenly rotating economy, the given process and the given operations of refining the intermediate products are repeated in the same way by the same methods. Hence, the cross-section image of the economy is much the same as the image over time. A very similar set of capital goods is created every moment, either in the present, in the past, or in the future. At the same moment, the entire range of various intermediate products of different degree of processing is being produced. Moreover, many processes at different stages of completion are actually on the
way. Some processes just mature into final consumption goods, others are almost at the end, some are in the middle, and also processes can be found that have just been initiated.


Figure No. 10 Spurious synchronization of the production processes

This picture of capitalistic stationary economy can be delineated by Figure No. 10, which is a simplified version of a three-dimensional model developed by Hayek (1941:117). ${ }^{23}$ At first glance, one may be easily trapped by a fallacy we may call the "Clark-Knight fallacy". Cutting this system of triangles at one particular moment, it seems that the production process is of negligible length, as if the entire process elapsed during one instantaneous moment the production and consumption are synchronous. However, such an interpretation is just apparent. Intermediate products never turn into consumption goods immediately without incurring any production time. The exact opposite is true. The processes just initiated will mature after a considerable period of time (see triangle 4), others, which are in a higher degree of completion, will lead up to final consumption goods earlier (e.g. triangle 2 or 3 ). It can be also deduced that a negligible part of processes that will mature in the very remote future have been also started.

Furthermore, the process that has just matured in final consumption goods was not and could not be initiated one instant moment in the past. Even though it is impossible to trace its entire history, the majority of operations on present consumption goods were made in the past. The relative importance of previous stages obviously depends on the degree of the roundaboutness of the process, which itself corresponded to the time preference of people. This also determined the productivity of the process and the eventual amount of the final consumption goods produced by this method. ${ }^{24}$

By focusing again on all the processes that are underway in present, it is obvious that only a small fraction of the original means of production (i.e. labour and land) are devoted for

[^9]completing and finishing final consumption goods. As can be seen, especially the labour force is allocated into various processes maturing at different dates in the future. This observation should also be consistent with the relative value of the various forms of output manufactured by labour. In other words, the total value of intermediate products at some given time should be much higher than that of the final consumption goods. We may say that the major part of the potential wealth of the society takes the form of products that will mature some time in the future rather than the form of finished consumption goods.

Another important fact is that although only a minority of workers are allocated in the final consumption stage, their output is notably higher than if the total labour force was devoted to creating output directly. The reason obviously lies in the higher productivity of the roundabout methods, whose fundamental advantage is that the labour and land, implemented in the past, are somehow "stored up" in the intermediate products that are to be finished within the given period (Böhm-Bawerk 1891; Wicksell 1977a). Another noteworthy observation, which is a logical consequence of the foregoing analysis, is that the richer the society and the more roundabout methods it uses, the higher the proportion of unfinished goods on the entire amount of goods actually in existence. ${ }^{25}$
When we extend the analysed period to (for instance) one year, it is obvious that total expenditures on unfinished goods of various forms are considerably higher than the total expenditures on final consumption goods, even though the price of the consumption goods (the vertical leg) is of the greatest height. The reason rests in the fact that future consumption goods (having the form of raw materials, semi-finished goods, goods in wholesale stocks, etc.) change hands many times before they mature into the form prepared for direct consumption (Hayek 1935a).

Several observations made so far, especially about the interest rate, can be illustrated by the neoclassical loanable funds model introduced to the Austrian analysis by R. Garrison (2001). ${ }^{26}$ Even though the theory of capital should be treated in the dynamic environment, this model may be helpful in disciplining one's reasoning as it can be easily trapped in vicious circles.

Garrison (2001:50) plotted a simple Marshallian diagram where the downward sloping investment curve intersects at one point with the upward sloping saving curve. This intersection depicts the equilibrium on the loanable funds market. Before proceeding to the equilibrium price and elucidating its connection to the Hayek triangle, let us briefly discuss forces that can be hidden behind both curves, since by drawing this simple diagram of supply and demand, the problem of the interest rate determination is not solved but rather established. As Fisher put it:

To say that the rate of interest is fixed by supply and demand is merely to state, not to solve the problem. Every competitive price is fixed by supply and demand. The real problem is to analyze the particular supply and demand forces. (Fisher 1930:46)

The upward sloping saving curve may represent the gradually increasing rate of time preference (or impatience), as more present goods are offered for longer processes. Thus, only an increase in the rate of interest will persuade a saver to postpone additional doses of

[^10]consumption to the future. Such an explanation is consistent with the first two causes outlined by Böhm-Bawerk. Nevertheless, a more technical approach will be introduced in Chapter 3.

The decreasing investment curve may be consistent with the third reason for the existence of interest and may reflect the diminishing marginal productivity of the roundabout processes. Every extension of the roundabout process will result in a higher output of consumption goods; however, the marginal increments gradually decline. Hence, if the interest rate is too high, it is not profitable to continue with further lengthening, it is not profitable to make much investment.


Figure No. 11, The loanable funds market and the Hayek triangle

As we can see in Figure No. 11, even though the investment curve is always above zero, which accounts for a non-negative marginal product approaching zero only with an infinite extension of the production process, ${ }^{27}$ this undue lengthening is blocked by the time preference that condensed the first two causes for interest. In other words, the lengthening is limited by the height of the interest rate ruling in the economy.

Interestingly, this pure Fisherian approach would not presumably be accepted by any major author writing in the Austrian tradition. Mises (1996), Rothbard (2004), but also Garrison (1979), who himself introduced this model into the Austrian theory, all adhere to the approach of Frank Fetter (1902; 1928) who accentuated the time preference as the sole determinant of the interest rate and who denied any role of the roundaboutness (i.e. productivity) in determining the interest rate. For these authors, the productivity is of secondary importance since its change may affect the interest rate only temporarily. In the near future, the interest rate should return to the level that is solely determined by the time preference. Moreover, according to these authors the founder of the Austrian capital theory - Böhm-Bawerk(1890; 1891) - was self-contradictory with regard to the fact that he persuasively demolished the (naïve) productivity theories in his first book, by reintroducing the productivity element in the second book under the disguise of the roundaboutness phenomenon (Fetter 1902). ${ }^{28}$

On the other hand, Hayek (1941; 1936b) favoured the productivity element, accepting the objections of Frank Knight (1936a; 1936b) especially against the undervaluation of future wants. Although Hayek (1945) finally modified his view by attributing the crucial importance to time preferences at the end of the unsustainable boom, his emphasis on the marginal productivity of investment in determining the interest rate is ubiquitous in his theory.

[^11]Chapter 3 of this dissertation will deal with this controversy in more detail. At this point, we will accept the compromise version, maybe most consistent with the original BöhmBawerkian exposition, which is also consonant with the Fisherian analysis. ${ }^{29}$ Hence, both elements - time preference and productivity - will retain their importance in determining the interest rate; yet, the loanable funds model as such will be utilised mainly for expositional reasons.

Nevertheless, even this simple approach suggests that the interest rate is a real phenomenon as it is determined by the flow of real saving and real investment. Moreover, the intersection of the two curves is of crucial importance. This equilibrium interest rate, at which saving is in line with investment, is usually called the natural rate of interest. The definition most often cited is the one of Knut Wicksell:

The rate of interest at which the demand for loan capital and the supply of savings exactly agree, and which more or less corresponds to the expected yield on the newly created capital, will then be the normal or natural real rate. (Wicksell 1977b:193)

Wicksell modified his definition originally given in his earlier work:
There is a certain rate of interest on loans which is neutral in respect to commodity prices, and tends neither to raise nor to lower them. This is necessarily the same as the rate of interest which would be determined by supply and demand if no use were made of money and all lending were effected in the form of real capital goods. It comes to much the same thing to describe it as the current value of the natural rate of interest on capital. (Wicksell 1936:102)

This interest rate should also reflect the agio between present goods and future goods. It must be consistent with the slope of the Hayek triangle (Figure No. 11). Suppose, for example, that the rate on the loanable funds market is lower than the one in the Hayek triangle, which means that the value difference between present consumption goods and present factors of production is higher than the interest rate on the loanable funds market. ${ }^{30}$ Then, there is a motivation to borrow on the loanable funds market and reap this profit opportunity. This demand on the loanable funds market will drive the interest rate up, which will eventually stabilize at the level ruling in various processes of the structure of production.

For some Austrian authors, the loanable funds market is just of secondary importance compared with the value difference between present goods and future goods (Salerno 2001). Hence, this model may be used to reflect this agio. Nevertheless, we will see that in analysing some features of the business cycle, it sheds more light on various complicated processes than the clumsy (but at least partly dynamic) Hayek triangle. We will therefore utilise both tools, keeping in mind their limitations and drawbacks.
Furthermore, a variable as important as the natural rate can be also found on the horizontal axis. As can be seen in Figure No. 11, for the given natural interest rate, the flow of real investment equals the flow of real saving. It is assumed that this amount of saving is exactly the one that is needed to keep the production process intact. In other words, the economy

[^12]creates precisely the amount of new capital that is required to maintain a continuous and undisturbed process of production, to preserve the given shape and size of the Hayek triangle. The length of the production process is optimal as the given time-extension of production is perfectly consistent with time preferences of people. We may say that at the margin, the first and the second ground for interest are in line with the third ground.
At the end of this section, let us take a closer look at the Hayekian triangle. We will develop a simple system of graphs, which will be useful in analyzing processes when the economy is being restructured either due to real or monetary causes.


Figure No. 12, Markets of production goods at different stage of completion

The processes inside the Hayekian triangle are depicted in Figure No. 12, which illustrates Hayek's idea that capital goods are mainly production goods in process $\left(\mathrm{Q}_{\mathrm{i}}\right)$ that finally mature into consumption goods $\left(\mathrm{Q}_{\mathrm{c}}\right)$. Each good in process $\left(\mathrm{Q}_{\mathrm{i}, 1} ; \mathrm{Q}_{\mathrm{i}, 2} ; \mathrm{Q}_{\mathrm{i}, 3} ; \ldots\right)$ has its own market with its own market price. At the same time, the price of the production good that is posited further from the final consumption is lower than the price of the production good that is closer to it. The price margins between goods in the different phase of completion reflect the slope of the Hayekian triangle. They are also consistent with the natural rate of interest.

The supply side of each market is represented by companies or entrepreneurs operating in different stages of production. Although the demand on each market is driven by firms producing in the subsequent stage, the key agents of the demand side on each market are to be found somewhere else, namely at the end of the entire production process. In identifying the major drivers of the demand for production goods, one has to realise the economic essence of these goods. As has been already demonstrated, production goods are nothing less and nothing more than future consumption goods. Hence, the demand for them originates on the part of individuals who demand future consumption goods. And the major demanders are people saving part of their income since saving is just the demand for consumption goods that will mature at some moment in the future.

## 4. DECREASE IN TIME PREFERENCE

The simple tools just developed suffice to describe the nature of the process that transforms the Hayek triangle from one shape to another (Figure No. 3 and No. 4) and shed some light on what is known as the change in the roundabout method of production.

There can be no doubt that every individual tries to adjust her time shape of consumption so as to maximize her utility. As was demonstrated by Fisher (1930), the specific stream of consumption of the individual depends on factors such as the time shape of her income, average level of her income, composition of income, degree of risk in different periods, and
on many others. At this stage, we are interested neither in the major determinants affecting the specific stream of consumption of an individual nor in the response of the individual to these factors. The main objective of this section is to find out whether the Austrian theory is well designed to analyse mechanisms in the structure of production and the roundabout processes, once a large number of individuals change their optimal allocation of consumption over time.
Consider a change in the optimal path of consumption such that a remarkable percentage of consumers reshuffle their stream of income on behalf of consumption in more remote periods in the future at the expense of consumption in present or in the near future. In other words, suppose that in the economy the relative demand between present consumption goods and future consumption goods is shifted toward the latter at the expense of the former. To put it plainly, people start to save more.
The reduced demand for present consumption goods will be first observed in the stages producing goods for immediate usage and in the stages very close to final consumption, whereas earlier stages creating production goods should experience an increase in demand as the new savings, representing demand for future goods, are channelled to this part of the production process. However, new savings would never find their way to the earlier stages of production if the interest rate did not decrease. In other words, the initial phase of this mechanism hinges on the assumption that increased saving will result in a decrease in the interest rate. This decline in the interest rate will encourage investment spending, as is depicted in Figure No. 13. If something blocks the smooth transformation of saving into investment, further steps of the mechanism cannot proceed, and the structure of production can never be fully transformed. ${ }^{31}$


Figure No. 13, Decrease in time preference in the loanable funds market

Nevertheless, if this process operates without disturbances, the inflow of saving into earlier stages will show up as an increase in the demand for production goods, and the resulting higher price will encourage supply to meet this expanded demand (Figure No. 14). The crucial question is, however, whether the economy is endowed with enough factors of production such that the supply can at least partly respond to higher demand or whether the supply of production goods is fixed due to the lack of essential factors of production. It should be stressed that there is no reason to believe that at the beginning of the processes there was a considerable amount of free material resources or unemployed labour force. Hence, at this stage, we assume that the economy operates at its potential or full-employment level where the economic scarcity is a prevalent feature throughout the economy. The analysis of an

[^13]economy with idle resources and labour force will be postponed to Chapter 2.
The answer to the question above is to be found at the end of the production process, in the consumption stage, or in the stages very close to final consumption which suffer from the diminution of the consumers' demand. Sooner or later the lower demand will cut prices of consumption goods (Pc). Together with the inevitable reduction in the amount of sales of the consumption goods, their lower prices should depress profits of the firms producing consumption goods. As a direct consequence, the factors of production in this stage of the production process will experience a fall in the value of their marginal product. This is how the market manifests that the factors of production are not worth as much as before in the consumption stage. The market also demonstrates that in some sectors of the consumption stage, consumers value inputs more than the resulting output, and lower profits or even losses are a direct proof that some of the factors of production should be released. ${ }^{32}$

As a result, here is to be found the source of factors of production that are eagerly demanded in the earlier stages of production. As is depicted in Figure No. 14, the decrease in the consumption demand should release labour and other resources from this stage of the production process. ${ }^{33}$ At the same time, a higher demand for labour in earlier stages of production may absorb it. If this truly happens, then the supply of consumption goods declines and the supply of production goods rises. However, this process is conditioned upon a well-functioning price system that ought to reflect not only the increase in savings via a lower interest rate but also a decrease in the consumption demand through a decline in the prices of consumption goods (Pc). ${ }^{34}$ In addition, the price system must also reflect an increase in the demand for production goods via a rise in the prices of production goods ( Pi ). As regards the labour markets, the structure of wages is affected in such a way that earlier stages of production should experience a rise in wages and stages closer to consumption a decline in wages, which is the straightforward signal for labourers to move from one sector to another.

At this moment, several observations deserve our attention because they are of fundamental importance. First, as can be seen on the loanable funds market, the natural rate of interest declined. Secondly, the slope of the Hayek triangle is lower as the demand for consumption goods plummets. Moreover, the triangle is also longer, which stems from the fact that a lower interest rate motivates entrepreneurs to increase the roundaboutness of the production process even by opening some very long processes of production, which were impossible to start before. This observation is consistent with the fact that the increase in the demand for production goods is more robust in stages of production furthest from the final consumption. ${ }^{35}$

[^14]

Figure No. 14, Decrease in time preferences and the structure of production

Thirdly, the profit margins, represented by the difference in value between goods at different stages of production, are also lower since there is a general tendency to equalize the profit rate with the interest rate. Fourthly, although it is not immediately obvious from this simple picture, the capital in the economy has not only changed its size but also its shape and structure because more factors of production are allocated to relatively more remote stages of production. Furthermore, some very remote stages of production have been even created anew.

However, as was stated by Hayek (1941:78), the (average) roundaboutness in the economy may be extended even if no individual process was lengthened as such. It may suffice to release necessary inputs from shorter processes (or industries closer to the consumption stage) and subsequently absorb them in longer processes, which will increase production. On technical grounds, every industry continues in the same line as before, using the same methods. Nonetheless, the total structure of production has been changed as more inputs are now invested in longer processes and in industries operating further from final consumption.

It may be convenient with respect to further exposition of the business cycle to elucidate processes in the labour market. It might seem suspicious that the total demand for labour is not diminished when the demand for consumption goods declines. Such reasoning would be perfectly consistent with the theorem of the derived demand. On the other hand, the classical
statement of John Stuart Mill, which was studied by Hayek (1941:436) in some detail in this connection, posits that demand for commodities is not the demand for labour.

Garrison (2001) developed an innovative approach that integrated both points of view. He stated that throughout the entire structure of production, there are two effects in operation the effect of derived demand and the time discount effect. The first one pushes the demand for labour downward, when the demand for consumption goods declines, and is strong in very late stages of production and in the final consumption stage. On the other hand, the time discount effect is affected by changes in the interest rate and is effective in the early stages of production that are highly sensitive to the interest rate.


Figure No. 15, Labour markets in various stages of the structure of production

The idea of time discount in the market for factors of production, which was thoroughly elucidated by Hayek (1941) and Rothbard (2004), can be immediately used in analysing various labour markets at different stages of production. The demand for labour in every labour market depends on the marginal value product of labour (MVP); however, in the stages very far from the final consumption, this value of marginal product must be discounted at the ruling interest rate because the output of labour will mature into consumption goods after a long period of time. The further the given stage is from the final consumption, the more that particular MVP is discounted.

Hence, every entrepreneur, who is maximizing profit, must equalize the ruling wage with the discounted marginal value product (DMVP), not just with the simple MVP. The system of graphs in Figure No. 15 illustrates this idea in detail. As can be seen in the picture, the schedules of MVP might be the same in all stages; yet, markets differ a lot in the position of DMVP - the further the stage from the final consumption, the lower the discounted marginal value product. Line AA, which connects the system of demands for labour and which represents the time discount effect, is consistent with the given slope and shape of the Hayek triangle. However, provided that the labour force is non-specific and relatively mobile, an identical wage must rule in the entire structure of production.

This simple schema may resolve the puzzle mentioned in Block (1990). A different discounted marginal value product at various stages of the structure of production might seem inconsistent with the assumption of identical wages that result from arbitrage across labour markets. However, the DMVP depends not only on the size of the time discount and on the position of the MVP but also on the amount of labour hired. Due to the diminishing marginal productivity of labour, the MVP and DMVP fall with more labour employed. As a result, we
must distinguish between the position of the entire DMVP curve and one particular point on this curve. The arbitrage of labour guarantees that the DMVP and wage are identical in all stages of the production process even though the position of the DMVP curves differs.
Now suppose that people start to save more. This will reduce the demand for consumption goods, and it will also decrease the natural rate of interest. As was demonstrated before, the price of consumption goods declines, which must immediately diminish the MVP. However, the impact on the DMVP, which is crucial for the demand for labour in each particular market, is ambiguous. As can be seen in Figure No. 16, the DMVP declines in stages close to final consumption, whereas it rises in the stages in which the MVP is very discounted. In the first case, the effect of derived demand dominates the time discount effect since the decrease in the interest rate is not strong enough to compensate for the decline in MVP. However, for stages very remote from the final consumption, the second effect is of crucial importance, leading to the fact that the DMVP moves in the opposite direction than the MVP.

We can also see that the connecting line AA changes its slope reflecting the new and lower natural rate of interest. It is obvious that a decline in the demand for consumption goods has an ambiguous effect on the demand for labour. The labour demand is lowered in stages very close to the consumption stage, but it is definitely raised in early stages of the production process; in stages that are producing production goods or capital goods of various forms that will mature in final consumption goods at some remote date in the future. The same evolution as for the demand holds also for wages - they are raised in early stages and diminished in the later stages.


Figure No. 16, Labour markets in various stages of the structure of production - decrease in time preference

As can be seen in the diagram, there is one particular stage in which the effect of derived demand is perfectly offset by the time discount effect. Its exact position can be identified by a simple model of present value: ${ }^{36}$

[^15]$P V=X /(1+i)^{n}, \quad$ where $X=p . q$
$p$ is the price of a final consumption good, $q$ is its quantity, and $i$ is the interest rate ruling in the system. The decline in consumption demand reduces price $p$, but also the interest rate $i$. It is obvious that the longer it takes for the given project to mature into final consumption goods (i.e. the higher the variable $n$ ) the more important is the effect of the interest rate on the present value compared with the effect of price. The particular $n$, for which both effects offset each other, can be derived as follows. The total differential of PV is given by:
\[

$$
\begin{align*}
& d P V=(\partial P V / \partial X) \cdot d X+(\partial P V / \partial i) \cdot d i  \tag{1.2}\\
& d P V=(1+i)^{-n} \cdot d X-n \cdot X \cdot(1+i)^{-n-1} \cdot d i \tag{1.3}
\end{align*}
$$
\]

Both effects compensate each other if the change in the PV is zero. Hence:

$$
\begin{align*}
& d P V=0 \\
& (1+i)^{-n} \cdot d X=n \cdot X \cdot(1+i)^{-n-1} \cdot d i \\
& d X / d i=n \cdot X /(1+i) \tag{1.6}
\end{align*}
$$

$n=[d X / X] .[(1+i) / d i]$

For instance, if the initial interest rate was $5 \%$ and declined by one percentage point and if the price of final consumption goods was reduced by five percent, then a straightforward calculation gives us that $\mathrm{n}=5.25$. Hence, the projects that will mature in five years or more will benefit from the increase in saving. From this stage further to the early stages, the demand for labour increases.

If the labour force is sufficiently mobile and unspecific, the situation in Figure 16 is not sustainable. A wage difference between stages will attract the labour force to early stages of production, and, on the other hand, it will motivate labourers to leave the stages that are very close to final consumption. This process allocates the labour force to longer processes and makes the lengthening of the structure of production possible.

The final picture of the economy might be close to Figure No. 17, which depicts a new state of rest. As can be seen in this picture, stages far from the final consumption stage attracted labour force at the expense of later stages. Some labourers could even find the job in very roundabout processes that were created anew owing to the decrease in the natural rate of interest (Garrison 2001). ${ }^{37}$
When the labour force is reallocated, the eventual size of the nominal wage, which is equal in all labour markets, is hard to determine. It is true that the height of the Hayek triangle is lower, so the nominal value of the product, out of which wages can be paid, is lower (Rothbard 2004). On the other hand, the interest rate is lower as well, so the fraction of output that can be attributed to labourers is higher (Hayek 1935a). These two effects go against each other; hence, the final size of the nominal wage cannot be determined by this simple model.

[^16]

Figure No. 17, Final equilibrium in the labour markets

A more important question is the size of the real wage. Since the prices of consumption goods declined, it might increase. However, what concerns us here is the size of the real wage when the new longer processes are completed and when they start to provide consumption goods. To answer this question, we will develop a simple model that is based on Figure No. 1 with two methods of different roundaboutness. This simple picture will not only compare the productivity of two processes of different length, but it will also clarify the behaviour of the economy in the transition period.

Suppose that the economy is endowed with five units of labour, and it uses only direct methods. Every labourer produces one unit of output after one period. The total amount of consumption goods in this simple economy is therefore five at time $t-1$. As can be seen in Figure No. 18, the restructuring to longer processes starts at time $t$. We assume that the labour force will be transferred to longer methods only gradually. At time $t$, one unit of labour is allocated to the process that will mature in five periods, i.e. to the roundabout process in Figure No. 1.
At time $t$, the output of consumption goods falls to four units since only four direct processes provide consumption goods. At time $\mathrm{t}+1$, an additional unit of labour is devoted to the roundabout process, and the total output of final consumption goods declines to three. Three units of labour make consumption goods in direct processes, one unit of labour works on the intermediate product II, and one unit of labour started to work on the intermediate product I.

At time $t+2, t+3$, and $t+4$, the remaining units of labour are allocated to long processes, and the output of consumption goods gradually falls to its minimum at time $t+3$. The cause of this decline has been already discussed, and it is quite easy to be read from the picture. The labour force is diverted from processes that give consumption goods directly, so their output falls. Furthermore, the longer and more productive processes, to which the labour has been allocated, have not provided final consumption goods yet - they have not matured.


Figure No. 18, Gradual restructuring of production processes

However, this simple diagram characterises only production, not consumption. Thus, the total consumption may differ from the total production of consumption goods. For example, the
actual consumption in the transition period can be higher than the actual production if in the periods before the transition started enough consumption goods have been saved and not consumed out of the output of that particular period. After all, it is the reduction in the consumption demand that initiated the lengthening of the production process.


Figure No. 19, Evolution of output of consumption goods, consumption, and savings

One possible path of consumption is depicted in Figure No. 19. Here, the reduction of the consumption demand occurred two periods before the methods of production started its restructuring. As can be seen, consumption is below actual output for three periods (from t-2 to $t$ ). Three units of consumption goods are gradually accumulated, and they are subsequently used in periods of minimum output ( $\mathrm{t}+2$ and $\mathrm{t}+3$ ). This simple picture also preserves a usually required assumption of (reasonable) consumption smoothing. However, more on this will be said in Chapter 3 in which we develop a more rigorous dynamic model. Nevertheless, the Hayek (1941) words about the unsold consumption goods that can be utilized, when the process is in transition, seem to be quite plausible.
As can be seen in Figures No. 18 and No. 19, at time $t+4$ the first roundabout process starts to provide final consumption goods, and the total output rises from one to 10 . The higher output and consumption might be then maintained also in the future periods. Furthermore, the increased demand for future consumption goods, which was reflected by higher saving in the past, is eventually met by a higher supply of these goods. Thus, in the Austrian theory, the price system is able to deliver the information from consumers to entrepreneurs, and it may accordingly shape the structure of the production process such that the production plans of firms are consistent with the intertemporal preferences of consumers.

Another important fact is that eventually only one fifth of the entire labour force is devoted to the production of final consumption goods. Initially, it was $100 \%$; however, in these short processes the total output of consumption goods was much lower than in the long one - at the aggregate level, five units in short processes and 10 in the longer, or one unit of output per
one unit of labour in the short and two units in the long methods. Furthermore, a decisive part of the labour force is now devoted to the production of various forms of the intermediate products I,II,III,IV, which will mature into consumption goods in different periods in the future. ${ }^{38}$ We can see that the majority of "wealth" in the economy has the form of the unfinished goods rather than final consumption goods. And finally, the cross-section picture of the economy at time $t+4$ may easily trap one's sound reasoning. It appears that the total output of consumption goods is produced within a single period out of the intermediate products I,II,III,IV of this particular sequence. However, Figure No. 18 demonstrates even more clearly than before the fallacious approach of Frank Knight about synchronous production. It can be perfectly seen that it takes five periods to produce final consumption goods, so the phenomenon of time is of the utmost importance in the production analysis and the theory of capital.

From time $\mathrm{t}+4$ onwards, the output of consumption goods is permanently higher. This allows us to answer the question stated above. In the end, the real wage should be definitely higher than before the transformation to more roundabout processes started. In technical terms, it is generally believed that the accumulation of capital raises the marginal product of labour, which should be reflected in the competitive markets in higher real wages. Nevertheless, in the Austrian capital theory, this phenomenon arises due to roundabout methods, which are more productive than shorter methods.
The Austrian approach can be compared with the standard neoclassical labour market model. Neoclassical labour economics uses an aggregative model, such as the one depicted in Figure No. 20. For simplicity, the labour supply is sketched as a vertical line; nonetheless, the demand for labour shifts outwards owing to the higher capital stock in the economy and the resulting greater marginal productivity of labour.

The standard approach seems to be rather sterile in exploring fundamental processes of reallocation of the labour force analysed above. We can only see that the total quantity of labour remained the same and the real wage went up. Hence, to acquire a more subtle insight, this aggregative approach might be replaced by a more detailed analysis that investigates separate labour markets at different stages of the production process. Otherwise, the essential insight could be lost in aggregation.


Figure No. 20, Increase in capital and the aggregate labour market

[^17]We may conclude that the reduction in the consumption demand and the resulting higher saving eventually benefited workers. However, the entire process is much more complicated than is usually illustrated by textbook growth models, as we will see below. The essence of the act of saving lies in the fact that factors of production are partly diverted from producing final consumption goods, and they are devoted to longer processes that will lead up to a higher amount of consumption goods, though after a longer period of time.

Böhm-Bawerk (1891: 101) in this connection raised a question whether the source of the accumulation of capital is saving or production. The foregoing analysis demonstrated that Böhm-Bawerk was right in saying that both were. Factors of production must be first saved - meaning not used in shorter processes - and then devoted to longer production processes that will mature after a longer period of time. We may also say that the act of saving necessarily means a lengthening of the period for which the particular factors of production are used in the production process (Hayek 1941). The role of consumption goods in the act of saving, which was so emphasised by Jevons and which is stressed even in modern economics, is that the reduction of demand for final consumption goods and their potential accumulation may bridge the period of transition when the processes of production are being restructured. In this respect, theories, such as the neoclassical growth models mentioned below, that assume that the saved consumption goods are immediately transformed into capital goods, are rather naïve. Such an easy transformation may hold for only an insignificant part of the entire capital stock. In the majority of cases, capital goods must be made anew with the help of the factors of production that are released from shorter processes.

The last important question that will concern us in connection with the change in the time preference is the further evolution of the economy when the structure of the production process was restructured to longer methods. Garrison (2001: 64) examined this question in some detail, and he predicted that after the reduction in the time preference, the economy could jump on the path of never-ending (secular) growth (or higher growth rate) in output of consumption goods, represented by Figure No. 21. ${ }^{39}$


Figure No. 21, Eternal growth after the decrease in time preferences

However, we added question marks to this picture since such a result seems to be highly improbable, especially if we recall and list again all the assumptions about the capital stated at the beginning. We may even ask whether the increase in output is not only temporary and

[^18]whether the economy will not return to the previous level.
Figures No. 18 and No. 19 suggest that the higher output is sustainable if the necessary amount of labour is steadily devoted to longer processes. When some part of the labour force (e.g. one unit) is moved back to the shorter process, in the period of this change and for the next three periods, the output will increase to 11 . However, in the periods that follow, the output will suddenly drop to one. ${ }^{40}$ On the other hand, if the labour force is permanently and totally allocated in longer processes, which means that the saving behaviour will remain the same as in the periods of transformation to more roundabout methods of production, the output will be sustainable at the higher level of 10 units. ${ }^{41}$

We may conclude that higher incomes lead both to higher consumption and higher saving, and therefore to higher capital, so a permanently higher level of output will be maintained. Nevertheless, this higher "saving rate" must be preserved, and the necessary investment must be made in order to keep the higher amount of capital intact - to keep the longer methods of production undisturbed.

However, if we stick to the assumption of diminishing marginal productivity of roundabout methods, it is unlikely that the growth can be eternal, as was claimed by R. Garrison (2001). Salerno (2001) in his critique of Garrison's approach stated that only a permanently decreasing time preference could lead to a permanently increasing output. This conclusion seems to be consistent with the predictions of standard growth models, as we see below.
Consider a simple Solow (1956) growth model with stationary population and zero technological progress. In this neoclassical framework, the capital goods and consumption goods are of the same type, hence it is a simple single-commodity model. ${ }^{42}$ What is saved and not consumed is immediately transformed into capital. Hence, compared with the Austrian model, the processes of capital restructuring and the essence of the creation of capital are hidden in this model in aggregation.

Point $\mathrm{k}_{1} *$ in Figure No. 22 represents the state of rest (the steady state) of capital, at which the depreciation of capital $(\delta k)$ is perfectly offset by new saving and investment ( $s_{1} y$ ). The level of consumption $\mathrm{c}_{1} *$ is sustainable and stable. Now, if people increase their saving rate (to $\mathrm{s}_{2}$ ), the consumption drops to $\mathrm{c}(0)$, and the accumulation of capital begins - the economy grows. However, due to the diminishing marginal productivity of capital, a higher saving rate as the source of growth gradually dies out, and in the end the economy finds a new steady state with higher capital $\left(\mathrm{k}_{2}{ }^{*}\right)$, output ${ }^{43}\left(\mathrm{y}_{2}{ }^{*}\right)$ and consumption $\left(\mathrm{c}_{2}{ }^{*}\right) .{ }^{44}$ This model may also uncover that the eventual real wage $w_{2}{ }^{*}$ exceeds the initial level $w_{1}{ }^{*}$ (Barro 2004: 41).
Hence, the conclusion of R. Garrison about the eternal (or secular) growth is unattainable if the marginal productivity of capital is diminishing, technological progress is zero, and population is stationary. The model could produce eternal growth only with constant marginal

[^19]product of capital, a property known from simple AK models. It seems that Salerno (2001) was right in saying that only a permanent and continuous decline in time preferences may trigger a permanent growth. Such a scenario is not totally impossible if the rate of impatience (or time preference) decreases with higher average level of income (Fisher 1930). However, with regard to the predictions of the Solow model, there is a limit for the level of consumption that can be permanently reached by ever-increasing rate of saving. Such a limit is the wellknown golden rule (Phelps 1961; 1965). Above that level, the economy over-accumulates capital, and the decrease in time preferences is no longer beneficial. This situation will be discussed in Chapter 3.


Figure No. 22, Impossibility of the eternal growth in the Solow model

The evolution of the key variables is sketched in Figure No. $23 .{ }^{45}$ As can be seen, the growth of the economy is only temporary since it will eventually find a new steady state. Furthermore, the picture of the real wage over this transition process will closely mimic the evolution of capital. Hence, in the end, it will be definitely higher. To get the evolution of the natural rate of interest, it may be more convenient to utilise the Ramsay-Cass-Koopmans model. ${ }^{46}$

[^20]
consumption, $\mathrm{c}(\mathrm{t})$


growth rate of output,


Figure No. 23, The evolution of the key variables after the increase in the saving rate



[^21]Figure No. 24, The evolution of the economy after the decrease in impatience in the RCK model

The idea of lower impatience in this model is represented by the decrease in the subjective discount rate $\rho$. The behaviour of the economy is sketched in Figure No. 24. The qualitative characteristics of the key variables are very similar to the Solow model. However, we also added the natural rate of interest, which gradually falls with the accumulation of capital and which converges to a new level of the subjective discount rate. ${ }^{47}$
This model will be further discussed in Chapter 3. Nevertheless, even the simple neoclassical models may shed some light on the process of accumulation of capital and the evolution of consumption over time. However, as they rely on the assumption of homogenous capital, some important issues are hidden. In the Austrian capital theory, we observed that the accumulation of capital, activated by the reduction in consumption demand, is always accompanied by a change in its structure (Lachmann 1956). The accumulation of capital necessitates the reallocation of the labour force to longer roundabout processes. In this particular respect, the neoclassical models are silent.

## 5. CONCLUSIONS

The Austrian theory of capital puts emphasis on the time element in the production process. As we have seen, in many respects it is superior to the neoclassical theory in which capital is considered a homogeneous mass. Furthermore, the Austrian theory of capital was mainly designed to explain the allocation of material resources that might be changing in the short run. The basic notion of capital in the neoclassical theory - the fixed capital - is not the core in the Austrian theory, because the key characteristics of capital are not present in this type of capital.
As was shown in this chapter, the centre of the theory of capital is the optimal allocation of various factors of production over time. Even though the neoclassical theory may stress the dynamic nature of various models, some ideas of high importance are neglected. On the other hand, the Austrian capital theory lacks a sophisticated mathematical apparatus of the modern growth theory. Thus, a natural evolution might be the integration of these two theories in one comprehensive model. Such cooperation might even help us understand unsettled issues in the business cycle theory that usually neglects the importance of capital creation.

[^22]
## REFERENCES

Barnett, William II and Block Walter 2006. On Hayekian Triangles. Procesos De Mercado:Revista Europea De Economia Politica; III(2): 39-141.

Barro, Robert J. 2004. Economic Growth., 2nd ed. Cambridge: MIT Press.
Block, Walter. 1990. The DMVP-MVP Controversy: A Note. The Review of Austrian Economics 4(1): 199-207.

Böhm-Bawerk, Eugen von 1890 [1884]. Capital and Interest. New York: McMillan and Co.

Böhm-Bawerk, Eugen von 1891 [1888]. Positive Theory of Capital. New York: G. E. Stechert \& Co.

Cass, David 1965. Optimum Growth in an Aggregative Model of Capital Accumulation. Review of Economic Studies 37(3): 233-240.

Fetter, Frank A. 1902. The "Roundabout Process" In the Interest Theory. The Quarterly Journal of Economics, 17(1): 163-180.

Fetter, Frank A. 1928 [1915]. Economics. New York: The Century Co.
Fillieule, Renaud 2005. The „Value-Riches" Model: An Alternative to Garrison's Model in Austrian Macroeconomics of Growth and Cycle. The Quarterly Journal of Austrian Economics 8(2): 3-19.

Fisher, Irving 1930. Theory of Interest. New York: The Macmillan Company.
Garrison, Roger 1979. In Defense of the Misesian Theory of Interest. Journal of Libertarian Studies 3(2): 141-150.

Garrison, Roger W. 2001. Time and Money, The Macroeconomics of Capital Structure, Routledge

Hayek, Friedrich A. von 1932. Reflections on the Pure Theory of Money of Mr. J. M. Keynes (continued), Economica 35: 22-44.

Hayek, Friedrich A. von 1935a [1931]. Prices and Production, 2nd edition, Augustus M. Kelly, Publishers New York

Hayek, Friedrich A. von 1935b. The Maintenance of Capital. Economica 2 (7): 241-276.
Hayek, Friedrich A. von 1936a. The Mythology of Capital, Quarterly Journal of Economics, 50(2): 199-228.

Hayek, Friedrich A. von 1936b. Utility Analysis and Interest. The Economic Journal 46(181): 44-60.

Hayek, Friedrich A. von 1941. The Pure Theory of Capital. Chicago: The University of Chicago Press.

Hayek, Friedrich A. von 1945. Time-Preference and Productivity: A Reconsideration. Economica 12 (45): 22-25.

Hülsmann, Jörg Guido 2001. Garrisonian Macroeconomics. The Quarterly Journal of Austrian Economics 4(3): 33-41.

Jevons, William S. 1957 [1871]. The Theory of Political Economy. New York: Sentry Press
Knight, Frank H. 1934. Capital, Time, and the Interest Rate. Economica 1(3): 257-286.
Knight, Frank H. 1935a. Professor Hayek and the Theory of Investment. The Economic Journal 45(177): 77-94.

Knight, Frank H. 1935b. The Theory of Investment Once More: Mr. Boulding and the Austrians. The Quarterly Journal of Economics 50(1): 36-67.

Knight, Frank H. 1936a. The Quantity of Capital and the Rate of Interest: I. The Journal of Political Economy 44(4): 433-463.

Knight, Frank H. 1936b. The Quantity of Capital and the Rate of Interest: II. The Journal of Political Economy 44(5): 612-642.

Koopmans, Tjalling C. 1963. On the Concept of Optimal Economic Growth. Cowles Foundation Discussion Papers 163, Cowles Foundation for Research in Economics, Yale University.

Lachmann, Ludwig M. 1978 [1956]. Capital and Its Structure. Institute for Humane Studies.
Menger, Carl. 2007 [1871]. Principles of Economics. Auburn, Alabama: Ludwig von Mises Institute.

Mises, Ludwig von 1996 [1949]. Human Action: A Treatise on Economics, 4th ed. San Francisco: Fox \& Wilkes.

Murphy, Robert P. 2003. Unanticipated Intertemporal Change in Theories of Interest. Ph.D. Dissertation, New York University.

Phelps, Edmund S. 1961. The Golden Rule of Accumulation: A Fable for Growthmen. The American Economic Review 51(4): 638-643.

Phelps, Edmund S. 1965. Second Essay on the Golden Rule of Accumulation. The American Economic Review 55(4): 793-814.

Potužák, Pavel 2007. Rakouská teorie hospodářského cyklu - pohled současné makroekonomie, diplomová práce, VŠE Praha.

Ramsey, Frank. P. 1928. A Mathematical Theory of Saving. Economic Journal 38(152): 543-559.

Romer, David. 2006. Advanced Macroeconomics, 3rd edition, McGraw - Hill, New York.
Rothbard, Murray N. 2004 [1962]. Man, Economy, and State. Ludwig von Mises Institute.
Salerno, Joseph T. 2001. Does the Concept of Secular Growth Have a Place in Capital-Based Macroeconomics?. The Quarterly Journal of Austrian Economics 4(3): 43-61.

Samuelson, Paul A. 1966. A Summing Up. The Quarterly Journal of Economics, 80 (4): 568-583.

Smith, Adam 2001 [1776]. Bohatství národů. Liberální institut, Praha.
Solow Robert M. 1956. A Contribution to the Theory of Economic Growth. The Quarterly Journal of Economics, Vol. 70 (1): 65-94.
de Soto, Jesús Huerta 2006 [1998]. Money, Bank Credit, and Economic Cycles. Ludwig von Mises Institute.

Strigl, Richard von 2000 [1934]. Capital and Production, Ludwig von Mises Institute.
Wicksell, Knut 1954 [1893]. Value Capital and Rent. Augustus M. Kelley Publishers.
Wicksell, Knut 1936 [1898]. Interest and Prices. Augustus M Kelley Publishers.
Wicksell, Knut 1977a [1901]. Lectures on Political Economy, Volume 1. Augustus M Kelley Publishers.

Wicksell, Knut 1977b [1906]. Lectures on Political Economy, Volume 2. Augustus M Kelley Publishers.

## Chapter 2

## The Dynamics of the Interest Rate in the Austrian Business Cycle Theory

## 1. INTRODUCTION

The recent financial crisis has brought about unprecedentedly low nominal interest rates for an unprecedentedly long period of time. At the same time, the inflation rates are stabilized at low yet positive levels. Nominal interest rates slightly above zero together with stable and positive inflation result in the fact that real interest rates observed in the majority of the most developed countries are very close to zero or they are even negative. Furthermore, since the real interest rates have been low for many years so far, the ex post and ex ante real interest rates cannot be far off from each other.
Long-lasting zero or even negative real rates accompanied by stable and low inflation seem to be puzzling for those monetary and business cycle theories that attribute rising inflation rates and economic fluctuations to the artificially depressed interest rates. One of the strongest proponents of this view is known as the Austrian business cycle theory developed by Mises (1976) and Hayek (1933; 1935) nearly one century ago.

Based on the Wicksell (1936; 1977b) theory of the natural rate of interest and the BöhmBawerk (1890; 1891) theory of capital, the Austrian theory states that the gap between the actual and the natural rate of interest should result in a baleful boom-bust cycle. Furthermore, if this interest rate imbalance is sustained for a sufficiently long period, the economy should suffer from continuously rising inflation finally resulting in a bitter hyperinflationary collapse, once this spiral gets out of control.
Although the economy went through the business cycle-like process over the last several years, currently entering a very fragile recovery phase, the inflationary pressures seem to be far lower than predicted by the Wicksellian and the Hayekian theory. Hence, this apparent puzzle is also studied in this chapter.
Chapter 2 proceeds as follows. The first two parts elucidate the dynamics of the interest rate over the business cycle predicted by the Austrian theory. They are especially focused on the transmission mechanism leading to the deviation of the market interest rate from the natural level, and the consequent reverse U-shaped behaviour of the market interest rate. The third part tries to clear up some of the misunderstandings and seeming inconsistencies that were pointed out by the critics of this theory. The next two parts investigate the dynamics of the money supply and other possible dynamics of the interest rate in the Austrian model. The sixth part questions the concept of the natural rate of output. The last part gives several suggestions of why the current evolution of the interest rate is at odds with the Austrian theory, mainly why the U-shaped evolution is not observed and the interest rates are currently stuck at unprecedentedly low levels. In particular, the analysis relaxes the assumption of an invariable natural rate of interest over the cycle and offers some of the reasons for its instability. The paper concludes with recommendations of great caution which the central bank has to keep in mind mainly at the beginning of the boom if forces of the business cycle are not to emerge - forces that finally make the behaviour of the natural rate of interest impossible to follow.
One important note deserves brief attention. Since the Austrian theory has never been developed into a rigorous and condensed mathematical model, the exposition in this paper will mainly follow verbal and occasional graphical reasoning. Simple graphs and numerical
examples are given especially to discipline one's mind as it can be easily trapped in vicious circles when analysing such a complicated phenomenon as the business cycle.

## 2. THE AUSTRIAN MODEL

The Austrian business cycle theory stands and falls with the Austrian theory of capital, outlined by Menger (2007), thoroughly developed by Böhm-Bawerk (1890; 1891), and refined by Hayek (1941) and Lachmann (1956). As it is not the main objective of this paper to go into deep intricacies of the capital theory, this part just briefly introduces simple tools usually used by the Austrian theorists.
The Austrian capital theory assumes that production of any economic good proceeds in time. Since no consumption good is created directly by labour, it usually takes some time to manufacture even the simplest one. Using a simile from biology, consumption goods gradually mature out of the unfinished goods called capital goods. A useful tool widely employed to describe this process was developed by Hayek (1935), and further refined by Rothbard (2004), Garrison (2001), and de Soto (2006), hence the name - the Hayek triangle. ${ }^{48}$

In Figure No. 1, the horizontal line represents the period of time that has to elapse for the least matured and unfinished goods (e.g. raw materials) to become fully matured consumption goods prepared for direct consumption. The length of the vertical leg measures the nominal value of consumption goods. By cutting the triangle at any point of time and erecting a vertical line, one can get the value of unfinished goods. It is obvious that the further the point from the eventual vertical leg, the lower the value of the unfinished goods. This simple approach resonates with the Austrian view that the value of goods in process (or capital goods) is directly derived from the value of consumption goods (Menger 2007); yet, the value of capital goods is always lower than the value of consumption goods due to the personal discount of future (Böhm-Bawerk 1891).

The third most important property of the triangle is its slope, reflecting the size of the personal discount of future - the higher the discount, the bigger the slope. According to the Austrian theorists, the personal discount of future should be also reflected in the market interest rate, so the slope of the Hayek triangle is (among other things) determined by the market interest rate. The reason is obvious. If the slope of the triangle exceeded the interest rate - in other words, if the value of the good in one stage of the production process significantly dwarfed the value of the less processed good in the preceding stage - the resulting above-average profit would motivate entrepreneurs to buy goods in one stage and sell them in the other till the profit rate would level with the ongoing market rate of interest. ${ }^{49}$ The direct effect would be an increase in the price of the good in the more remote stage and a decrease in price in the later stage making the difference in values and consequently the slope of the triangle consistent with the market interest rate.

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Figure No. 1, The Hayekian triangle

Figures No. 2 and No. 3 demonstrate that the lower the interest rate, the longer the production process. This property is derived from the theory of Böhm-Bawerk (1891), who demonstrated that lower personal discount of future and hence the lower interest rate allow the factors of production to be locked and used in longer production processes. In the Fisher (1930) terms, the lower the time preference (or impatience), the longer the particular consumer is prepared to wait till the production process provides consumption goods.

Figures No. 2 and No. 3 are inaccurate as regards the fact that they suggest that shorter processes provide more consumption goods. However, it is generally believed (Böhm-Bawerk 1891; Hayek 1941) that exactly the opposite is true. For the given level of technology and quantity of labour, higher output of consumption goods may be produced only if the factors of production are employed for a longer period of time. The fundamental reason is the requirement of efficiency; the given output of consumption goods is produced by the shortest possible method of production. The higher output necessarily requires a longer production process, in the Böhm-Bawerk (1891) terms - a more roundabout process.


Figure No. 2, Production process for a high interest rate


Figure No. 3, Production process for a low interest rate

The picture would be more consistent with the Böhm-Bawerkian theory if the Hayekian triangle was plotted in real terms. Figure No. 4 demonstrates that in real terms longer processes after completion produce larger output than shorter methods. As can be seen, the picture is also consistent with the assumption that the increase in the roundaboutness has its limits. Every increase in the time period for which the factors of production are tied up in the production process raises the eventual output of consumption goods, yet at a decreasing rate. In other words, the analysis assumes a diminishing marginal productivity of the roundabout methods of production (Böhm-Bawerk 1891; Hayek 1941).


Figure No. 4, Decreasing marginal productivity of the roundabout process

The transition period between two processes with different roundaboutness will be discussed later on. Nonetheless, it is more convenient to analyse the theory in nominal terms. Hence, the Hayek triangle depicted on Figure No. 1, reflecting the market interest rate, will be utilized in this article.
R. Garrison (2001) enriched the Austrian exposition by the loanable funds market model. ${ }^{50}$ The interest rate, which is an integral part of the slope of the Hayek triangle and determines the difference in value of goods at different stages of completion, is to be consistent with the interest rate that equilibrates supply and demand on the loanable funds market.

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Figure No. 5, The loanable funds market and the Hayek triangle

Figure No. 5 depicts one of the most important variables in the Austrian model - the natural rate of interest ( $\mathrm{IR}_{\mathrm{nat}}$ ). The adjective "natural" comes from the assumption that it is determined purely by real forces. As is suggested in the figure above - by the flow of investment (I) and the flow of saving (S). ${ }^{51}$ The increasing saving function can be derived from a rising marginal rate of time preference, the decreasing investment function stems from the diminishing marginal productivity of capital. This rather Fisherian (1930) explanation of the natural rate is consistent with Hayek (1941) theory, whereas Mises (1996) and Rothbard (2004) strictly rejected that productivity is of any importance in the explanation of the interest phenomenon. ${ }^{52}$ Although this obvious controversy among Austrian authors would require further investigation, this paper will stick to the Hayekian and Fisherian explanation of the natural rate of interest. ${ }^{53}$

A variable as important as the natural rate can be read also on the horizontal axis. For the given natural interest rate, the investment equals saving. It is assumed that this amount of saving is exactly the one that is needed to keep the production process intact. In other words, the economy creates precisely such the amount of new capital that is required to maintain continuous and undisturbed process of production and to preserve the given shape and size of the Hayek triangle.

So far, the processes inside the Hayek triangle have not been discussed. Figure No. 6 illustrates Hayek's idea that capital goods are mainly production goods in process $\left(\mathrm{Q}_{\mathrm{i}}\right)$ that finally mature in consumption goods $\left(\mathrm{Q}_{\mathrm{c}}\right)$. Each good in process $\left(\mathrm{Q}_{\mathrm{i}, 1} ; \mathrm{Q}_{\mathrm{i}, 2} ; \mathrm{Q}_{\mathrm{i}, 3} ; \ldots\right)$ has its own market with its own market price. The price of the production good that is posited further from the final consumption is lower than the price of the production good that is closer to it. The price margins between goods in the different phase of completion reflect the slope of the Hayekian triangle. They are also consistent with the natural rate of interest.

The supply side of each market is represented by companies or entrepreneurs operating in different stages of production. Although the demand on each market is driven by firms producing in the subsequent stage, the key agents of the demand side on each market are to be found somewhere else, namely at the end of the entire production process. In identifying the

[^25]major drivers of the demand for production goods, one has to realize the economic essence of these goods. Production goods are nothing less and nothing more than the future consumption goods. Hence, the demand for them originates on the part of individuals who demand future consumption goods. In addition, the major demanders are people saving part of their income since saving is just the demand for consumption goods that will mature at some moment in the future. ${ }^{54}$


Figure No. 6, Markets of production goods at a different stage of completion

### 2.1 A DECREASE IN TIME PREFERENCE

The simple tools just developed suffice to describe the nature of the process that transforms the Hayek triangle from one shape to another (Figure No. 2 and No. 3) and to shed some light on what is known as the change in the roundabout process of production.
There can be no doubt that every individual tries to adjust her time shape of consumption to maximize her utility. As was demonstrated by Fisher (1930), the specific stream of consumption of the individual depends on many factors, such as the time shape of her income, average level of her income, composition of income, degree of risk in different periods, and many others. At this stage, we are interested neither in the major determinants affecting the specific stream of consumption of an individual, nor in the response of the individual to these factors. The main objective of this section is to find out whether the Austrian theory is well designed to analyze mechanisms in the structure of production and the roundabout processes when a large number of individuals change their optimal allocation of consumption over time.

Consider a change in the optimal path of consumption such that a remarkable part of consumers reshuffles their stream of income on consumption in more remote periods in the future. In other words, suppose that in the economy the relative demand between present consumption goods and future consumption goods is shifted toward the latter at the expense of the former. To put it plainly, people start to save more. ${ }^{55}$
The reduced demand for present consumption goods will manifest first in the stages producing goods for immediate usage and in the stages very close to final consumption, whereas earlier stages manufacturing production goods should experience an increase in the demand as the new savings, representing demand for future goods, are channelled to this part of the production process. However, new savings would never find their way to the earlier stages of production if the interest rate did not decrease. In other words, the initial phase of the mechanism hinges on the assumption that increased saving will result in a decrease in the interest rate. This decline in the interest rate encourages investment spending, as is depicted in Figure No. 7. If something blocks the smooth transformation of saving into investment,

[^26]further steps of the mechanism cannot proceed and the structure of production can never be transformed.

Nevertheless, if this process operates without disturbances, the inflow of saving into earlier stages will show up as an increase in the demand for production goods, and the resulting higher price will encourage supply to meet this expanded demand (Figure No. 8). The crucial question is, however, whether the economy is endowed with enough factors of production such that the supply can at least partly respond to higher demand or whether the supply of production goods is fixed due to the lack of the essential factors of production.


Figure No. 7, Decrease in the time preference in the loanable funds market

The answer to this question is to be found at the end of the production process, in the consumption stage or in the stages very close to final consumption, which suffer from the fall in the consumers' demand. Eventually, the lower demand will cut prices of consumption goods (Pc). Together with an inevitable reduction in the amount of sales of the consumption goods, their lower prices should depress profits of the firms producing consumption goods. As a direct consequence, factors of production in this stage of production, especially labour, will experience a fall in the value of their marginal product. This is how the market manifests that the factors of production are not worth as much as before in the consumption stage. The market also demonstrates that in some sectors of the consumption stage consumers value inputs more than the resulting output, and lower profits or even losses are a direct proof that some of the factors of production should be released.
Here is to be found the source of factors of production that are eagerly demanded in the earlier stages of production. As is depicted in Figure No. 8, the decrease in the consumption demand should release labour and other resources from this stage of the production process. At the same time, the higher demand for labour in earlier stages of production may absorb it. If this truly happens, then the supply of consumption goods will decline and the supply of production goods will rise. However, this process is conditioned by the well-functioning price system that should reflect not only the increase in savings via a lower interest rate but also a decrease in the consumption demand through a decline in the prices of consumption goods $(\mathrm{Pc})$. In addition, the price system must also reflect an increase in the demand for production goods via a rise in the prices of production goods ( Pi ). The structure of wages should be also affected in such a way that earlier stages of production will experience a rise in wages and stages closer to consumption a decline in wages, which is a straightforward signal for labourers to move from one sector to another.

At this moment, several observations deserve our attention. First, as can be seen on the loanable funds market, the natural rate of interest has declined. Second, the slope of the

Hayek triangle is lower as the demand for consumption goods plummets. Moreover, the triangle is also longer, which stems from the fact that lower interest rate motivates entrepreneurs to increase the roundaboutness of the production process even by opening some very long processes of production, which were impossible to start before. This observation is consistent with the fact that the increase in the demand for production goods is more robust in stages of production furthest from the final consumption. ${ }^{56}$ Third, the profit margins, represented by the differences in value between goods at different stages of production, are also lower since there is a general tendency to equalize profit rate with the interest rate. Fourthly, although this is not immediately obvious from the simple picture below, the capital in the economy has not only changed its size and shape but also its structure because more factors of production are allocated into relatively more remote stages of production. Furthermore, some very remote stages of production have been even newly created.


Figure No. 8, Decrease in the time preference and the structure of production

Finally, unless consumers abruptly change attitudes toward the time shape of their stream of consumption, the new structure of production is sustainable. It will provide higher output of consumption goods after completion, as was posited by the Austrian theory of capital mentioned at the beginning.

[^27]So far, it has been demonstrated that the movement of the interest rate reflects changes in real factors on the market. At the same time, the smooth functioning of the market system has been preserved. A slightly different issue can be found by opening a question about whether the interest rate may clear the market when an imbalance in the intertemporal structure of production emerges.
Suppose that for the given interest rate, the investment exceeds savings. This fact is reflected by the excess of supply over demand in the markets for production goods. On the other hand, the consumption goods markets suffer from the lack of supply. If the price system works reasonably well, prices of production goods should decline with a sharper fall observed in the earliest stages of production, whereas the prices of consumption goods should follow the opposite path. The ratio of prices between consumption goods and production goods ( $\mathrm{Pc} / \mathrm{Pi}$ ) will increase, which is consistent with the fact that the interest rate in the loanable funds market rises, gradually equilibrating saving and investment.



Figure No. 9, Imbalance in the structure of production and its elimination

As can be seen in Figure No. 9, the response of the price system will eliminate the initial imbalance both in the structure of production and in the loanable funds market. Resources are reallocated such that the excess supplies or demands are removed.
The foregoing paragraphs suggested that the role of the interest rate should not be underestimated, since it orchestrates the entire process of the capital restructuring. It mirrors the slope of the system of demand functions in different stages of the production process, and it will eventually decide which of these stages will expand, which will shrink, which will be newly created, and which of these will completely disappear.

In this connection, Hayek wrote:
But to think of interest only as a direct cost factor is to overlook its main influence on production. What is much more important is its effect on prices through its effect on demand for the intermediate products and for the factors from which they are produced. (Hayek 1935:83)

The analysis suggests that it is the interest rate, which transmits the key information about changes in the intertemporal markets. It represents the key guideline in the market economy, the only link among many individuals as to the fact of how scarce resources should be optimally allocated over time to maximize utility and profit.
If this signal is disturbed or blurred by reasons that do not stem from the real economy, it is highly unlikely that the intertemporal allocation of resources will not be affected simultaneously with the real economy itself. The key question is whether the sequence of events thus triggered is sustainable and whether the path of the process originally initiated by non-real (e.g. monetary) factors will not be eventually reversed.

### 2.2 MONEY AND THE INTEREST RATE

Although the natural rate of interest has been introduced as a variable equilibrating real phenomena - saving and investment in the loanable funds market - neither investment nor saving is traded in kind in the real world. Flows of saving and investment come on the market in the form of money; hence, it is sometimes difficult to identify a certain exchange as an act of saving or investment. Furthermore, it is inconceivable that a change in the realm of money, i.e. the supply of money or the demand for money, will not affect the loanable funds market. Only in a hypothetical barter economy, where investment is by definition equal to saving, problems, such as the ones discussed later in this chapter, would not emerge.

One of the most important disturbances, which the loanable funds market might face from the monetary part of the economy, is a change in the money supply. Let us assume that the central bank injects some amount of generally accepted medium of exchange in the money market. By using any tool at its disposal, this injection should end up as new reserves in commercial banks. Although it is sometimes conceivable that the story stops here, in normal times the optimal ratio of reserves to deposits in commercial banks is somewhat disturbed, which motivates them to re-establish a more profitable relation. The most straightforward method on the part of the commercial banks is to offer more loans to their clients, either to the old or to the new ones.
This action will undoubtedly increase the supply in the loanable funds market. However, assuming a stable investment demand function, new loans will be accepted only for a lower interest rate. Hence, monetary expansion leads to an overall decrease in the interest rates. ${ }^{57}$ It should be also stressed that the interest rate in the loanable funds market falls below the natural rate because the real forces determining its level have not changed (see Figure No. 10). Thus, the monetary expansion should at least for some time generate a negative gap between the actual market interest rate and the natural rate unless the time preference decreases hand in hand with the money supply expansion.

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Figure No. 10, Monetary expansion in the loanable funds market
Although the natural rate of interest is unchanged at this stage of the process, the information about the scarcity of the factors of production and about the relative demands for present and future consumption goods is conveyed by the market interest rate, which was somewhat lowered. Hence, a lower interest rate is an explicit signal for the producers of the production goods that the demand for production goods has increased at the expense of the demand for consumption goods. The net present value of business projects that are highly sensitive to the interest rate will become positive. The responsiveness is clearly the strongest in case of projects that will ripen in a relatively distant point of time in the future. As a result, a new profitable space to produce more capital goods has emerged, as well as the opportunity to open new stages of the production processes, which would be never lucrative if the interest rate did not decline.

In our graphical model, this decrease in the interest rate is depicted similarly as the decline in the time preference (Figure No. 11). An increase in the demand for production goods raises their prices - relatively more in the furthest stages from the final consumption - and the price ratio between consumption goods and production goods $(\mathrm{Pc} / \mathrm{Pi})$ falls. This fact is reflected in a lower slope of the Hayek triangle in which the creation of new stages of the production process very remote from the final consumption can be clearly seen.
One obvious question deserves our attention. Can a higher demand in the early stages of production be readily met by a higher supply? As was demonstrated before, this was the case when the time preference declined. However, after the monetary expansion, the loanable funds market model suggests (Figure No. 10) that the flow of real saving has been decreased rather than increased (to point A). Moreover, this model depicts an excess of investment over (voluntary) saving (distance AB ). Unlike in the case of the decline in the time preference, the consumer demand seems to be intensified - a logical corollary of the credit expansion and the interest rate decrease.

As a result, the right upper diagram in Figure No. 11, depicting the consumption goods market, would be more in line with the loanable funds market if the demand curve was shifted outwards, which would consequently increase the price of consumption goods as well. However, even in that case, the slope of the curve connecting all demand curves in the structure of production (the dash-dot line) should be flatter than before since it reflects a lower interest rate in the loanable funds market.


Figure No. 11, Monetary expansion and the structure of production

What is especially important in Figure No. 11 is the direction of arrows on the quantity axis. As can be perfectly seen, all are inclined to the right, to the expansion of production in every stage of the process, yet the strongest tendency is clearly observed in the earliest stages. Hence, the question whether the supply can readily meet the demand, when all demand curves are increasing, does not seem to be answered in the affirmative.

As a first approximation of the whole problem, Hayek (1935) assumed that the economy operates at the potential (full-employment) level - there are no idle resources that can be readily mobilised to satisfy the increased demand. ${ }^{58}$ In such a case, the supply will be raised only in the stages that have the biggest power to attract necessary factors of production. Figure No. 11 suggests that these are the stages very remote from the final consumption. They enjoy not only a widespread increase in demand but another strong impulse in this phase of the production process arises due to the lowered interest rate. Furthermore, it is very likely that in the real world more than the proportional amount of new loans is channelled to industrial sectors and other capital-intensive branches of the economy. Nevertheless, the key economic reason lies in the fact that these sectors are highly sensitive to changes in the interest rate because they are producing capital goods rather than the goods very close to final consumption.

The foregoing analysis suggests that the earlier stages of production should attract factors of production (mainly labour and other unfinished and capital goods) at the expense of later stages and stages very close to final consumption. With the newly employed factors of production new capital formations are initiated, and the economy is on the way to more roundabout processes through capital restructuring as if the time preference was decreased.

[^29]Nonetheless, contrary to the decrease in the time preference, which resulted in a lower demand for consumption goods and consequently in a reduced supply of these goods, in the case of the monetary expansion, the demand for consumption goods is rather increased. Yet, the supply of consumption goods is paradoxically reduced as the factors of production have been attracted to earlier stages of production. ${ }^{59}$ This relative lack of consumption goods that was brought about by the monetary expansion is known as the "forced saving" phenomenon (Hayek 1932) - in Figure No. 10 represented by the distance AB. This type of saving makes the new investment possible to materialize. For some time, it supports the formation of new capital.

The key question, which immediately springs to one's mind, is whether this process is sustainable, whether the new capital formation can be finished and whether the new more roundabout processes of production will finally provide higher output of consumption goods. However, this does not seem very probable as the forces of supply and demand in the consumption stage are not consistent with each other - the consumers' demand is boosted, yet the supply of consumption goods is partly reduced due to the transfer of resources to earlier stages of production. At the same time, the loanable funds model displays that the interest rate is below its natural level.

The Austrian theory of the business cycle predicts that the interest rate gap is not sustainable, and the market interest rate should eventually return to the (initial) natural level. With the increase in the interest rate, the new longer roundabout processes will collapse as they cannot be completed, and the economy will move to a painful recessionary phase (see Figure No. 12). However, the main objective of this chapter is not to describe the Austrian business cycle theory in deep detail and to discuss why the new capital formations are abandoned due to lower profitability, but to elucidate the reverse process of the interest rate to its natural level, even though the impact of this interest rate reversion on the real economy will necessarily accompany the analysis.

The next part of the article compares major contributions of F.A. Hayek, L. Mises, and M. Rothbard to the theoretical explanation of this interest rate reversion process, and it will also point out some of the inconsistencies that could emerge in the analysis of this phenomenon.
Hayek (1935) considered the interest rate mainly as the price margin between two successive stages of production. The lower the price margin, the longer and more roundabout methods of production the economy can afford. The monetary expansion depresses these price margins and enables the lengthening of the structure of production. In other words, it leads to more capital demanding methods of production. New structure can be achieved only by attracting factors of production previously employed in the late stages of production by offering higher wages and other forms of income. Through this channel, the newly created money is obtained by the owners of various factors of production, mainly by workers. It is highly improbable that workers, now in the role of consumers, would dramatically change the time shape of their consumption in favour of the more remote one, in favour of the future consumption goods, whose production has been just initiated in the form of various production goods.
It is rather the opposite that can be expected since the interest rate has been decreased. The enlarged demand for consumption goods on the part of workers will then lead to a substantial upward pressure on prices of consumption goods further supported by the outflow of resources to earlier stages of production. The increase in the price of consumption goods will eventually result in a rise in the price margins $(\mathrm{Pc} / \mathrm{Pi})$ back to the pre-monetary-expansion level. This explanation is precisely at the core of the Hayekian approach to the interest rate

[^30]reversion. In effect, the Hayek triangle changes its slope, and the earlier stages of production are abandoned since the higher interest rate decreases the demand for their products. As can be seen in Figure No. 12, the slump in prices is especially remarkable in the earliest stages of production, i.e. in the stages that experienced the highest price-increase at the beginning of the monetary expansion.


Figure No. 12, The reversion process of the interest rate and the road to recession

As Hayek put it:
At the same time incomes of wage earners will be rising in consequence of the increased amount of money available for investment by entrepreneurs. There can be little doubt that in the face of rising prices of consumers' goods these increases will be spent on such goods and so contribute to drive up their prices even faster. These decisions will not change the amount of consumers' goods immediately available, though it may change their distribution between individuals. But-and this is the fundamental point-it will mean a new and reversed change of the proportion between the demand for consumers' goods and the demand for producers' goods in favour of the former. The prices of consumers' goods will therefore rise relatively to the prices of producers' goods. And this rise of the prices of consumers' goods will be the more marked because it is the consequence not only of an increased demand for consumers' goods but an increase in the demand as measured in money. All this must mean a return to shorter or less roundabout methods of production if the increase in the demand for consumers' goods is not compensated by a further proportional injection of money by new bank loans granted to producers. (Hayek 1935:89)

This point of view is perfectly consistent with Hayek's earlier approach:
If, however, the fall in the rate of interest is due to an increase in the circulating media, it can never lead to a corresponding diminution in the price margin, or to a readjustment of the two sets of prices to the level of an equilibrium rate of interest which will endure. In this case, moreover, the increased demand for investment goods will bring about a net increase in the demand for consumption goods; and therefore the price margin cannot be narrowed more than is permitted by the time-lag in the rise of consumption goods prices - a lag existing only as long as the process of inflation continues. As soon as the cessation of credit inflation puts a stop to the rise in the prices of investment goods, the difference between these and the prices of consumption goods will increase again, not only to its previous level but beyond, since, in the course of inflation, the structure of production has been so shifted that in comparison with the division of the social income between expenditure and saving the supply of consumption goods will be relatively less, and that of production goods relatively greater, than before the inflation began. (Hayek 1933:217)

It is obvious that the key role in Hayek's explanation is played by the ratio of prices of consumption goods to the prices of producers' goods. This ratio will decide the maximum length of the process of production the economy can eventually afford. For Hayek, the evolution of the general level of prices is by no means as important as the behaviour of the structure of relative prices, which substantially distinguishes his approach from modern business cycle theories.

At this point, let us explore the real forces that are effective when the economy is being restructured back to shorter methods. The labour force was first allocated to long methods of production, yet the reversion process attracts the labour force back to the late stages. The core of this reversion lies in the fact that workers earn new money in the form of higher labour incomes. They start to demand more present consumption goods even though they are engaged in the production of the future consumption goods. There is an obvious mismatch in the intertemporal structure of the demand and supply. The demand for consumption goods is boosted, but the production process was shaped such that it supplies production goods, which will mature into consumption goods in the future. This inconsistency between the production plans and the consumers' demand will drive up not only the prices of consumption goods but also the ratio of prices between present goods and future goods $(\mathrm{Pc} / \mathrm{Pi})$. Thus, the interest rate rises.

As far as the allocation of labour among various processes is concerned, it should be stressed that the future consumption goods are being produced in the early stages of the production process in the form of capital goods. These processes require the cooperation of labour also in the future in order to mature in final consumption goods. And this process takes time. Yet, the labour force (and time) is not available, as it tends to move to stages very close to final consumption where it is attracted by the strong demand for present consumption goods. As a result, there are not enough labourers to keep the Hayek triangle, which was expanded by the monetary expansion and by the artificial lowering of the interest rate, intact. In other words, the labour force is not large enough to meet the high demand for consumption goods and simultaneously to operate in the very long methods of the production process. Longer processes must be therefore abandoned since they cannot compete for the labour force with the late stages. As can be seen in Figure No. 12, the structure of the intertemporal demand curves benefits the sectors close to the final consumption stage. The early stages are thus on the way to liquidate their capital and to release the labour force on the market.

We encounter a paradoxical situation - the insufficient amount of the labour force required to maintain the expanded capital structure of the economy will lead to unemployment in the sectors very remote from the final consumption stage. As there is not enough labour to cooperate in the subsequent stages, the labour force in the early stages is very difficult for the entrepreneurs to keep employed because the (discounted) value of the marginal product of labour is drastically falling due to the drop in the demand for output of capital-creating industries. If the late stages are not able to rapidly absorb the released labour force, the economy will experience a period of massive unemployment. Since the economy requires more labour than is available, part of the labour force will be laid off due to the general mismatch between the intertemporal demand and supply.

As can be seen, the time dimension of the entire production process was too extended, so there are not enough resources and time to complete all the initiated processes. The scarce resources will be attracted to sectors with the largest demand - to the consumption stage and to stages close to final consumption - and the very long processes will be abandoned. In other words, people are too impatient to wait for the consumption goods that are being produced in the early stages and that will mature in the remote future. As a result, the symptom of the crisis is not insufficient consumption, but too much consumption - the demand for present consumption goods harms investment and eliminates creation of the new capital structures.
Not just the labour force is reduced in the early stages. As is indicated in Figure No. 12, some capital structures are abandoned as well. As a result, the same conclusion can be made to capital as was made to labour. Since there are not enough resources to finish all newly initiated capital formations, part of the capital stock will be lost. We encounter a paradoxical situation of the simultaneous existence of the surplus of capital and the lack of capital (Hayek 1941). This situation was described by older authors such that too much circulating capital was converted into the fixed capital, and now there is not enough circulating capital to finish the fixed capital structures (Wicksell 1976b). As a result, large amounts of unused capital formations are accompanied by insufficient reserves of variable capital. Capital is both in short supply and in excess supply. Yet, according to the Austrian theory, this vision is not general enough. It puts too much emphasis on specific forms of capital and on its durability and mobility. The Austrian approach would rather stress the artificial time extension of the production processes and the creation of various capital goods on the path to final consumption goods that will never mature in the latter due to the insufficient supply of resources in the economy and the misallocation of these resources (Hayek 1941:428).

As a result, the key reason for the paradox of the simultaneous existence of too much capital and too little capital lies in the mismatch between the intertemporal structure of the demand for the present goods and the future goods, and the supply of the present goods and the future goods caused by the monetary expansion. The monetary forces brought a noise into the price system. The artificially lowered interest rate has given a false signal to the allocation of resources, which was not consistent with the planned intertemporal allocation of income of the free acting people. It is again the price system, though, that will re-establish the consistency between the consumption plans of people and the production plans of firms. Yet, this re-allocation of resources will be accompanied by losses of various capital formations that did not have to take place if the economic resources were not deflected by the false price signals originating from the monetary side of the economy.


Figure No. 12B, Restructuring of the production processes after the monetary expansion

Figure No. 12B illustrates this dynamic process in a simple diagram. It builds upon the tools developed in Chapter 1, in Figure No. 18. Suppose that the economy is in the general intertemporal equilibrium. Between time $\mathrm{t}-5$ and $\mathrm{t}-1$, it provides a smooth flow of consumption goods of 10 units. In every period, each unit of labour is allocated to a different stage of the production process. For example, in period t-4 one unit of labour creates intermediate product (capital good) II from intermediate product I, another unit produces capital good I, the third unit in the same period completes the entire process and offers ten units of consumption goods, etc. As can be seen, in each period, the economy employs five units of labour even though some labourers might be voluntarily unemployed as they are looking for a better job. In period $\mathrm{t}-3$, the same processes are repeated again. It may be assumed that higher productivity of the division of labour results in the fact that each labourer works at the same stage of the production process in every period. As a result, in period $\mathrm{t}-3$ the worker is not shifted from the production of the intermediate product II, which occupied him in the previous period, to the creation of the intermediate product III. Workers are specialized - the same labourer creates intermediate product II in period t-3, and the intermediate product III in period $\mathrm{t}-3$ is produced by a different labourer. This division of labour is obviously not displayed in the simple diagram, even though it surely contributes to a higher productivity of roundabout methods. Five units of labour may produce 10 units of consumption goods every period if the production process is separated and extended over five periods. As was shown in Chapter 1, very short processes that take only one period are able to create one unit of consumable output per worker, which yields total output of five units every period. Thus, the longer processes provide twice as much consumption goods as the direct (i.e. very short) methods.

Now suppose that the monetary expansion lowers the interest rate at time t0. As was said above, this reduction in the interest rate motivates entrepreneurs to initiate longer methods of production. Suppose that in period t0 the old process that would supply 10 units of consumption goods in five periods is not started, and it is replaced by a longer process that will provide 15 units in six periods. Roundabout methods are more productive, yet they take more time and require cooperation of labour in the future. Suppose that the structure of production is only gradually re-formed. As a result, the second process will be changed to a longer method in period $t+1$, the third process in period $t+2$, etc. ${ }^{60}$ If all these processes are completed, the flow of consumable output may be depicted in panel (a) of Figure No. 12C. It rises from 10 to 15 , yet every sixth period the output drops to zero since there are only five units of labour, whereas the duration of the longer process is six periods. Furthermore, at least one unit of labour must be specialized in the production of two types of intermediate products (or an intermediate product and the final refining to the matured consumption good). Nonetheless, in 30 periods, the total output of this longer process might be $(30 / 6) \times 15 \times 5=$ 375 , which gives 75 units of consumption goods per worker in 30 periods, and $75 / 30=2.5$ units per worker each period. This figure therefore exceeds the output of the shorter method, which provides $(30 / 5) \times 10 \times 5=300$ units in 30 periods, 60 units per worker in 30 periods, and 2 units of consumable output per one worker in every period.

Moreover, if the monetary expansion attracts labour that was previously unemployed or if it speeds up the matching process between workers and jobs, the economy may use six units of labour rather than five, and the total output of consumption goods in 30 periods will rise to $(30 / 6) \times 15 \times 6=450$. The total output in every period will then increase to $450 / 30=15$ from $375 / 30=12.5$, and there will be no drop in the flow of consumption goods.

[^31]However, the production processes lengthened by the monetary expansion are not sustainable, since people are too impatient to wait for the output. As can be seen in Figure No. 12B, the output of consumption goods drastically drops in period $t+4$ because the process that should provide consumption goods in this particular period was diverted to a longer method that will mature next period. ${ }^{61}$ The economy in period $t+4$ creates an enormous amount of capital goods (intermediate products I,II,III,IV,V); yet, the flow of consumption goods is missing. This drop in consumable output that will result in the decline in real consumption might be considered the forced saving. People did not save consumption goods in the past to bridge this period of low supply of consumption goods. The monetary expansion diverted factors of production to longer methods, and the creation of capital goods is at the maximum in period $t+4$; however, people suffer from the lack of consumption goods.


Figure No. 12C, Evolution of the output of consumption goods if the process is not reversed (a) and if it is reversed (b).

[^32]Suppose for simplicity that the consumption demand is reinforced in period $t+4$, when the newly created money is fully earned by labourers. A sudden increase in the prices of consumption goods that follows attracts factors of production to processes that provide consumption goods in a very short time. The longer processes are abandoned or not renewed, or the capital in the early stages is not replaced, since entrepreneurs are motivated by very high profits to move the factors of production to the consumption stage in order to provide consumption goods as soon as possible.
Suppose that all processes in period $t+5$ are drastically shortened to direct methods. The only exception is the process that is about to provide consumption goods in this particular period. As a result, the output of consumption goods in period $t+5$ rises to 17 units. As can be seen, the monetary expansion may bring about a lengthening of the capital structure that is ended by fully matured consumption goods. Yet, this conclusion applies only to some processes in the economy, not to all of them. People are too impatient to allow completion of all the lengthened processes (Hayek 1935).

In period $\mathrm{t}+5$, the major part of the factors of production is attracted to shorter methods. However, this reallocation takes time, so only a fraction of the labour force is able to provide consumption goods immediately in direct processes. This is the reason the output in period $\mathrm{t}+5$ is not 19 , but only 17 - two units of labour provide neither the capital goods nor the consumption goods, since it might be difficult to absorb so much labour in the short processes, or labour might be entirely unemployable owing to the lack of complementary factors of production.

What is even more important is the fact that the creation of various capital goods (intermediate products I,II,III,IV) between period $t+1$ and $t+4$ was a pure waste of resources and time because the longer processes, in which they were produced, will never be completed. The second process that would mature in period $\mathrm{t}+6$ in 15 units of consumption goods may serve as an example of this economic loss.

Furthermore, the output of consumption goods drops in period $t+6$ from 17 to four units. One unit of labour is still out of the production process. The units of labour that are employed give a very low output of consumption goods since the production processes in the economy were shortened. In our example, we assume along with Hayek (1935) that the reversion is to even less roundabout methods than prevailed before the monetary expansion. The reason lies in the huge demand for present consumption goods that motivate entrepreneurs to resort to methods that provide consumption goods very quickly. The eventual evolution of output of consumption goods after the monetary expansion is displayed in panel (b) of Figure No. 12C.

Another observation is that there is a significant lag between the moment of the monetary expansion and the moment the output of consumption goods is affected. Even though the lengthening of the capital structure took place immediately in our model, higher output of consumption goods is delivered later. Moreover, the flow of consumption goods is not smooth, because their output eventually drops, and the pattern closely resembles a typical sinusoid wave of the business cycle. Nothing signalled that the economy was on an unsustainable path, and it might seem that the smooth economic development was unexpectedly interrupted by a sudden crisis. Yet, the monetary expansion provoked processes that deflected the economy from its long run intertemporal (dynamic) equilibrium. The business cycle pattern is just a necessary response to this shock, as the economy is moving out of its equilibrium and as it is consequently and gradually returning to its equilibrium, which is characterised by the consistency between the intertemporal plans of consumers and entrepreneurs.

It should be also stressed that since the economy suffers from massive losses of the capital structures, the marginal productivity of labour in the crisis is definitely lower than before the crisis - the real wage defined as the amount of real goods that might be purchased by the average labourer falls. This effect along with the rigidity in nominal wages may stay behind very high unemployment figures observed during the recession. The misallocation of resources triggered by the monetary expansion therefore reduces the well-being of people in the society.
Figure No. 12B also uncovers that the recession has its genetic code built in the boom phase of the cycle. In other words, the present state of the economy is much more important for the future performance than is generally believed. However, the answer is not to be found in the simple autoregressive process of the total output of final goods or in any other macroeconomic aggregate. As we have seen, it is rather the other way round.

The answer to this question is hidden in the amount of capital goods of various forms, and in the (intertemporal) consistency of plans among entrepreneurs at different stages of the production process and of the intertemporal consumption plans of the consumers. If this consistency is disturbed, the economic boom must end up in the recession, even though not one in a million is able to recognise that in the present, the economy is on the unsustainable path. In other words, the creation of the majority of goods consumers enjoy in the present was initiated in the past. Similarly, the output of consumption goods in the future critically depends on the amount of present capital goods and on the consistency of various production plans among thousands of entrepreneurs in the free market economy. These plans are coordinated both intra-temporally and inter-temporally by the price system. If the price system is disturbed and the coordination is therefore upset, the output of consumption goods in the future is endangered even though the macroeconomic figures might suggest that the economy is booming in the present. As can be seen in Figure No. 12B, the apparent abundance of capital goods (period $t+4$ ) and consumption goods (period $t+5$ ) is not sustainable, and the boom ends up in the bust in period $t+6$. However, the recession does not come due to the insufficient consumption demand or investment demand. It is the outcome of the misallocation of resources triggered by the monetary expansion. This misallocation is manifested as the excess of consumption demand at the top of the boom accompanied by the unfinished formation of capital, which results in the simultaneous abundance of "fixed" and shortage of "circulating" capital.

Let us return to the main topic of this chapter - the reversion of the interest rate during the business cycle. Hayek (1935) recognised the important contribution of his predecessor L. Mises. However, his big master was criticised for the excessive reliance on changes in the purchasing power of money in investigating the business cycle phenomenon rather than on the relative price analysis thoroughly explored in the foregoing discussion.
L. Mises is considered the founder of the Austrian business cycle theory. Hence, his approach resembles the more elaborated Hayek's theory. In his first opus, Mises (1976) analysed the effects of changes in the quantity of money on the interest rate. According to Mises, the money expansion firstly decreases the interest rate and triggers more roundabout methods of production. However, sooner or later the so-called subsistence fund ${ }^{62}$ is exhausted, which is manifested by the lack of consumption goods and by an increase in their prices. This means (almost by definition) a rise in the interest rate since the price-difference between consumption goods and production goods expands. Mises suggested that the increase in the interest rate is further reinforced by a decline in the objective purchasing power of money.

[^33]Let us document his approach in the following passage:
The increased productive activity that sets in when the banks start the policy of granting loans at less than the natural rate of interest at first causes the prices of production goods to rise while the prices of consumption goods, although they rise also, do so only in a moderate degree, viz., only in so far as they are raised by the rise in wages. Thus the tendency towards a fall in the rate of interest on loans that originates in the policy of the banks is at first strengthened. But soon a counter-movement sets in: the prices of consumption goods rise, those of production goods fall. That is, the rate of interest on loans rises again, it again approaches the natural rate. (Mises 1976:363)
This counter-movement is now strengthened by the fact that the increase of the stock of money in the broader sense that is involved in the increase in the quantity of fiduciary media reduces the objective exchange-value of money. Now, as has been shown, so long as this depreciation of money is going on, the rate of interest on loans must rise above the level that would be demanded and paid if the objective exchange-value of money remained unaltered. (ibid.)

In his later work, Mises (1996) heavily relied on changes in the purchasing power of money in analysing the dynamics of the interest rate over the business cycle. His reasoning closely resembles that of Fisher (1930) as he assumed that an expected decline in the purchasing power of money will result in the increase in the interest rate and vice versa. The component of the interest rate, which responds to changes in the purchasing power of money, Mises called a price premium.
Mises (1996) claimed that the size of the initial cut in the interest rate is closely related to the relative amount of money that is injected into the credit markets compared with the amount of money flowing into the markets for consumption goods. According to Mises, if the initial injection of money was poured solely into the consumers' markets, the interest rates would not decline. ${ }^{63}$ Nevertheless, once the interest rate is lowered, the well-known process of the business cycle is triggered. The rise in the quantity of money will eventually depress the general purchasing power of money (i.e. increase the price level). This widespread upsurge in prices will be manifested in higher interest rates owing to the price premium. Finally, a higher interest rate should upset the process initiated by the monetary expansion, and the economy is on the path to recession.
However, the key problem is whether the price premium does not raise only the nominal interest rate leaving the real interest rate at the artificially lowered level. At this moment, the explanation that the interest rate increases due to the fall in the purchasing power of money is not sufficient. If the nominal interest rate does not keep pace with inflation, the real interest rate may decline below its natural level, which will result in further lengthening of the structure of production. Mises (1996) himself mentioned this problem. It will be discussed in more detail in the section investigating the dynamics of the money supply.

The explanation of the interest rate reversion cannot therefore hinge only on the general increase in prices. The more important ingredient is to be found in Hayek's price margins, roughly (in aggregate) expressed as a ratio between prices of consumption goods and production goods $(\mathrm{Pc} / \mathrm{Pi})$. The real interest rate, which is nothing more and nothing less than a specific relative price, is reflected in the (logarithmic) price difference between goods at various stages of processing regardless of the evolution of the general price level. Although this ratio declines after the expansion in the money supply, it is eventually raised due to a substantial upsurge in the demand for consumption goods when newly-created money is

[^34]acquired by workers in higher wages. In this phase of the business cycle, prices of consumption goods are rising at a higher rate than prices of capital goods because factors of production were attracted to the production of the latter at the expense of the former. It is the relative scarcity of the present consumption goods compared with the quantity of production goods - future consumption goods being in the present at a different stage of processing that will cause the reversion of the real interest rate back to its previous level.


Figure No. 13, The dynamics of the interest rate in the Misesian system

The entire process can be illustrated in Figure No. 13. Firstly, the monetary expansion reduces the real interest rate (ratio $\mathrm{Pc} / \mathrm{Pi}$ ) from $4 \%(102 / 98)$ to $2 \%(102 / 100)$. However, economic forces, which were thoroughly elucidated in the foregoing analysis, start to operate such that the real interest rate will return to $4 \%(104 / 100)$ in the future. The monetary expansion may also increase the general price level (e.g. by $10 \%$ ), which will result in the rise of the nominal interest rate to $14 \%$. Since the process of production proceeds in time, the nominal interest rate is to be calculated by comparing the initial value of the unfinished goods at time $t_{0}$ (i.e. 100) with the value of the more processed goods at time $t_{1}$ (i.e. 114). ${ }^{64}$

Rothbard (2004), another important proponent of the Austrian business cycle theory, built upon the findings of the previous two authors. However, only a few new ideas can be found in his exposition. Rothbard's approach to the dynamics of the interest rate may be documented by the following words:

The owners of the original factors, with their increased money income, naturally hasten to spend their new money. They allocate this spending between consumption and investment in accordance with their time preferences. Let us assume that the time-preference schedules of the people remain unchanged. This is a proper assumption, since there is no reason to assume that they have changed because of the inflation. Production now no longer reflects voluntary time preferences. Business has been led by credit expansion to invest in higher stages, as if more savings were available. Since they are not, business has overinvested in the higher stages and underinvested in the lower. Consumers act promptly to re-establish their time preferences-their preferred investment/consumption proportions and price differentials. The

[^35]differentials will be re-established at the old, higher amount, i.e., the rate of interest will return to its free-market magnitude. As a result, the prices at the higher stages of production will fall drastically, the prices at the lower stages will rise again, and the entire new investment at the higher stages will have to be abandoned or sacrificed. (Rothbard 2004:996)

Rothbard (2004) also mentioned one important idea. It has been assumed so far that the (real) interest rate will eventually return to its (previous) natural level. This assumption de facto implies that money is neutral in the long run since it does not affect the natural rate of interest. The Austrian theory could accept this as a first approximation. Yet, by investigating this phenomenon in more detail, the Austrian authors usually extend their analysis by the following findings. The injection of money into the economy never enters all markets at the same time and with the same intensity. Hence, prices of various goods are affected unequally. Even in the long run, the resulting structure of relative prices might be different compared with the pre-expansion period. The pouring of money into the economy non-uniformly changes incomes and wealth of various people because the effects depend on the proximity of the particular individual to the stream of newly injected money. Some people gain, others may lose. Since people differ in their time preferences, the aggregate saving curve will end up in a different position than it was before the money injection.
It would be quite surprising if the second blade of scissors in the loanable funds market investment - did not change its position as well. If the Austrian explanation of the economywide fluctuations is at least partly correct, business cycle will undoubtedly bring about losses of capital on a large scale. Hence, the final schedule of the marginal productivity of capital is not known. As a result, since the monetary disturbances and the triggered business cycle will affect both curves in the loanable funds market, the eventual level of the natural rate of interest is impossible to determine:

A precise re-establishment of the old price-ratios between production goods and consumption goods is not possible, on the one hand because the intervention of the banks has brought about a re-distribution of property, and on the other hand because the automatic recovery of the loan market involves certain of the phenomena of a crisis, which are signs of the loss of some of the capital invested in the excessively-lengthened roundabout processes of production. (Mises 1976:364)

It may be concluded that money is not neutral even in the long run. However, this long run non-neutrality of money substantially differs from the well-known New Keynesian theory of hysteresis. The New Keynesian theory developed an idea that monetary restriction or expansion can affect the natural rate of unemployment - the relaxed monetary policy may decrease the natural level as it is attracted to a lower actual unemployment rate, whereas tighter monetary policy may leave scars on the economy in the form of permanently higher unemployment figures (Mankiw and Romer 1991). The Austrian theory by no means implies that the monetary expansion could permanently reduce not only the actual but also the natural rate of interest, and by this action support faster economic growth with lower unemployment rates, as some of the New Keynesians might suggest. As was demonstrated above, monetary expansion may disturb the equilibrium of the economy, so the eventual level of the natural rate is undetermined. At the same time, monetary expansion temporarily deflects the interest rate from its natural level and triggers the business cycle process rather than the sustainable economic growth. This brings about losses of capital rather than its new creation. It is highly unlikely that such a process would reduce long-term unemployment. The opposite evolution of the natural rate of unemployment is quite reasonable to expect.


Figure No. 14, The indeterminacy of the eventual level of the interest rates

The evolution of the interest rate in the Austrian business cycle theory is depicted in Figure No. 14. The diagram shows that at time $t_{0}$, the monetary expansion decreases both the nominal and the real rate of interest below the natural level. At one moment in the future, the process is reversed even though it is not known how long it will take before it comes. Furthermore, it is quite reasonable to expect that the nominal interest rate starts to grow earlier than the real interest rate $\left(t_{1}<t_{2}\right)$ as its growth is fuelled by one more source - price inflation. ${ }^{65}$ Nevertheless, the eventual level of both rates cannot be determined, yet the nominal interest rate will presumably exceed its initial level as it may include a higher inflation rate. The primary reason of the final indeterminacy of the market interest rate (full line) is the unknown path of the natural rate of interest (dashed line) after the money supply injection. As a result, in the Austrian theory, it is the actual market interest rate that is attracted to the natural level even though the deflection of the former from the latter brought about by the monetary shock may lead to the change in the natural rate of interest.

### 2.3 HAYEK STRIKES BACK - THE RICARDO EFFECT

In the late thirties and early forties Hayek refined his theory and introduced a novel argument that should have supported his previous findings. This new approach, however, was soon subjected to a sharp critique that led to final victory of the Keynesian revolution. ${ }^{66}$
The analysis begins at the top of the boom phase of the business cycle in which the consumers' demand is reinforced by higher earnings of the primary factors of production (especially labour). Markets for consumption goods experience a positive demand shock, which leads to the increase in their prices. At this stage of the process, Hayek assumed that the increase in the prices of consumption goods exceeds the rise in wages. Thus, the real wages, defined by Hayek as the ratio between nominal wages and the price of goods produced by the given labour, fall. Consistent with the foregoing analysis, Hayek claimed that the price margins (in his later works called the profit margins) rise. In this phase of the business cycle, Hayek proved that this particular development of the profit margins necessarily results in the

[^36]shortening of the methods of production and in the substitution of labour for capital since entrepreneurs abandon the most roundabout methods of production initiated at the beginning of the boom. Hayek's novel argument was designed to give additional reasons for the straightening of the Hayek triangle at the end of the boom and at the outset of the recession.
Hayek claimed again that the increase in the consumers' demand at the top of the boom would lead to a reduction in the investment spending. However, he added one important assumption - the entrepreneurs cannot borrow money in credit markets. Hence, he analysed a situation in which the profit margins grow, while the access to new loans is limited. By this assumption, Hayek bypassed the influence of the loanable funds market, and he focused solely on the processes within a single firm.

In this respect, the key question is which methods of production are to be used by entrepreneurs when the profit margins go up. Let us introduce Hayek's own numerical example, which will be further extended by additional calculations. Consider a hypothetical firm that has three different methods at its disposal for manufacturing a given output. At the beginning, all three methods earn the same profit rate; hence, the firm is indifferent to which of these should be used. The first method is characterised by a very low profit margin, say $1 \%$ (e.g. the entrepreneur buys inputs for 100 and sells output for 101), but the rate of turnover of this method is rather high - the firm can repeat this production process six times per year, from buying inputs to selling the final output since the final output matures just in two months. As a result, the profit rate on the annual basis is about $6 \%$ (precisely $6.15 \%$ ). The second method turns over the firm's capital once a year, and it has the profit rate of about $6 \%$ as well (e.g. inputs are bought for 95 at the beginning of the year, and the output is sold at the end of the year for 101). And finally, the profit margin of the third method is very high, $80 \%$, however, the turnover is very long - 10 years (e.g. the firm buys inputs for 56, and it sells the output 10 years later for 101). Yet, the profit rate on the annual basis of this highly roundabout method of production is also $6 \%$.

As was stated above, the cycle is at the top of the boom, and the methods of production were lengthened owing to the artificial lowering of the interest rate. Now, the newly created money is earned by workers that are trying to restore their real consumption somewhat decreased by the shift of resources to earlier stages of production. ${ }^{67}$ At this stage of the process, the strong consumers' demand flows on the markets for consumption goods, leading to a rise in their prices of, say, $5 \%$ (from 101 to 106). This increase in prices of consumption goods cannot leave the profit margins and the profit rates of the methods of production unaltered.

The profit margin of the first method rises from $1 \%$ to $6 \%$ (106/100), but the profit rate on the annual basis skyrockets to $42 \%$ (without compounding to $36 \%=6 \times 6 \%$ ). The second method registers an increase to $11 \%$ (106/95) and the third method, the most roundabout method, will be almost unaffected - the profit rate on the year basis will rise just by 0.6 pp to $6.6 \% .{ }^{68}$

Such abrupt changes in the relative profitability of various methods of production should induce entrepreneurs to use the methods that reach the final output as fast as possible. Hence, firms will be motivated to reduce the roundaboutness of their production process. Hayek (1942a) suggested various ways to speed up the production process - firms may substitute the labour-intensive methods for labour-saving methods, they can overuse the capital goods at the firms' disposal, or they might not renew the worn-out capital units.
This effect, which Hayek called the Ricardo effect, supports the fundamental conclusion of the Austrian theory - an upsurge in the consumers' demand will lead to a reduction in the

[^37]length of the production process, which can be represented by the shortening of the Hayek triangle. The core reason lies in the fact that an increase in the profit rates favours those methods of production that generate consumption goods in shorter periods of time.

At the end of this section, let us mention one important objection raised by N. Kaldor (1942), which will form the basis of the next part of this chapter. In Kaldor' point of view, it is quite absurd to assume that the increase in the rates of profit should result in the reduction of the capital invested rather than in the capital expansion. After all, it is the rise in profitability (the marginal productivity of capital) that should primarily lead to an increase in the investment spending. One possible solution to this criticism is suggested in the following part.

## 3. CONFUSION IN THE AUSTRIAN BUSINESS CYCLE THEORY

In this section, we will try to clarify some confusions, misunderstandings, and contradictions that can be detected in the Austrian analysis. At first glance, these misunderstandings might seem trivial. However, since the original authors did not use much graphical or even mathematical apparatus, they provoked discussion between the adherents and the critics of the Austrian theory.

In brief, the Austrian business cycle theory and the resulting dynamics of the interest rate can be portrayed by the following chain of implications:

1) Monetary expansion induces a decrease in the interest rate, which motivates entrepreneurs to increase investment spending.
2) However, massive investment demand should sooner or later raise the interest rate back.
3) And finally, the higher interest rate depresses the investment spending and triggers recession.

No deep scientific investigation is needed to uncover the fact that the foregoing statement is a clear example of circular reasoning. In particular, the implications do not distinguish between the movement along the (investment) curve and the shift of the entire curve. The first confusion is in the second sentence, yet the biggest problem is to be found in the logical links between the sentences as such. In the following paragraphs, we will try to explain the sources of this confusion.

In Figure No. 10, it can be clearly seen that the monetary expansion lowers the interest rate. This interest rate cut results in the growth in investment. However, a more precise conclusion should be that the decline in the interest rate increases the quantity of investment. No shift of the entire investment curve is initiated. Hence, there can be no subsequent increase in the interest rate, as is suggested in the second statement. In other words, the second statement is by no means implied by the first statement.
The fundamental problem rests in the fact that the Austrian theory is a dynamic theory, whereas the neoclassical loanable funds market is a static model. Hayek (1937) in this respect identified one important fact - the investment initiated today will require additional capital investment in the future if the entire process of capital formation is to be completed. This simple idea is depicted in Figure No. 15.


Figure No. 15, Investment that demands further investment

Other important Hayek's insights can be found in the following paragraphs:
Anything which will lead people to expect a lower rate of interest, or a larger supply of investible funds, than will actually exist when the time comes for their utilization, will in the way we have suggested force interest rates to rise much higher than would have been the case if people had not expected such a low rate. (Hayek 1937:176)

An increase in the rate of investment, or the quantity of capital goods, may have the effect of raising rather than lowering the rate of interest, if this increase has given rise to expectation of greater future supply of investible funds than is actually forthcoming. (ibid.)

Hayek's words suggest that the interest rate tends to go up when the expected flow of saving (or better to say, an expected increase in the supply of loanable funds that can be brought about even by the monetary expansion) does not arrive. The unfounded increase in the investment, which will eventually lack necessary savings, will initiate an abrupt rise in the interest rate when this error is realised. We will return to this phenomenon later on.
The second and more important problem in the sound reasoning is in the implication between statement No. 2 and No. 3 above. As was already mentioned, N. Kaldor subjected Hayek's theory to a critique. If the profitability of firms grows at the end of the boom, this fact should undoubtedly result in the outward shift of the investment curve. As is obvious in Figure No. 16, this will increase the interest rate. Nonetheless, the total quantity of the capital invested can never decline (as is suggested by statement No. 3) unless the saving curve has a perverse downward sloping shape. ${ }^{69}$ According to Kaldor, higher profitability can never induce the elimination of capital and the consequent recession, as was continually asserted by Hayek in his articles. In graphical terms, it is highly improbable that higher profit margins could ever

[^38]straighten the Hayek triangle and erase the earliest stages of production (see the grey area in Figure No. 12).


Figure No. 16, Ricardo effect in Kaldor's reasoning

At first glance, it does not seem that the Austrian theory could overcome Kaldor's critique. However, fortunately for Hayek's theory, Kaldor omitted one important step in the Austrian reasoning. It is undoubtedly true that a rise in prices of consumption goods leads to a higher profitability. Nevertheless, this upsurge in prices is caused by a reinforced consumers' demand, thoroughly elucidated in the foregoing sections. This resurrection of the demand for present consumption goods represents the end of the forced saving phenomenon. In other words, higher consumption demand reduces saving in the economy. Obviously, this must result in the increase in the interest rate. Consistently with the rise in the interest rate, the profit margins ( $\mathrm{Pc} / \mathrm{Pi}$ ) go up along with higher prices of consumption goods. Figure No. 17 may illustrate this process.


Figure No. 17, The reversion of the interest rate in the Austrian theory

As can be seen in the picture, the increase in the interest rate reduces rather than raises the amount of investment. This rise in the interest rate manifests itself in the shortening of the Hayek triangle and in the reduction of the length of production methods. This abrupt shortening, accompanied by massive losses of the artificially created capital especially in the very early stages of the production process, is identified as the recession (Figure No. 18). Furthermore, the marginal product of capital rises, as can be seen by the upward movement along the investment curve. In other words, profit margins grow, which closes the logical chain of the Austrian reasoning.


Figure No. 18, Straightening of the Hayek triangle: A road to recession

The foregoing analysis suggested that high profit margins are consistent with the reduction of the capital stock, once we realise that the Austrian analysis predicts a decrease in saving (i.e. the inward shift of the saving curve) at the top of the boom rather than an outward shift of the investment curve. However, the key question is what phenomenon moves the saving curve inwards, as only this movement protects the Austrian theory from the Kaldor critique.
The first solution may be traced in Mises's theory. So far, it has been assumed that the loanable funds market is a model with real savings, real investment, and the real interest rate. However, since in the real world the intermediation between saving and investment comes in the form of money, the initial increase in the supply of credit, brought about by the pure monetary expansion on the part of the banking sector, undoubtedly leads to an extension of the supply of real loanable funds. ${ }^{70}$ The presence of higher real supply in the credit market cuts the real interest rate, which works as a signal to entrepreneurs that more real resources may be used in the creation of new capital. Sooner or later, as Mises stated, the monetary expansion will be reflected in the lower purchasing power of money, which reduces the supply of real loanable funds to the previous level. The major reason lies in the fact that the higher price level reduces the purchasing power of funds obtained by entrepreneurs and diminishes the amount of real resources that can be attracted by them. This type of the reversion process is illustrated in Figure No. 19.

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Figure No. 19, Reversion of the interest rate based on the decline in the purchasing power of money

The question is, however, whether this movement of the supply curve, representing the supply of real loanable funds on the market, is justifiable. As was analysed earlier, the increasing price level may push up only the nominal interest rate, leaving the real rate unchanged. It is therefore possible that the monetary expansion would depress the real interest rate permanently. As a result, the crucial reversion dynamics of the interest rate will never occur. Figure No. 20, where the variables are expressed in nominal terms rather than in real terms, represents this process.


Figure No. 20, Monetary expansion and the Fisher effect

As can be seen in the figure, monetary expansion depresses the nominal interest rate from IR1 to IR2. A gradually increasing price level, which results from the monetary expansion, will provoke expectations on the part of both savers and entrepreneurs. Savers are willing to offer lower saving in expecting lower purchasing power of money in the future (the saving curve moves to the left). On the other hand, entrepreneurs are prepared to invest more for the given nominal interest rate foreseeing the fact that in the future the debts can be repaid with
depreciated money (the investment function shifts outwards). Both tendencies push the nominal interest rate upwards to IR3. This process, known as the Fisher effect, moves the nominal interest rate one-to-one with inflation, yet the real interest rate can be stuck at an artificially lower level. The problem is that there might be no mechanism that would compensate for the initial increase in the money supply. As a result, the investment can be permanently boosted since the real interest rate remains at a lower level.
The solution to this apparent puzzle might be found in the mainstream New Keynesian/monetarist theory. The foregoing analysis did not sufficiently explore the time horizon of the entire process and the dynamics of the money supply; namely, it did not distinguish between the one time increase in the money supply and the rise in the money supply growth. In what follows, four effects will be explicitly separated. Figure No. 19B demonstrates three of them.


Figure No. 19B, Monetary expansion and the reversion of the interest rate in the standard theory.

The economy starts at point $A$ at which the interest rate is at the natural level ( $r=r_{\text {nat }}$ ) and output is at the potential level $\left(\mathrm{Y}=\mathrm{Y}^{*}\right)$. Suppose that the inflation rate is zero, so the nominal interest rate is equal to the real interest rate $(i=r)$. In the first round, the increase in the nominal and the real money supply $\left(\mathrm{M}_{\mathrm{s}, 2} / \mathrm{P}_{1}>\mathrm{M}_{\mathrm{s}, 1} / \mathrm{P}_{1}\right)$ reduces the interest rate. This effect is known as the liquidity effect, and it is reflected by a movement from point A to point B ( $r_{\text {LiQ }}$ $<r_{\text {nat }}$ ). This reduction may support investment spending and subsequently output of the economy. The resulting higher incomes raise the demand for money, which may increase the interest rate. This effect is known as the income effect ( $r_{1}>r_{\text {LiQ }}$ ). It partly offsets the increase in investment spending; yet, output is still larger than before the shock, as can be seen by comparing point C and A . The partial reversion therefore comes from the money market, and it is consistent with the liquidity preference theory of interest. The interest rate is below the natural level ( $\mathrm{r}_{1}<\mathrm{r}_{\mathrm{nat}}$ ), and the output is above the potential ( $\mathrm{Y}_{1}>\mathrm{Y}^{*}$ ). According to Wicksell (1936), this negative interest rate gap should result in the increase in the price level. This
conclusion is consistent with the modern theory that predicts that in the long run the positive output gap should result in a higher price level.

Thus in the third step, the price level rises and the LM curve shifts back to the original position $\left[\mathrm{LM}_{3}\left(\mathrm{M}_{\mathrm{S}, 2} / \mathrm{P}_{2}\right)=\mathrm{LM}_{1}\left(\mathrm{M}_{\mathrm{S}, 1} / \mathrm{P}_{1}\right)\right]$ along with the reversion of the real money supply. Output is returning to its potential level, and the interest rate moves back to its natural level. The third effect might be called the price level effect. It is again derived from the money market since the nominal demand for money rises (or the real money supply declines) with a higher price level. This theory is consistent with Figure No. 19, even though it heavily relies on the Keynesian liquidity preference theory. On the other hand, the real sector is represented by the IS curve, $\mathrm{Y}^{*}$ curve, and the natural rate of interest. Thus, this simple "IS-LM-Y*" model, thoroughly discussed in Chapter 4, integrates the real and the monetary part of the economy.

The problem of the permanently increasing price level is not reflected here, because the model does not take into account the dynamics of the money supply. If the growth rate of the money supply permanently rises along with the inflation rate and the expected inflation rate, the fourth effect - the Fisher effect - may arise. However, this inclusion will separate the interest rate for the LM curve, which is constructed for the money market and the nominal interest rate $i$, and the IS curve, which is derived from the goods market and the real interest rate $r$.


Figure No. 19C, Permanent increase in the money supply growth and the Fisher effect

The permanently increasing money supply ( $\mathrm{Ms}_{\mathrm{S}, 2}>\mathrm{M}_{\mathrm{S}, 1} ; \mathrm{g}_{\mathrm{M} 2}>0 \%$ ) and the price level ( $\mathrm{P}_{2}>$ $\left.\mathrm{P}_{1} ; \pi_{2}=\pi_{2}{ }^{\mathrm{e}}=\mathrm{g}_{\mathrm{M} 2}>0 \%\right)$ are fixing the LM curve $\left(\mathrm{M}_{\mathrm{s}, 1} / \mathrm{P}_{1}=\mathrm{M}_{\mathrm{s}, 2} / \mathrm{P}_{2}=\ldots\right)$; yet, the IS curve is shifted outwards due to the higher expected inflation ( $\pi^{\mathrm{e}}>0$ ) as its position depends on the real interest rate that was lowered by the increase in the expected inflation. As can be seen in Figure No. 19C, there might be a period of booming economy ( $\mathrm{Y}_{2}>\mathrm{Y}^{*}$ in panel (a); $r_{2}<r_{\text {nat }}$ in panel (b)), in which the Mundell-Tobin effect is effective (point D ) since the positive inflation rate and the resulting higher nominal interest rate lead to a lower real demand for money, lower real interest rate, and higher output. Point $D$ is then consistent with the analysis
in Figure No. 20. In the long run, the output ends at the initial real potential level, and the real interest rate at the natural rate of interest (point E ). The movement from point D and E is accompanied by an additional increase in the price level. In the end, the nominal interest rate is higher (" $\mathrm{inata}_{\text {na }}$ " $>{ }^{\prime} \mathrm{i}_{\text {nat, }, 1}$ "), and the real demand for money is lower. Thus, the permanent monetary expansion ( $\mathrm{g}_{\mathrm{M} 2}>\mathrm{g}_{\mathrm{M} 1}=0 \%$ ) reduces one real variable - real money balances. This reduction was brought about by a temporarily higher inflation rate that exceeded the expansion in the money supply. ${ }^{71}$ On the other hand, in the long run the real output and the real interest rate are fixed at their natural levels.

This analysis may resolve the puzzle of the interest rate reversion suggested in Figure No. 19 and No. 20 that were based on the price level changes. As can be seen, it must be distinguished between the price level effect, which is the outcome of the one-time increase in the money supply, and the Fisher effect, which results from the permanent money supply expansion. However, this analysis heavily relies on the liquidity preference theory because all reversion processes originate from the increase in the demand for money. As such, the interest is considered the monetary rather than the real phenomenon, even though the long-run equilibrium of the economy is at point E , at which the real interest rate is at the initial natural level that is solely determined by real forces. On the other hand, this analysis lacks the investigation of the capital structure and of the relative prices between consumption goods and capital goods.
Hence, the Austrian analysis requires one more element for the explanation of the rise in the real interest rate and the eventual reversion of the processes initiated by the monetary expansion. This element is to be found in the growth in the consumers' demand at the end of the boom. This upsurge in demand leads to the exhaustion of the forced saving phenomenon and to the return of price margins ( $\mathrm{Pc} / \mathrm{Pi}$ ) back to their initial level. Exactly at this moment, the economy starts to suffer from the lack of saving. The artificially initiated capital structures cannot be completed, because resources are attracted back to the stages closer to consumption.


Figure No. 21, Dynamics of the interest rate in the Austrian business cycle theory (I)

[^40]The supply of real loanable funds gradually decreases, as the newly created money flows to the hands of consumers...


Figure No. 22, Dynamics of the interest rate in the Austrian business cycle theory (II)

As was discussed before, the more important factor in the entire reversion process is the dynamics of the relative prices (in this case $\mathrm{Pc} / \mathrm{Pi}$ ) rather than the movement of the general price level. This reversion process of the interest rate is depicted in Figure No. 21 and No. 22.
An alternative graphical representation of the interest-rate reversion might be based on Hayek (1937). Entrepreneurs, erroneously led by a decrease in the interest rate, expect a sufficient amount of real saving in the future. Yet, as was demonstrated before, monetary expansion cannot guarantee this permanently. Hence, the relative lack of real saving results in an increase in the interest rate that overshoots the level that would have been established in the market if the false signal in the form of the artificially lowered interest rate had not been introduced. However, it is not immediately obvious why the expectations about the future amount of real savings are to be wrong. One possible explanation can be found in the monetary part of the economy. Changes in the money supply disturb the only signal entrepreneurs may act upon - the interest rate. As a result, this deformed signal leads to a behaviour on the part of entrepreneurs that has no support in the real economy.


Figure No. 23, Alternative representation of the interest rate reversion

The cut in the interest rate brought about by the monetary expansion can be considered a regulation of price that leads to an excess of investment over (voluntary) saving. However, sooner or later, robust economic forces are activated in the economy, which eventually make this intervention ineffective. The initial decrease in the real interest rate will bring about a drop in real savings, leading to a steep increase in the interest rate - IR3 in Figure No. 23.
For some economists, even this type of analysis may not be satisfactory. Especially the fact that a rise in consumption and profitability should reduce investment might be totally fantastic at least for Keynesians. However, the Austrian capital theory may offer the answers.
Suppose that we accept the Keynesian argument that the upsurge in consumption demand and in overall profitability would lead up to a general increase in investment. Let us analyse consequent implications of the Austrian model. In this model, a rise in investment will be manifested by an increase in the roundaboutness of the production process. However, the initiated capital structures require resources not only in the form of the original means of production (labour, land), but also in the form of complementary variable capital. Assuming that resources are scarce, which is a reasonable assumption especially at the top of the boom, ${ }^{72}$ resources can be obtained only at the expense of other stages of production; namely those operating very close to final consumption. Hence, the initiated capital structures start to attract more resources from the very late stages of production. This shift should consequently reduce the supply of present consumption goods, as the initiated longer processes will provide consumption goods in the remote future. As can be seen, the Keynesian analysis leads to a paradoxical conclusion - the upsurge in the consumption demand (i.e. demand for present goods) finally brings about a reduction in the supply of consumption goods (i.e. the supply of present goods) since resources are shifted to the production of future consumption goods.

It would be quite difficult to find a more significant failure of the working of the market system. Moreover, if the increase in the demand for consumption goods results in the reduction of their supply, the entire process cannot be sustainable, as both tendencies push the prices of consumption goods upwards. With higher prices of consumption goods, the profit margins skyrocket because the ratio $\mathrm{Pc} / \mathrm{Pi}$ increases beyond all limits. ${ }^{73}$

Sooner or later, the whole process must be upset since very high profit margins and the deficient supply of present consumption goods will motivate entrepreneurs to shift a considerable part of the factors of production back to the production of final consumption goods. Hence, the upsurge in the consumption demand can never lead to the growth in capital in the economy; it can never provoke the lengthening of the Hayek triangle. ${ }^{74}$
The foregoing paragraphs suggest that the Austrian theory is endowed with sufficient tools to disprove the traditional Keynesian idea. Furthermore, taking the Keynesian argument ad absurdum, its fantastic content will be immediately unmasked. As was shown above, Keynesians assume that the increase in consumption demand should induce higher investment spending. As a result, factors of production are then attracted to early stages of production, where they earn incomes for their services. It is reasonable to expect that new incomes will be mainly used for consumption. However, at the same time, only a negligible part of the initiated processes mature in final consumption goods. This must necessarily lead to an imbalance between the supply and demand in the market for consumption goods, to a subsequent increase in their prices and eventually to a higher profitability of stages that are posited very close to final consumption.

[^41]By following the Keynesian argument, the increase in profitability leads to the additional rise in investment spending, which in the Austrian model results in further lengthening of the structure of production. As a result, more and more factors of production are attracted to processes that produce no or just a small amount of final consumption goods in the near future. By using the reductio ad absurdum argument, the Keynesian analysis necessarily implies that the continuing flow of demand for consumption goods leads to a zero supply of these goods, which is impossible. This analysis could only work if incomes earned by various factors of production that have been attracted to early stages were completely saved (Hayek 1935). Yet, no reasonable economic argument entitles us to assume that no part of the new income will be used for consumption. It is as inconceivable as the argument that the increase in consumption demand will provoke lengthening of the structure of production.

Finally, we may state one fundamental conclusion: An upsurge in the consumption demand will always bring about a partial destruction of capital in the economy, especially during the business cycle. The entire process is orchestrated by the interest rate, or more precisely - by its reverse (or U-shaped) dynamics.

## 4. THE DYNAMICS OF THE MONEY SUPPLY

So far, we have analysed the reversion process of the interest rate over the business cycle. The primary shock came from the monetary part of the economy; namely from the increase in the total money supply. Although we have not stressed that explicitly, it has been assumed that the primary increase was just a one-off shock to the money supply. However, what will be the dynamics of the interest rate if the central bank carries on with the monetary expansion?
When the reverse upward movement of the interest rate is activated, the continuing monetary expansion may reinforce the downward pressure on the interest rate. The crucial question is what increase in the money supply is necessary to overcome the reverse dynamics of the interest rate, to overcome the shortening of the Hayek triangle, and to avoid the inevitable destruction of the initiated capital structures? In other words, what size of the flow of the money supply is needed to complete the artificially lengthened processes of production? What are the necessary doses of the money supply injection that would generate new rounds of the forced saving?
One relevant answer is that it may suffice to increase the money supply by injecting the same amounts of money (e.g. the series of the money supply would be 1000; 1100; 1200; 1300...), Another might be that the constant growth rate of the money supply is adequate to permanently cut the interest rate (e.g. $10 \%$; i.e. $1000 ; 1100 ; 1210 ; 1331 \ldots$ ). The third option is that the money supply must accelerate (i.e. the growth rate of the money supply must gradually go up, e.g. from $3 \%$ to $4 \%, 5 \% \ldots$.

Since the Austrian theory has never been developed into a rigorous mathematical model, it is quite difficult to decide. However, Hayek suggested that:

We shall now assume that it does so, not at a constant, absolute rate, but at such a rate as is necessary to maintain the increased volume of real investment. This will mean a constant percentage increase in the total flow (and quantity) of money, because, if before it needed a 1 per cent addition to attract the additional resources to investment, after the total money stream (and general prices) will have risen by 1 per cent, it will need an increase of 1,01 per cent to produce the same effect and so on. (Hayek 1969:280)

Thus, if we accept Hayek's point of view, money supply must accelerate to keep the interest rate at lower levels. This conclusion can be demonstrated in several diagrams. Figure No. 24, which uses variables in real terms, illustrates that the first round effect of the monetary injection is neutralised by the consequent increase in the consumers' demand. The second round injection of the money supply is thus needed to push the real interest rate down, where the second-round increment must sufficiently exceed the initial one to depress the interest rate again.


Figure No. 24, Accelerating monetary expansion


Figure No. 25, Accelerating monetary expansion and the Fisher effect

Figure No. 25 depicts the same process with nominal variables. If the central bank is determined to fix the nominal interest rate at the artificially lower level, the inflationary process is triggered in the economy. As was discussed earlier, market nominal interest rates tend to increase along with the inflation due to the Fisher effect, as can be seen from the shifts of both the investment and saving curves. The persisting upward pressure on the nominal interest rate forces the central bank to accelerate the injection of money into the economy, which will further fuel the price inflation. Even this simple diagram raises the well-known
question of whether the central bank is capable of controlling the nominal interest rate, ${ }^{75}$ or whether the interest phenomenon is determined by real forces.
One additional impulse should be mentioned in connection with the process just studied. The permanent flow of money into the economy will eventually induce a heavy pressure in the markets for consumption goods, and the profit margins $(\mathrm{Pc} / \mathrm{Pi})$ will consequently rise. If the central bank fixes the nominal interest rate (and commercial banks follow this policy), entrepreneurs are motivated to borrow money to reap the profits that have emerged in the form of the difference between the price margins and the interest rate in the banking system. Through the new loans created in the banking system, the entrepreneurs continually acquire new money, and the process reinforces itself. In the end, new money is obtained by consumers, and the entire process of increasing prices of consumption goods, upsurge in the profit margins and new loans on the part of entrepreneurs may begin anew unless the central bank changes its monetary policy by the interest rate increase to curb this inflationary process. The graphical representation of the foregoing analysis is sketched in Figure No. 26.


Figure No. 26, Hayek triangle and the accelerating monetary expansion

At this moment, we encounter the topic that will concern us in the next section - the endogenous money supply. However, it can be concluded at this stage that monetary forces can never overcome the forces arising in the real economy. The real forces will eventually reverse the movement triggered by the monetary impulse. Hayek wrote in this connection:

As we have seen, any delay by monetary means of the adjustments made necessary by real changes can only have the effect of further accentuating these real changes, and any purely monetary change which in the first instance deflects interest rates in one direction is bound to set up forces which will ultimately change them in the opposite direction. (Hayek 1941:393)

One last question that will be investigated in this section examines the height to which the money supply can eventually rise. Nowadays, the majority of central banks explicitly or implicitly target inflation, and price stability is officially one of their major objectives. In this framework, the central bank is forced to increase the interest rate to stop the inflationary process and to cool off the economy even at the expense of a tough recession. However, it

[^42]would be a fatal error to think that the cause of the recession is to be found in the tightening of the monetary policy. A careful analysis would rather suggest that the primary source of the recession is the unsound manipulation of the interest rate at the beginning of the boom, namely cutting it down to boost the economic growth. The consequent rise in the interest rate is just a necessary reaction if the economy should not end in a hyperinflationary disaster.

Let us consider a framework where no such brake, which will eventually stop the accelerating flow of money into the economy, comes into operation. Is it possible that the Hayek triangle will be permanently lengthened? In other words, can the continually rising monetary expansion endlessly lengthen the capital structure of the economy by creating an undying source of the forced saving? It is obvious that the answer can be hardly in the affirmative, since the ultimate frontier is the total collapse of the currency. This hyperinflationary disaster is frequently stressed by Mises (1996) and Rothbard (2004), but also Hayek opened a similar question:

It has always been an open question to me as to how long a process of continued inflation, not checked by a built-in limit on the supply of money and credit, could effectively maintain investment above the volume justified by the voluntary rate of savings. It may well be that this inevitable check only comes when inflation becomes so rampant-as the progressively higher rate of inflation required to maintain a given volume of investment must make it sooner or later-that money ceases to be an adequate accounting basis. (Hayek 1969:282)

However, many forces may be detected that will undoubtedly block further lengthening of the structure of production, far earlier than the ultimate collapse of the currency occurs. The first brake is the increase in the risk premium, which usually arises during galloping inflation. Higher risk premium then discourages long-term investment in favour of the shorter ones. The shorter methods of production are then preferable to roundabout methods even if the inflow of money into the system continues. Higher inflation can also create fictitious profits when firms keep their accounting books in historical costs and do not properly revalue their capital, undermining the real value of its amortization. In such a case, the firms, acting upon the misleading information about nominal profits, consume part of their capital instead of keeping it intact.

Even if we neglect the risk premium and the historical costs accounting, one strong impulse for capital destruction will undoubtedly emerge when the inflation rate reaches relatively high levels. So far, mainly the money supply dynamics has been analysed. However, economic disturbances may also arise owing to the demand for money. In the situation of rapidly growing prices, it is quite reasonable to assume that people will hasten to reduce their real money balances, which are continuously eroded by high inflation rates. Hence, the dissolved real money balances may expand the flow of consumption demand. As a result, the consumption stages of production will experience an additional source of demand leading to an upsurge in their profitability. This phenomenon can therefore bring the lengthening of the capital structure to a standstill since the resources will be attracted back to the consumption stages even if the monetary expansion feeding the demand in the early stages of production continues. To put it differently, a continual monetary expansion may change the attitudes of people towards holding money. By reducing the optimal level of their real money balances, people may further reinforce the overall consumption demand, which will destroy part of the capital stock in the economy.

We can conclude this section with the statement that even the accelerating monetary expansion does not have enough power to suppress the reversion dynamics of the interest rate studied in this paper.

## 5. FURTHER DYNAMICS OF THE INTEREST RATE IN THE AUSTRIAN THEORY

It has been consistently assumed throughout that the initial impulse for the business cycle and the consequent dynamics of the interest rate are to be found in the behaviour of the central bank and the monetary policy as such, especially in the interest rate cut below the natural level. This stream of reasoning, which can be found in the works of Mises (1996) and Rothbard (2004), will be modified in the following parts. Specifically, we will investigate a different evolution of the interest rate that can be also observed over the business cycle.
Hayek (1933) in his works admitted that the initial impulse for the business cycle might arise in the real economy. Following his reasoning, let us assume that the economy is hit by a positive technological shock, which is a modern analytical counterpart of the older assumption of a new invention that suddenly arises in the economy.
In the Böhm-Bawerk theory, the new invention can further improve the superiority of present goods over future goods, and it makes the time shape of the income stream steeper (Fisher 1930). Both will increase the premium of present goods over future goods. In the well-known Fisherian analysis, new invention will improve the investment opportunities in the economy, which will lead to an increase in the natural rate of interest at least for the period till the possibilities of the new invention are fully exhausted. ${ }^{76}$ Hayek (1941) in his work follows the Fisherian or even the Knightian analysis by stressing the importance of the marginal productivity of capital, which in the Austrian theory is defined as the additional final output of consumption goods that results from a marginal rise in the roundaboutness of the production process. A positive technological shock, or a new invention, will therefore increase the schedule of the marginal productivity of capital. ${ }^{77}$
In the loanable funds market model, this improvement will be manifested as the outward shift of the investment curve. For the given shape of the saving curve, the natural rate of interest must necessarily increase. However, according to Hayek (1933), if no reaction is activated in the monetary part of the economy, there is no reason to expect that the phenomenon of the business cycle should ever be triggered because the voluntary saving is consistent with higher investment.

As can be seen in Figure No. 27, the initial natural rate of interest is no longer consistent with the equilibrium in the loanable funds market, since at this level investment exceeds voluntary saving. To clear the market, the natural rate of interest must go up. It is obvious that the increase would be more significant for a steeper saving curve. As the interest rate rises, two phenomena start to operate. First, part of the investment is detracted because only the most profitable ones can survive with a higher interest rate. Second, a higher interest rate attracts new voluntary savings. In the end, more investment projects are initiated compared to the situation before the invention. We may also add that the number of the investment projects opened will crucially depend on the willingness of people to forego part of their present consumption.

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Figure No. 27, Positive technological shock in the loanable funds market


Figure No. 28, Positive technological shock with perfectly inelastic saving curve - the Hayek triangle representation

A little bit puzzling could be the attempt to illustrate such a change of the natural rate of interest in the Hayek triangle model because it was mainly designed to depict changes in the saving rate. Nonetheless, since the interest rate increases, it is obvious that the triangle becomes steeper. At the same time, it is quite reasonable to assume that the new invention allows the given output of consumption goods to be produced (with the given amount of primary factors of production) more rapidly than before. Hence, the triangle will also become shorter. ${ }^{78}$ If the saving curve was perfectly inelastic, in other words, if the increase in the interest rate did not persuade people to postpone their consumption, the positive technological shock would transform the Hayek triangle in a way that can be seen in Figure No. 28. On the other hand, if people respond to changes in the interest rate, its rise will somewhat reduce their present consumption, which allows partial increase in the roundaboutness of the production process depicted in Figure No. 29.

[^44]

Figure No. 29, Positive technological shock with elastic saving curve - the Hayek triangle representation

In Chapter 1 of this dissertation, higher productivity of longer methods was partly explained by the application of inventions that could not be used in shorter methods due to insufficient saving. These inventions were called endogenous since they were present in the economy, yet they were not used owing to the lack of capital. In other words, it was known that better technology and different techniques associated with longer methods may increase the output of consumption goods, however, these methods would require more waiting. On the other hand, the technological shock presented in Figure No. 27 and No. 28 suddenly arises "outside the economy". The new technology was not known, and it allows the production time to be shortened. In other words, it speeds up the production process, and consumption goods may mature earlier than before the positive technological shock. These new inventions might be therefore called exogenous.
The foregoing analysis suggests that the positive technological shock raises the natural rate of interest. However, the model intentionally disregarded the monetary part of the economy. In the real world, saving and investment are traded in the form of money. Can we assume that money will not play any significant role in this transmission? In other words, will the response of the banking sector be purely neutral to the increase in the natural interest? Will the banking system follow the evolution of the natural rate of interest by increasing the market interest rates in the economy?
A hypothetical schema in Figure No. 30 shows two possible responses of the banking system to the increase in the natural rate of interest. This schema represents a simplified balance sheet of commercial banks initially holding a $100 \%$ reserve ratio. In situation (a), banks hold 100 in reserves to fully back demand deposits, whereas loans of the volume of 200 are financed by time deposits that may not be immediately withdrawn by clients. ${ }^{79}$ This situation can be also interpreted as follows: the investment is financed by voluntary saving, and the market interest rate is equal to the natural level, which is, for example, $5 \%$.
Consider a positive technological shock increasing the investment demand that leads to a higher demand for loans in the banking sector (e.g. from 200 to 300). At this stage, the demand of 300 exceeds the supply of 200, and the interest rate in the banking sector should go up. If this happens (situation b), part of the investment demand is repelled (e.g. from 300 to 250 ), and presumably some savings are attracted (from 200 to 250 ). As we can see, the

[^45]voluntary saving is again in balance with investment. This equilibrium has been brought about by an increase in the natural (and market) interest rate from $5 \%$ to (say) $7 \%$.

However, a different response from the banking sector is also conceivable. Let us assume that the banking sector keeps the market interest rate unaltered, and it fully accommodates the increased demand for loans by creating new demand deposits (situation c). In this case, the phenomenon of the forced saving emerges since the investment (300) exceeds voluntary savings (200), and the market interest rate (5\%) does not keep pace with its natural counterpart (7\%). ${ }^{80}$ The imbalance between voluntary saving and investment is possible due to an elastic supply of money. Specifically, money supply increases from 100 to 200, as banks ceased to hold $100 \%$ reserves - the reserve-deposit ratio falls from $100 \%$ to $50 \%$ ( $=100 / 200$ ).
(a) $\mathrm{IR}=5 \%$
(b) $\mathrm{IR}=7 \%$


Figure No. 30, Endogeneity of money in the Austrian model

It can be perfectly seen that in situation (c), the interest rate fails to transmit the information about the increase in the investment demand. It would be surprising if this failure had no repercussions in the real economy.

We have suggested two extreme responses of the banking sector. In the real world, the reaction of the banking sector may be at any point of this hypothetical interval. Hayek (1933) assumed that once the demand for loans increases, commercial banks seldom raise their interest rates sufficiently high and rapidly enough. Their typical response is much closer to situation (c) as they rather increase their credit capacity instead of the interest rate. If there are banks that do not provide new loans during the general expansion in credit, they will accumulate more reserves from other banks. Moreover, higher demand for credit may be lessened only by the increase in the interest rate. Yet, such strategy would lead to a

[^46]disadvantage in the competition with the banks that reduced their reserve ratio and expanded loans granted to entrepreneurs. Thus, the accumulation of reserves and the competition from other banks may persuade the more conservative banks to follow the "mainstream" behaviour. As a result, the credit expansion will become general. The market interest rate is then stuck below the natural level - it does not sufficiently increase to keep pace with the natural rate of interest.

So far, we have completely disregarded the response of the central bank. However, it is not very far-fetched to assume that the central bank will reinforce the entire process rather than dampen it. Since in the modern world central banks fix their interest rates at some predetermined level, the commercial banks may keep their reserve ratio intact simply by borrowing high-powered money directly from the central bank. As a result, the increased demand for loans is then transmitted throughout the banking system directly to the central bank, which eventually feeds up the monetary expansion.

Although the central bank may follow some policy rule in setting the policy rates, the biggest weight is usually put on the output gap and inflation, not on credit or monetary aggregates. Since only these aggregates provide the vital information about the fact that the demand for loans has been somewhat intensified, it would require an enormous precision in the central bank predictions to uncover that the natural rate of interest increased. ${ }^{81}$ As will be demonstrated later, the output gap and higher inflation are rather the outcome of the entire process. They both come into existence especially due to the inability of the central bank to keep track with the natural rate of interest. In other words, by increasing the policy rates after the positive output and inflationary gaps have occurred, it is too late for the central bank to stop the course of events known as the business cycle.

Hence, it is quite reasonable to assume that under the current banking and monetary system, the market interest rate does not respond sufficiently to the increase in the natural rate of interest, either due to the accommodative behaviour of the commercial banks or due to the lag in the monetary policy response.

The consequent dynamics of the interest rate may be described by the following system of figures. The positive technological shock shifts the investment demand outwards. For simplicity, Figure No. 31 assumes that the saving curve is vertical. Nonetheless, higher investment demand is fully satisfied by the inflow of money from the banking sector. At this stage, it is not at all important whether the primary source of the newly injected money is the central bank, the commercial banks, or both. What is important is the fact that part of the investment is financed by newly created money rather than by voluntary saving.

As can be seen in the picture, the natural rate of interest exceeds the market rate, which is equivalent to the statement that the forced saving phenomenon has emerged in the economy. Forced saving allows the capital structure to be artificially lengthened, as is obvious in Figure No. 32.

[^47]

Figure No. 31, Endogeneity of money in the Austrian model


Figure No. 32, Investment demand being accommodated by the injection of money - Hayek triangle representation

It should be perfectly clear that also in this case, the business cycle process has been triggered. The structure of production is artificially lengthened since more investment has been made than is justified by voluntary saving. The only difference lies in the fact that the money supply is endogenous rather than exogenous because the banking system, for one reason or another, is fixing the interest rate.

One more fact deserves our attention. Call and Cochran (2000) identified that the economic boom in the process just described is rather peculiar. ${ }^{82}$ It contains two components, indistinguishable in the official statistics, yet the Austrian theory provides us with the necessary insight to disentangle them. The first part of the faster growth of the economy can be attributed to the positive technological shock (new invention), and it can be considered sound and genuine. However, the second part of growth has been provoked by artificial credit expansion that has accommodated the increased demand for loans. Thus, the second part is not supported by voluntary saving. We may also say that the growth of the economy is faster than it would have been if no inflow of money into the system had ever arisen. It seems that

[^48]this analysis is quite consistent with official statistics about the economic boom since they usually suggest that investment is far above the average in this phase of the business cycle. The Austrian theory would predict similar behaviour as the credit expansion is mainly channelled to the formation of new capital structures because the largest demand for credit comes from the sectors that experienced the increase in the marginal productivity of capital. ${ }^{83}$

Nevertheless, as has been thoroughly analysed in the previous sections, the artificial boom cannot be maintained forever. Sooner or later, the newly created money ends up in the hands of consumers. This will terminate the forced saving source of the lengthening of the capital structure in the economy. Resources will be attracted back to the consumption stages of the production process, and the new capital structures will never be completed. This phase of the business cycle seems to be also consistent with the data as they suggest that in the recession, the investment spending somewhat collapses. The same prediction is made by the Austrian model.

The crucial source for the capital shortening is to be found in the reverse dynamics of the interest rate that will gradually move to its natural level. Only the additional round of credit expansion can overturn this reverse process. However, it would be rather surprising if the central bank did not play a significant role in this process. The increased demand for cash is usually observed in booms, so banks that exposed themselves to a lower reserve-deposit ratio due to the credit expansion can hardly continue with this policy without the backing from the central bank. In other words, a relaxed monetary policy is needed to keep the credit expansion underway. Otherwise, the commercial banks ought to stop granting new loans, as their exposition would become too fragile. At this point, it is not at all necessary to add that only an accelerating credit expansion may keep the market interest rate below the natural level. Thus, if the central bank raises its interest rate, the commercial banks will follow this policy, and the boom will be terminated. The consequences of the credit expansion are depicted in Figure No. 33 and No. 34.


Figure No. 33, The reverse dynamics of the interest rate

[^49]

Figure No. 34, Eventual recession

We have demonstrated that a genuine growth of the economy might end up in a painful recession if the initial boom was intensified by the inflow of money from the banking system that was sluggish in raising the interest rates when the investment demand increased. At the same time, it is extremely difficult for the monetary authorities to detect from the official statistics that the growth in output is not sustainable. Their own (in)action may be the source of this unsustainability even if the interest rate policy is considered and praised as being passive.
One last dynamics of the interest rate remains to be analysed. This dynamics can be found in one of the most difficult books of Hayek (1941). Let us again consider a positive technological shock. This time, both the saving curve and the amount of money in the economy are assumed to be fixed. Not only does the central bank keep the monetary base constant but also the entire super-structure of deposits erected on it by commercial banks will be invariable.

A positive technological shock shifts the investment curve outwards and raises the natural rate of interest. Since we assume a vertical saving curve and a fixed amount of money, the amount of investment cannot increase, and the rise in the marginal productivity of capital will only result in the upsurge in the market (and natural) interest rate.

However, one important source may exist that will allow entrepreneurs to increase the amount of resources at their command. This source has been already mentioned when analyzing the limits of the monetary expansion in the lengthening of the structure of production. New investment may be also financed out of the real money balances held by people. In this case, it is critical to distinguish between saving, which represents part of the flow of income that has not been consumed, and money balances, which represent the most liquid part of the stock of assets accumulated by households over a certain period of time. Money balances as a stock are increased either by the reduction of the flow of consumption or saving. On the other hand, each flow may be increased without reducing the second flow when people dissolve part of their money balances.

As a result, money balances may become the source of financing increased investment spending when the money supply is fixed, and people are reluctant to reduce present
consumption. As was stated above, positive technological shock raises the interest rate. If the demand for money is sufficiently elastic with respect to the interest rate, people may rearrange their portfolio of assets by reducing their money balances. By buying interest-bearing assets without decreasing present consumption, people transfer part of their money balances to entrepreneurs that offer higher interest rates. This stream of money allows firms to attract resources, and the capital structure may be lengthened.
At the same time, it is important to realise that with the decrease in the demand for money, the demand for consumption goods is not necessarily reduced. Hence, this situation is similar to the pure credit expansion. The investment spending increases even though the demand for consumption goods remains unaltered. Early stages of production attract resources; yet, if there are no idle resources in the economy, these can be obtained only at the expense of stages that are close to final consumption. Even in this situation, the phenomenon of forced saving arises, although it is quite inaccurate to call this saving "forced". It is just the result of voluntary decisions made by people rather than of the manipulation with the money supply. In the monetary economy, money is a loose joint that may create an imbalance between investment and saving (Hayek 1941:408). This imbalance may arise not only due to changes in the money supply but also owing to the alteration in the demand for money. ${ }^{84}$
This situation resembles the theory analysed in Figures No. 32 and No. 33. The only difference lies in the fact that the excess of investment over saving is financed by the released money balances rather than by the monetary expansion. Nevertheless, the economy must undergo a sequence of events closely resembling the business cycle. Money is used to start more roundabout methods of production at the expense of final consumption goods. In the end, money is earned by the original factors of production (mainly labour), and unless new incomes are saved in full, markets for consumption goods experience an increase in the consumption demand. The resulting higher profitability attracts resources back to the production of consumption goods. Since the economy does not have enough factors of production to simultaneously finish the newly initiated capital formations and to satisfy the intensified demand for consumption goods, some sectors in the economy must lose this struggle for resources.

As was demonstrated before, the winning side is always the consumption stage. The early stages could continue with their production only if the source of the demand for their output did not dry up. In this case, the source originated in the released money balances. However, it is quite inconceivable that another round of the release will be initiated. Compared with the volume of money that may be delivered by the monetary expansion, the amount of real money balances held by people is limited. The willingness to further dissolve money balances may be also confined by the fact that the higher demand for goods in the economy will raise prices, which by definition erodes the real money balances. In other words, higher prices necessitate more assets to be held in the form of money, and the money-holdings source of the expansion will therefore dry up.
Hence, we may conclude that the source of investment in the form of released money balances is rather mild, and the business cycle thus triggered would be quite moderate. Its strength critically depends on the interest-elasticity of the demand for money. The higher the sensitivity, the higher amount of money holdings may be released in the first round. If the money demand was perfectly elastic, a case known as the liquidity trap, the business cycle thus triggered could be considerable.

[^50]The entire process of the business cycle based on the elastic money demand function is illustrated in the following system of diagrams. Figure No. 35 shows that the positive technological shock drives the natural rate of interest up (from IRnat to IR'nat). However, if the demand for money is sensitive to the interest rate, people release part of their real money balances $\left(\Delta \mathrm{Md}_{\mathrm{r}}\right)$, and the market interest rate $\left(\mathrm{IR}_{\mathrm{mkt}}\right)$ does not keep pace with the natural level. Yet, part of the investment spending is not financed by the flow of saving (by the reduction of consumption of present goods).


Figure No. 35, A positive technological shock and a partial release of money balances

Figure No. 36 demonstrates that this release of money balances allows partial lengthening of the structure of production by initiating roundabout methods of production that would not be otherwise allowed by a pure flow of saving. However, as can be seen in Figure No. 37, sooner or later, the market interest rate is raised to the natural level when the flow of money enters the consumption goods markets. The economy loses part of the capital, formation of which started owing to the released money holdings. Figure No. 38 depicts the fact that higher interest rate cannot leave the structure of production unaltered, and the eventual outcome is a partial destruction of capital in the economy and recession.


Figure No. 36, A positive technological shock and a partial release of money balances


Figure No. 37, Interest rate reversion


Figure No. 38, Eventual recession

The foregoing analysis suggests that the business cycle may occur even with a fixed money supply. Austrian authors usually identify the cause of the boom-bust cycle in the elastic supply of money, which is a necessary outcome of the central banking system and fractional reserves. Abolishing one or the other is the essential condition to get rid of this highly unfortunate phenomenon. However, our analysis has shown that if the demand for money is significantly sensitive to the changes in the interest rate, business cycles may occur even in the world without the central bank.

## 6. THE NATURAL OUTPUT

So far, it has been assumed that the economy used all its resources in full. In more modern terms, it operated at its potential, or natural, or full-employment level. ${ }^{85}$ Hayek (1941) in his later works relaxed this assumption. His analysis begins in a state of the overall abundance of

[^51]factors of production and consumption goods. Yet, our analysis will not follow his original reasoning, since he developed this idea for a positive technological shock and released money holdings.

We will explore consequences of the monetary expansion in the economy operating far below the potential level. Answering the question of how it is possible that some resources are not fully utilised will be postponed for a while. As was demonstrated earlier, monetary expansion will cause an increase in the demand for production goods, which can be met only by transferring resources from very late stages of production-from stages that are rather close to final consumption. However, if the economy suffers from a massive unemployment of not only the labour force but also of other complementary factors of production, especially in the form of "floating" or circulating capital, new capital formations may be initiated without harming the continuous flow of goods on consumption markets. No resources need to be released from the stages of production that provide consumption goods in the near future or even in the present.

There can be no doubt that by absorbing previously idle factors of production, the total output of capital goods may be extended. In terms of the Austrian theory of capital, more roundabout methods of production are initiated. The factors of production earn incomes for their services in the lengthening of the structure of production (i.e. in extending the supply of future consumption goods). These incomes are then mainly used for purchasing present consumption goods. Meanwhile, we assume that there are many unsold consumption goods in the stocks of firms. Hence, the higher demand may be satisfied without pressing the producers of consumption goods to increase their output. At this moment, the extended number of working people is fully satisfied by the extended supply of consumption goods.
However, sooner or later, as new incomes are earned in all stages of production and the demand increases, the stock of previously unsold consumption goods must be exhausted. At the same time, higher demand for consumption goods may be transmitted even to the earliest stages of production, leading to a further lengthening of the structure of production. Hence, a new need for resources is felt in all stages of production. If some factors of production are still unemployed, they may be absorbed not only in processes creating new capital goods but also in final consumption stages. Nonetheless, at this moment it is quite reasonable to assume that the flow of consumption goods is not as robust as before, especially if the monetary expansion proceeds by channelling new funds to early stages of production, where new capital goods are being formed and where the majority of previously idle resources are employed. Hence, a critical moment must arise at which the extended number of workers, who were initially idle, must cope with more or less constant (or at a decreasing rate maturing) supply of consumption goods.

It also has to be stressed that the newly initiated processes of capital formation will provide consumption goods only after some period of time, and this process cannot be hastened. On the other hand, the demand for present consumption goods is further enhanced out of the incomes of workers that have been previously unemployed.
It is perfectly conceivable that the continuing monetary expansion will eventually reduce the unemployment to its natural level, and full employment will arise. Here comes the moment when the structure of production is lengthened to its maximum point since there are no additional idle resources to be used. At this point, the economy reaches its potential level. At first glance, it seems that the monetary expansion has eliminated unemployment and rescued the economy from the recession.
The crucial question is, however, whether the allocation of labour and other resources, created by the continuing monetary expansion, is sustainable. The newly initiated capital formations
and the structure of the production process must be so erected that it will be maintained by an inflow of resources in the future. In other words, the system of demands and supplies and the structure of prices created by the monetary expansion must be consistent with the price structure that will be established when the money injection ceases. As a result, the flow of production goods in every stage of the production process (i.e. the flow of future consumption goods) must be met by the flow of the demand for these goods in the form of voluntary saving, even without the (life-)support provided by the central bank.
However, it is very likely that this ideal match has not been set up by the monetary expansion. The new money was predominantly channelled to early stages, where the production apparatus was expanded. On the other hand, the demand, which is fuelled by incomes here generated, is directed mainly on the markets for present consumption goods. Hence, we observe a mismatch rather than a match between the supply and demand for present and future consumption goods.

Since the production process proceeds in time, the newly initiated more roundabout methods of production require further resources for completion. It is obvious that only fully completed processes may provide final consumption goods. However, once the full employment state of the economy is achieved, early stages may obtain necessary resources only at the expense of later stages - at the expense of production of final consumption goods. As we demonstrated before, factors of production may be attracted to early stages only by additional rounds of the money injection. If this happens, the given number of workers has to be satisfied with a lower amount of consumption goods. Thus, we have arrived at a well-known picture of the economy that is producing more future consumption goods than is justified by voluntary saving, whereas consumers demand more present consumption goods or goods that will mature in the near future.

As we saw in the preceding chapters, unless the monetary expansion accelerates, resources must be eventually attracted back to stages that are closer to final consumption. Paradoxically, the economy would require much more factors of production than the society is endowed with to maintain the artificially lengthened structure of production intact. At the same time, the major parts of incomes of these factors of production should be used on the purchases of future goods rather than on present goods, otherwise new resources will be required to keep pace with the increased consumption demand.
It is an indisputable fact that the monetary expansion has the power to fully employ all resources in the economy. However, in the end, the allocation of resources thus created cannot be sustained, and the newly erected capital structures must collapse. Many projects in the early stages of production will be abandoned since there are not enough resources to complete them, as the factors of production of all kinds are in greater demand in stages closer to final consumption.

Together with this abandonment of capital, the complementary labourers are released on the labour market. However, it would be quite naïve to presume that they will be immediately absorbed in the late stages of production. We have to realise that the economy loses large amounts of capital, which results in the sharp decline in the marginal productivity of labour. These workers are thus employable only for very low real wages or they may be entirely unemployable.

In the end, massive unemployment brings about a depressed demand for final consumption goods, leading to their excessive hoarding in the stocks of firms. As a result, the recession will become widespread, eventually affecting all stages of production. It is quite interesting to see that the economy may fully employ its resources, yet if the allocation provoked by monetary expansion is not in line with the (inter-temporal) preferences of consumers, such situation
ends up in a wide-scale unemployment of these resources. As was demonstrated before, the unsustainability of the "potential" output stems from the fact that there are not enough resources to keep the lengthened structure of production intact unless people abruptly change their preferences in favour of future consumption goods.
Other points deserve some attention as well. The eventual recessionary state of the economy closely resembles the state we started with - massive unemployment of resources of various types, unsold stocks of consumption goods, etc. Yet, it seems that the monetary expansion is not the best tool to fully employ these resources again. At the same time, much of the capital stock is lost, so the potential output itself is surely lowered in the recession. This observation is quite in line with recent studies about the potential output after the financial crisis. ${ }^{86}$

As we can see, the monetary expansion may move the economy to the potential level; however, owing to the general mismatch in the allocation of resources, this situation is not sustainable. The economy ends up with a lower level of potential output than before the monetary expansion started due to the partial destruction of capital. Furthermore, it would be naive to assume that the banking system is not affected by this process of capital abandonment. It is very likely that balance sheets of banks will be disrupted by the overall collapse in values of various collaterals. However, the second round effects of the capital destruction are not at the centre of our investigations. ${ }^{87}$

## 7. THE NATURAL RATE OF INTEREST

The last question that will be explored is connected with the evolution of the natural rate of interest over the business cycle, depicted in Figure No. 14. The natural rate of interest in the economic boom presumably rises, as was suggested by the previous analysis, especially if the boom itself was provoked by a positive technological shock. However, if the central bank (or the entire banking system) does not follow the behaviour of the natural rate of interest, and the market interest rate increase is somewhat delayed, the genuine economic growth is fuelled by additional monetary and credit expansion. Then powerful economic forces are triggered that will eventually reverse the process of economic expansion since the artificial part of the economic boom is not sustainable.

We modelled that there is a tendency for the market interest rate to return to the natural level. However, what is even more important is the overall restructuring of the economy after the money-induced boom. This process is accompanied by a significant loss of many capital structures and massive unemployment. Hence, it would be naive to insist that the natural rate of interest remains unaltered in this turbulent process. Nevertheless, are there any tendencies that the pure economic analysis may propose, or is the evolution of the natural rate of interest hidden in obscurity?

If we look at the recent behaviour of the market interest rate and at the data about the price inflation, both would seem quite puzzling for the theory here developed. The interest rates have been depressed to zero for a very long time, and the price inflation is insignificant. However, the Austrian theory predicts that the economy should suffer from a galloping inflation if the interest rates are so low - if they are below the natural level.

One possible explanation, which immediately springs to one's mind, is that the natural rate of interest must be close to zero as well. The Austrian theory predicts accelerating price inflation, not if the interest rate is low, but only if the gap between the market interest rate and its natural counterpart is negative enough. Yet, if there is no such gap, no inflation will arise.

[^52]Since nowadays we observe very low inflation even in the environment of negligible interest rates, one must immediately conclude that the natural rate of interest is close to zero as well. ${ }^{88}$

Is there any explanation within the Austrian model which would deal with the fact that in the recession the natural rate of interest declines to such a low level? Some solutions may be detected. One can be found on the supply side of the loanable funds market, the second on the demand side.

A very low natural rate of interest suggests that the time preferences of people are rather low. At the first glance, it is quite difficult to find any reason accounting for the fact that people value present goods almost the same as future goods in the recession. Yet, a thorough investigation made by Fisher (1930) may provide us with some insights.

If people expect that the time shape of their flow of income will be flat or even decreasing in the recession, their time preference may be somewhat reduced. However, the permanent income hypothesis suggests that the recession is connected mainly with a slump in the transitory income rather than in the permanent part. People should therefore expect that their income will rise in the foreseeable future. From this point of view, in the recession, when incomes are relatively low, people should reduce saving in order to smooth their consumption flow. Thus, the interest rate should go up rather than down. The descending tendency of the interest rate may be observed only if people expect that the recession will be prolonged or if they expect that it will become even deeper. These scenarios would really cause a decreasing time shape of the expected income path.
The second reason for a decline in the time preference is closely related to the first one. Recession and a depressed income usually bring about higher uncertainty regarding the path of future income. As a result, people may save more just for precautionary reasons, which will cause a decline in the interest rate in the market. On the other hand, an economic slump is by definition connected with lower income. Fisher (1930) suggested that the time preference may be negatively related to the level of income. With a lower income, the time preference should increase, pushing the interest rates upwards.

Higher risk in the recession may induce another phenomenon. People may dispose of the risky assets and reallocate their portfolio to safer assets. As a result, the interest rate on the former will go up, whereas it will decrease on the latter. The natural rate of interest is a theoretical concept, thus it is quite difficult to pick up the most representative counterpart from the real world. Moreover, in the recession the spread between the safe rate and the risky rate of interest is extended making this choice even harder. If we select the safe interest rate, then it presumably declines in periods of higher risk premium, and the recession may be a good example of such situation.

This brings us to another important fact. It is perfectly conceivable that the market rate of interest is far above zero even if the central bank sets its interest rate close to zero. Such monetary policy is just a measure to heal balance sheets of commercial banks, unfavourably hit by the recessionary process. In other words, by borrowing money from the central bank for almost zero and lending it for a much higher market interest rate, commercial banks may improve their profitability and stability, adversely affected in the crisis. At the same time, the overall demand for loans in recession is not very high, so no significant amounts of money may be finally injected into the real economy.

[^53]In the last sentence, the second blade of the loanable funds market was mentioned - the demand for loanable funds. Since the recession is usually associated with massive losses of physical capital, the willingness of firms to extend their capital stock is frozen. As the investment spending represents the flow into the capital stock, it may be significantly reduced when the demand for capital falls. The depression of the investment spending may be also provoked by commercial banks that are reluctant to lend money if the overall value of collateral in the economy declines. Hence, we can find many reasons why the investment curve in the loanable funds market is depressed in the recession. The sluggish investment demand may then decrease the natural rate of interest to very low levels.

So far, we have identified various reasons for a very low natural rate of interest that can originate either on the supply side or on the demand side of the loanable funds market. Finally, we may also imagine a situation in which the loanable funds market is totally unaffected by changes in the monetary policy. This situation is the well-known liquidity trap envisioned by Keynes (1936). Once the demand for money is infinitely elastic, which may arise in the case of very low nominal interest rates, the massive injections of the money supply do not enter the loanable funds market, as all money is immediately absorbed in the money holdings of the individuals. The supply curve in the loanable funds market will not be shifted to the right, because the money supply increase, pushing the curve to the right, will be instantaneously counterbalanced by a one-for-one increase in the demand for money, moving this curve back.

We indicated at the beginning of this section that the inability of the central bank to follow the natural rate of interest may trigger the business cycle. Over the business cycle, the structure of production could be so disorganised that the eventual level of the natural rate of interest is hidden in the dark. This may cause a serious problem for the central bank in pursuing practical monetary policy, and it raises the question whether the interest rate is the best tool the central bank should use in conducting the monetary policy. This paper suggested that the answer might be not in the affirmative.

## 8. CONCLUSIONS

This paper outlined the dynamics of the interest rate in the Austrian model of the business cycle. The first part suggested that the effort of the central bank to keep the interest rate below the natural level cannot be successful, as the economic forces will eventually reverse the interest rate back. It was also shown that the return of the interest rate is always accompanied by the business cycle phenomenon.

The next part thoroughly investigated the nature of the forces that push the interest rate back to the natural level. However, many confusions in the analysis have arisen, so the third part tried to clear them up. In the fourth part, the limits of the power of the monetary policy were elucidated. The fifth part, on the other hand, relaxed the assumption that the initial source of the business cycle was on the part of the central bank. The sixth and the seventh part reintroduced the monetary policy as a powerful means to affect the overall performance of the economy. However, many doubts have been raised whether this policy should be used even in the recessionary state of the economy.
Since the real and monetary parts of the economy are so interconnected, the central bank response to events in the real economy can never leave the real variables, on which the central bank based its action, unaffected. As a result, conducting the monetary policy even in a good faith may provoke more damage than anyone could have imagined before the action was taken.

As our understanding of the complicated processes in the economy is rather limited, the institutional framework, in which the central banks nowadays operate, seems to give them too much power and discretion. It is based on the illusory idea that the knowledge has so advanced over the last thirty years that the economy may be fine-tuned without any considerable risk of failure.

## REFERENCES

Barnett, William II and Walter Block 2006. On Hayekian Triangles. Procesos De Mercado:Revista Europea De Economia Politica III(2): 39-141.

Böhm-Bawerk, Eugen von 1890 [1884]. Capital and Interest. New York: McMillan and Co.

Böhm-Bawerk, Eugen von 1891 [1888]. Positive Theory of Capital. New York: G. E. Stechert \& Co.

Cassel, Gustav 1928. The Rate of Interest, the Bank Rate, and the Stabilization of Prices. The Quarterly Journal of Economics 42(4): 511-529.

Cochran, John P. 2004. Capital, Monetary Calculation, and the Trade Cycle: The Importance of Sound Money. The Quarterly Journal of Austrian Economics 7(1): 17-25.

Cochran, John P. and Call, Steven P. 1998. Free Banking and Credit Creation: Implications for Business Cycle Theory. The Quarterly Journal of Austrian Economics 3(3): 29-40.

Cochran, John P. and Call, Steven P. 2000. The Role of Fractional-Reserve Banking and Financial Intermediation in the Money Supply Process: Keynes and the Austrians. The Quarterly Journal of Austrian Economics 1(3): 35-50.

Cochran, John P.; Call, Steven P. and Glahe, Fred R. 1999. Credit Creation or Financial Intermediation?: Fractional-Reserve Banking in a Growing Economy. The Quarterly Journal of Austrian Economics 2(3): 53-64.

Cochran, John P., Call, Steven P. and Glahe, Fred R. 2003. Austrian Business Cycle Theory: Variations on a Theme. The Quarterly Journal of Austrian Economics 6(1): 67-73.

Eggertsson, G.B. and Woodford, M. 2003. The Zero Bound on Interest Rates and Optimal Monetary Policy. Brookings Papers on Economic Activity 2003(1): 139-211.

European Commission 2009. "Impact of the Current Economic and Financial Crisis on Potential Output". European Economy Occasional Papers 49, June, Brussels.

Fillieule, Renaud 2005. The „Value-Riches" Model: An Alternative to Garrsion's Model in Austrian Macroeconomics of Growth and Cycle. The Quarterly Journal of Austrian Economics 8(2): 3-19.

Fisher, Irving 1930. Theory of Interest. New York: The Macmillan Company.
Friedman, Milton 1968. The Role of Monetary Policy. The American Economic Review 58(1): 1-17.

Furceri, Davide and Mourougane, Annabelle 2009. The Effect of Financial Crises on Potential Output: New Empirical Evidence from OECD Countries. OECD Economics Department Working Papers, No. 699, OECD publishing.

Garrison, Roger W. 2001. Time and Money, The Macroeconomics of Capital Structure, Routledge

Hayek, Friedrich A. von 1933 [1929]. Monetary Theory and the Trade Cycle, Jonathan Cape, London

Hayek, Friedrich A. von 1935 [1931]. Prices and Production, 2nd edition, Augustus M. Kelly, Publishers New York

Hayek, Friedrich A. von 1931. The „Paradox" of Saving. Economica 32: 125-169.
Hayek, Friedrich A. von 1932. A Note on the Development of the Doctrine of 'Forced Saving'. Quarterly Journal of Economics 47(1): 123-133.

Hayek, Friedrich A. von 1937. Investment that Raises the Demand for Capital. The Review of Economic Statistics 19(4): 174-177.

Hayek, Friedrich A. von 1939. Profits, Interest and Investment. London: George Routledge \& Sons Ltd.

Hayek, Friedrich A. von. 1941. The Pure Theory of Capital. Chicago: The University of Chicago Press.

Hayek, Friedrich A. von 1942a. The Ricardo Effect. Economica 9(34): 127-152
Hayek, Friedrich A. von 1942b. [Professor Hayek and the Concertina-Effect]: A Comment. Economica 9(36): 383-385.

Hayek, Friedrich A. von 1969. Three Elucidations of the Ricardo Effect. The Journal of Political Economy 77(2): 274-285.

Hülsmann, Jörg Guido 2001. Garrisonian Macroeconomics. The Quarterly Journal of Austrian Economics 4(3): 33-41.

Kaldor, Nicholas 1942. Professor Hayek and the Concertina-Effect, Economica 9(36): 359382.

Keynes, John M. 1936. The General Theory of Employment, Interest and Money. Harcourt, Brace and Company.

King, Robert G. and Plosser, Charles I. 1984. Money, Credit, and Prices in a Real Business Cycle. The American Economic review 74(3): 363-380.

Kirzner, Israel 2011 [1993]. The Pure Time-Preference Theory of Interest: An Attempt at Clarification. In The Pure Time-Preference Theory of Interest, ed. Herbener, Jeffrey M., 99-126. Ludwig von Mises Institute

Lachmann, Ludwig M. 1978 [1956]. Capital and Its Structure. Institute for Humane Studies.
Mankiw, N. Gregory and Romer, David, ed. 1991a. New Keynesian Economics: Volume 1. MIT Press.

Mankiw, N. Gregory and Romer, David, ed. 1991b. New Keynesian Economics: Volume 2. MIT Press.

Menger, Carl 2007 [1871]. Principles of Economics. Auburn, Alabama: Ludwig von Mises Institute.

Mises, Ludwig von 1976 [1912]. Theory of Money and Credit. The Foundation for Economic Education

Mises, Ludwig von 1996 [1949]. Human Action: A Treatise on Economics, 4th ed. San Francisco: Fox \& Wilkes.

Rothbard, Murray N. 2004 [1962]. Man, Economy, and State. Ludwig von Mises Institute.
Rothbard, Murray N. 1963 [2000]. America's Great Depression. Ludwig von Mises Institute
de Soto, Jesús Huerta 2006 [1998]. Money, Bank Credit, and Economic Cycles. Ludwig von Mises Institute

Wicksell, Knut 1936 [1898]. Interest and Prices. Augustus M Kelley Publishers.
Wicksell, Knut 1977a [1901]. Lectures on Political Economy, Volume 1. Augustus M Kelley Publishers

Wicksell, Knut 1977b [1906]. Lectures on Political Economy, Volume 2. Augustus M Kelley Publishers

Woodford, Michael 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press.

## Chapter 3

## The Austrian Theory of the Natural Rate of Interest: A Neoclassical Critique

## 1. INTRODUCTION

This chapter explores the Austrian approach to the theory of interest. It is mainly focused on the pure time preference theory (PTPT) first developed by Frank Fetter [1915] (1928), later extended by Ludwig von Mises [1949] (1996) and Murray Rothbard [1962] (2004), and firmly defended by Walter Block (1978; 1990), Roger Garrison (1979; 2011), Israel Kirzner (2011), and Jeffrey Herbener (2011). The key idea of this theory is that the phenomenon of interest depends solely on the intertemporal subjective valuations of acting agents, whereas objective factors, such as productivity, are of secondary or even no significance.
The major objective of this chapter is to reconsider the Misesian approach and to stress some inconsistencies within this theory. Pellengahr (1996) and Murphy (2003) clearly demonstrated that the pure time preference theorists confused two meanings of time preference, which led them to inconsistent conclusions. This paper builds upon Pellenghar's and Murphy's research.
The first part briefly outlines foundations of the Austrian theory of interest invented by Böhm-Bawerk (1891). The following section presents a negative response of Mises to BöhmBawerk's work and his own positive solution. Section 2.2 introduces the key distinction between two meanings of time preference. Part 2.3 develops a simple Fisherian apparatus that is designed to expose the Böhm-Bawerkian theory and the two meanings of time preference in graphical terms. In this section, Rothbard's defence of Mises's pure time preference theory is analysed in great detail, and it is demonstrated that the PTPT is not consistent. Section 2.4 supports the previous findings with the help of a simple mathematical model. It is explicitly shown that the time preference may take on any value, and the role of the elasticity of substitution is presented in such a way that it extends the original approach of Böhm-Bawerk.
In section 3, an objective element is introduced into the theory of interest. With the help of the three famous Fisherian examples, it is demonstrated that the productivity of capital may be the crucial determinant of the interest rate in the economy, regardless of the size of time preference of people. This section demonstrates that a negative interest rate might exist simultaneously with a positive time preference.

In section 3.1, the response and defence of modern protagonists of the pure time preference theory are studied. It is shown that the critical controversy between the Austrian and the neoclassical approach arises in the definition of interest as such. Concepts of nominal interest and real interest are introduced, which are mainly designed to pin down crucial differences between these two schools of economic thinking. An attempt is made to reconcile the two approaches. However, it is suggested that the PTPT is just a special theory within a more general neoclassical framework as the real approach seems to be the superior one.
Part 3.2 explores original works of the founder of the Austrian theory of interest - Eugen von Böhm-Bawerk. It is demonstrated that both the nominal and the real approach to the theory of interest might be found in his magnum opus. Furthermore, three types of roundabout processes are presented which illustrate inconsistencies in the PTPT and demonstrate that such inconsistencies might originate in the incomplete analysis in Böhm-Bawerk's works.
Section 3.3 uncovers a surprising observation that the real and nominal approaches have strong interconnections to the dynamic efficiency discussion. It is shown that a positive value difference between output and expended inputs that the authors of the PTPT consider to be at the centre of the interest theory and that can be, according to them, explained only by an a
priori existence of positive time preference, may emerge in the economy with constant money supply and constant marginal productivity of capital only if such an economy is dynamically efficient.

The fourth part builds upon the foregoing sections as it presents how the interest rate might be determined under various theoretical environments. One section examines relative importance of productivity and time preference, the second one explores the impact of specific shapes of the income stream. Both sections, however, arrive at the same conclusion that the pure time preference is not the sole determinant of the natural interest, and negative interest may even prevail in the economy with many creditors.

The last part discusses a more rigorous dynamic approach. In the first section, we explore an optimum behaviour of a representative consumer in the finite and infinite planning horizons in the discrete time. It is explicitly shown that all consumption will not be postponed to the end of the planning horizon even if the time preference is zero and real interest is positive. In the next section, optimum consumption behaviour is briefly studied in a continuous time framework under very specific theoretical conditions. The last part then thoroughly investigates the behaviour of the natural rate of interest in continuous time within a very simple dynamic general equilibrium model. All these parts support fundamental findings presented in more elementary sections of the first half of the study. All findings suggest that the pure time preference theory cannot be considered the major candidate for the explanation of the phenomenon of interest in the free market economy.

## 2. INTEREST AS A VALUE DIFFERENCE BETWEEN PRESENT GOODS AND FUTURE GOODS

Modern neoclassical theory of interest is largely based on path-breaking ideas about the intertemporal choice first comprehensively explored by the authors of the Austrian school. ${ }^{89}$ Although the modern theory is much more mathematical and rigorous than the works of the original Austrian authors, both approaches make a very strong statement that the interest phenomenon cannot dispense with the time element.

The emphasis put on time and the fact that man always considers both present and future can be traced back to the works of Carl Menger:
[E]conomizing men generally endeavor to ensure the satisfaction of needs of the immediate future first, and that only after this has been done, do they attempt to ensure the satisfaction of needs of more distant periods, in accordance with their remoteness in time. (Menger 2007:154)

However, it was the founder of the modern theory of capital, Eugen von Böhm-Bawerk, who offered a fundamental explanation of the interest phenomenon:

PRESENT goods are, as a rule, worth more than future goods of like kind and number. This proposition is the kernel and centre of the interest theory which I have to present. (BöhmBawerk 1891:248)

Böhm-Bawerk posited that the interest exists because people value present goods more than future goods. In other words, people are prepared to exchange more future goods to obtain a

[^54]lower amount of present goods, or alternatively, they only forego present goods for the compensation of a larger amount of future goods.

Let us first present how Böhm-Bawerk elucidated this value difference (or agio as he called it) that accounts for the existence of interest. Interestingly, it can be shown that all three grounds expounded in the following paragraphs may be traced in modern economic growth theories, where the interest rate itself, though, is usually of secondary importance.

The first cause of the value advantage of present goods over future goods lies in the fact that people are usually better provided for in the future rather than in the present (Böhm-Bawerk 1891:249ff). This statement holds on average in a growing economy and on an individual basis for the majority of people over their lives (before they retire). However, it seems to be much less tenable in a stationary economy. Nevertheless, if people are better endowed with consumption goods in the future, then, according to the theory of diminishing marginal utility, ${ }^{90}$ the additional present consumption good has a higher marginal utility than the future consumption good. As a result, the marginal present good is valued more than the marginal future good of the same type and quality.

Böhm-Bawerk further developed this idea. Apparently, it may happen that present is better provided for than future. However, in this case present goods can be stored and moved to the future. This can obviously never be done with future goods, which gives the present goods additional advantage. Moreover, between the moment the present goods are stored and the time at which the future goods are available, a new, initially unanticipated, want may emerge, which can be satisfied only with the present good (1891:250-252). However, the last argument, as well as the first one, holds only for the goods that are easily storable. As will be seen in the next section, (present) perishable goods are less likely to acquire such an advantage.

The second cause of the agio between present goods and future goods rests in the inner inclination of man to systematically underestimate his future wants. Böhm-Bawerk (1891:253) stated that from the present perspective, future wants can be underestimated for the three following reasons.

The first lies in an incomplete imagination about future wants man has somehow built in his mind. The second one reflects a defect of will that makes him prefer present satisfaction even at the expense of future uneasiness or unpleasantness. The third one is connected with the uncertainty of future life as one never knows whether the gratification from future goods will ever arrive (Böhm-Bawerk 1891:254-255). ${ }^{91}$
According to Böhm-Bawerk, this undervaluation is effective regardless of the relative provision of goods between present and future. Compared to the first reason, the second cause is effective even in a stationary economy. Hence, it might form a stronger basis for the existence of the interest phenomenon.

So far, we have discussed two reasons for the existence of a positive premium on present goods against future goods. The third cause, which was put forward by Böhm-Bawerk, introduced a productivity element into his theory. Böhm-Bawerk postulated that present goods are technically superior to future goods (1891:260). This seemingly strong statement is based on several observations. The first and the most fundamental one is that if the factors of production are employed in time-consuming roundabout methods, instead of used directly in

[^55]the production of final consumption goods; they are, as a rule, more productive in the sense that they provide higher output of consumption goods.

However, this fact alone can surely not provide present goods any technical superiority. By locking present consumption goods for some time in the stock, future output of consumption goods will not increase by a wave of a magic wand. The more proper reasoning rests in the fact that if man possesses some amount of present consumption goods, he has an advantage compared to having the same amount of future consumption goods. By having present goods, he can release factors of production from processes that provide consumption goods directly or in a very short time. He may use them instead in the roundabout processes that take longer time, but that will provide higher output of consumption goods after completion.

Since man usually prefers a larger amount of goods to a lower amount of goods, the given amount of present goods must be valued more than the given amount of future consumption goods simply to the fact that the given amount of present goods may provide higher output of future goods.

The three causes of interest can be summarized by the words of Böhm-Bawerk himself:
The difference in the circumstances of provision between present and future; the underestimate, due to perspective, of future advantages and future goods; and, finally, the greater fruitfulness of lengthy methods of production. (Böhm-Bawerk 1891:273)

### 2.1 TIME PREFERENCE AS A SOLE DETERMINANT OF THE INTEREST

This section briefly describes the critique of the foregoing theory put forward by Ludwig von Mises. It also presents Mises's own positive theory. In his exposition, Mises first focused on the second cause for the existence of interest. He explicitly rejected the Böhm-Bawerkian explanation based on the lack of will and incomplete imagination (1996:486). According to Mises, these psychological grounds are too weak to be generally valid. The explanation of interest phenomenon must be established on a more fundamental attribute of human action.

Second, Mises (1996:528), and also Fetter (1902:177) before him, pointed out that BöhmBawerk persuasively demolished older productivity theories in his Capital and Interest (1890). However, in the second book, Positive Theory of Capital (1891), he reintroduced the productivity element back to his theory of interest under the disguise of roundabout methods. In Section 4 we will study this objection in more detail.
Mises developed his specific theory of interest by extending the pure time preference approach of Frank Fetter who explicitly rejected productivity of capital in the explanation of interest. Fetter postulated that only time preference is the sole determinant of the interest phenomenon. Moreover, it was Frank Fetter himself who introduced the term time preference into the economic analysis:
We are dealing here with a case of time-preference. The food is preferred at one time rather than another, in this case at present rather than in the future. An extreme case has been cited for purposes of illustration, but it is possible every day and almost every hour to observe cases involving the same kind of preference, that for present goods as compared with an equal amount of like future goods, and other cases where like goods are preferred in the future rather than at present. (Fetter 1928:236)

Nonetheless, as can be seen in the foregoing passage, Fetter argued that occasionally it might be the future good that has a higher value than the present good of the same type and quality. Thus, time preference may operate even in the opposite direction. Although Mises accepted

Fetter's theory of the pure time preference, he reasoned for only one direction. For Mises, time preference can never be negative, it is always positive - present goods are always and everywhere valued more than future goods. ${ }^{92}$

Mises's reasoning started with the observation that time is scarce:
Man is subject to the passing of time. He comes into existence, grows, becomes old, and passes away. His time is scarce. He must economize it as he economizes other scarce factors. (Mises 1996:101)

Another building block in the Misesian theory is the observation that valuation either about present or about future always takes place in the present:
The judgments of value which determine the choice between satisfaction in nearer and in remoter periods of the future are expressive of present valuation and not of future valuation. They weigh the significance attached today to satisfaction in the nearer future against the significance attached today to satisfaction in the remoter future. (Mises 1996:499)

A similar idea can be also found in Hayek (1941:418), but an explicit statement of this type is much older and can be traced back to F. Fetter:

There is no such thing as a future desire; there are only present desires for either present or future goods. (Fetter 1928:239)

The scarcity of time for an economizing man together with the postulate that valuation of an acting man is always made in the present led Mises to the following conclusion:
Satisfaction of a want in the nearer future is, other things being equal, preferred to that in the farther distant future. Present goods are more valuable than future goods. (Mises 1996:483)

This statement is so fundamental in his system that he called time preference "a categorical requisite of human action" (1996:494). Similar analysis can be also found in Rothbard's magnum opus:
A fundamental and constant truth about human action is that man prefers his end to be achieved in the shortest possible time. Given the specific satisfaction, the sooner it arrives, the better. This results from the fact that time is always scarce, and a means to be economized. The sooner any end is attained, the better. Thus, with any given end to be attained, the shorter the period of action, i.e., production, the more preferable for the actor. This is the universal fact of time preference. At any point of time, and for any action, the actor most prefers to have his end attained in the immediate present. Next best for him is the immediate future, and the further in the future the attainment of the end appears to be, the less preferable it is. The less

[^56]waiting time, the more preferable it is for him. (Rothbard 2004:15) ${ }^{93}$

To prove the validity of the above statement, Mises offered an indirect proof. He asked what would happen if man preferred future satisfaction of want to present satisfaction. He replied that in such a case, the act of consumption would never take place as it would be always postponed to the future:
The very act of gratifying a desire implies that gratification at the present instant is preferred to that at a later instant. He who consumes a nonperishable good instead of postponing consumption for an indefinite later moment thereby reveals a higher valuation of present satisfaction as compared with later satisfaction. If he were not to prefer satisfaction in a nearer period of the future to that in a remoter period, he would never consume and so satisfy wants. He would always accumulate, he would never consume and enjoy. He would not consume today, but he would not consume tomorrow either, as the morrow would confront him with the same alternative. (Mises 1996:484) ${ }^{94}$

Mises further referred to the fact, thoroughly analysed by Fisher (1930) before him, that survival to the future and the enjoyment of wants in the future always require that present needs are satisfied first. ${ }^{95}$ It seems that this is a direct support to his theory. However, Mises stressed that such statement holds only for situations in which the bare life is endangered. His ambition was to develop a theory that would apply to all forms of human action, not only to those where basic physiological needs are at stake. Hence, Mises concluded that:
Time preference is a categorial requisite of human action. No mode of action can be thought of in which satisfaction within a nearer period of the future is not-other things being equalpreferred to that in a later period. (Mises 1996:484)

At first glance, this theory seems to be very strong - man must always prefer the given want to be satisfied now; otherwise, it will be postponed forever as time elapses. It surely surmounts the original (psychological) reasoning of Böhm-Bawerk for the second ground of interest. However, for Mises, the idea that people prefer satisfaction in the present rather than in the future is a necessary and sufficient condition for the value premium possessed by present goods over future goods. It is necessary and sufficient for the explanation of interest. Neither the first ground nor productivity is required for the existence of interest in the economy.
This pure interest, usually known as the natural interest, which originates in the time preference "as a category inherent in every human action," Mises called the originary interest:

[^57]Originary interest is the ratio of the value assigned to want-satisfaction in the immediate future and the value assigned to want-satisfaction in remote periods of the future. It manifests itself in the market economy in the discount of future goods as against present goods.
(Mises 1996:526)

In Mises's point of view, originary interest can never fall to zero or below zero. It must be always positive because the time preference is positive, and the time preference is positive due to the fact that present satisfaction is always preferred to future satisfaction. Mises even claimed:

We cannot even think of a world in which originary interest would not exist as an inexorable element in every kind of action. (ibid.:527)

Originary interest cannot disappear as long as there is scarcity and therefore action. (ibid.:528)

We will see, however, that even though present satisfaction might be always superior to future satisfaction, this does not imply, as Mises frequently claimed, that also present goods possess such superiority over future goods.

Mises's favourite example was that a present apple can never be valued less than a future apple (ibid.:532). However, it was even Carl Menger (2007:151) who said that future goods might be sometimes valued more than present goods - ice in summer as a future good compared with ice in winter, if present is winter - is a typical counterexample of the superiority of a future good because 10 cubes of present ice in winter might be readily exchanged for only one future cube of ice in the summer. This example is quite widespread as it appeared also in the writings of Böhm-Bawerk (1891:245), Fetter (1928:238), and Fisher (1930:41).

Mises rejected the example with ice as an exception to his time preference theory. His response to the foregoing objection was that ice in winter represents a different good from ice in summer due to the different production methods by which ice in various seasons might be produced.

The second seeming exception is presented by the case of perishable goods. They may be available in abundance in one season of the year and may be scarce in other seasons. However, the difference between ice in winter and ice in summer is not that between a present good and a future good. It is the difference between a good that loses its specific usefulness even if not consumed and another good which requires a different process of production. Ice available in winter can only be used in summer when subjected to a special process of conservation. It is, in respect to ice utilizable in summer, at best one of the complementary factors required for production. It is impossible to increase the quantity of ice available in summer simply by restricting the consumption of ice in winter. The two things are for all practical purposes different commodities. (Mises 1996:489)

Pellengahr (1996:41) pointed out that Mises in this case diverted his exposition of defining the essence of goods to objective facts, such as the methods of production, which is quite alien in his predominantly subjective approach to the economic science.

Rothbard agreed with Mises that ice in winter and in summer represent different goods. However, he developed this peculiar idea on subjective grounds rather than objective grounds (Pellengahr 1996:45). Because ice in winter satisfies a different want than ice in summer, they
represent different goods despite their physical identity. The following two passages introduce Rothbard's position:

Time preference may be called the preference for present satisfaction over future satisfaction or present good over future good, provided it is remembered that it is the same satisfaction (or "good") that is being compared over the periods of time. Thus, a common type of objection to the assertion of universal time preference is that, in the wintertime, a man will prefer the delivery of ice the next summer (future) to delivery of ice in the present. This, however, confuses the concept "good" with the material properties of a thing, whereas it actually refers to subjective satisfactions. Since ice-in-the-summer provides different (and greater) satisfactions than ice-in-the-winter, they are not the same, but different goods. In this case, it is different satisfactions that are being compared, despite the fact that the physical property of the thing may be the same. (Rothbard 2004:15)

We must keep in mind the vital fact that the concept of a "good" refers to a thing the units of which the actor believes afford equal serviceability. It does not refer to the physical or chemical characteristics of the good. We remember our critique of the popular fallacious objection to the universal fact of time preference-that, in any given winter, ice the next summer is preferred to ice now. This was not a case of preferring the consumption of the same good in the future to its consumption in the present. If Crusoe has a stock of ice in the winter and decides to "save" some until next summer, this means that "ice-in-the-summer" is a different good, with a different intensity of satisfaction, from "ice-in-the-winter," despite their physical similarities. (Rothbard 2004:69) ${ }^{96}$

In this connection, it seems that Rothbard gave equality between the good and the given intensity of satisfaction. However, such interpretation is quite alien to the original work of Carl Menger, who first investigated relationships between wants and goods. It should be stressed that Menger explicitly distinguished between goods and wants (or want satisfaction). The following passage may illustrate his position:
Things that can be placed in a causal connection with the satisfaction of human needs we term useful things. If, however, we both recognize this causal connection, and have the power actually to direct the useful things to the satisfaction of our needs, we call them goods. If a thing is to become a good, or in other words, if it is to acquire goods-character, all four of the following prerequisites must be simultaneously present: 1. A human need. 2. Such properties as render the thing capable of being brought into a causal connection with the satisfaction of this need. 3. Human knowledge of this causal connection. 4. Command of the thing sufficient to direct it to the satisfaction of the need. (Menger 2007:52)

Hence, not only common sense but also the economic logic requires claiming that good is not a want satisfaction. Goods are means for the satisfaction of human wants. Moreover, Manger's theory of marginal utility is based on this distinction because different units of the same good may satisfy different needs. In such a case, the given good cannot be considered as a different good once additional wants are gradually satisfied with additional units of the same good. It is still the same good that just acquires a lower marginal utility as it is used for the satisfaction of a lower need. Applying Rothbard's idea to this analysis would make the theory of marginal utility completely senseless. In this connection, Murphy (2003:119) clearly stated that both Mises's and Rothbard's points of view are inconsistent since they use a different

[^58]typology of goods and wants for the inter-temporal and for the intra-temporal analysis. In the intra-temporal analysis, the same goods may be used to satisfy different needs. In the intertemporal analysis, once the physically same good satisfies a different want, it is qualified as a different good. ${ }^{97}$

### 2.2 TWO SENSES OF TIME PREFERENCE

Murphy (2003) identified a more fundamental problem in the pure time preference theory developed by Mises and Rothbard. He clearly demonstrated that Mises confused two approaches to time preference. The first meaning of time preference may be associated with the value premium possessed by present goods as against future goods. This premium or agio was elucidated by Böhm-Bawerk by three famous causes: 1) better provision in the future relative to present, 2) underestimation of future wants, and 3) technical superiority of present goods over future goods.
However, the second meaning of time preference is connected only with the second ground for interest. Future utility (or want) is undervalued or discounted compared with present utility, regardless of whether the other two causes are effective or not. So the key idea of Mises that "Satisfaction of a want in the nearer future is, other things being equal, preferred to that in the farther distant future. Present goods are more valuable than future goods" confuses two meanings of time preference. The first part of this statement refers to the second meaning of time preference, whereas the second part describes the first meaning. Thus, the implication between the first and the second sentence is too fast.
As a result, time preference means either that the given want is preferred to be satisfied sooner rather than later or that present goods are valued more than future goods. Since "want" and "good" is not the same thing, we must distinguish between the two concepts of time preference.

Murphy (2003) explicitly demonstrated that man may prefer present satisfaction to future satisfaction, ${ }^{98}$ or as he wrote that man may discount future satisfaction (or utility), but this does not necessarily imply that present goods are valued more than future goods. It is the main objective of this section to further develop this idea.

Consider an individual who is eager to satisfy four hypothetical wants - I,II,III,IV. These wants gradually descend in importance, hence want I is more urgent than II, II is more urgent than III, etc. ${ }^{99}$ Let us assume that these wants are initially unsatisfied in the present, and the

[^59]acting man foresee that they will be ungratified also in the future. Now suppose that the individual receives one unit of good A that is able to satisfy each want both in the present and in the future. Because the satisfaction of the given need is preferred in the present rather than in the future (i.e. positive time preference in sense two is effective), the first unit of good A will be used in the present for the satisfaction of want I. Wants II,III,IV in the present and I,II,III,IV in the future will remain unsatisfied. The individual will only give up present satisfaction of want I if he is offered to receive, for example, two units of good A in the future. In such a case, two wants (I,II) may be satisfied in the future, which could be preferable to the present satisfaction of want I. ${ }^{100}$ This representative individual exhibits positive time preference in sense one. He prefers the present good to the future good because he is prepared to exchange one present good for at least two units of future goods.
However, suppose that the initial endowment of this person is not $[0,0]$, as in the first example, but rather [3,1]. In other words, the individual owns three units of good A in the present and only one unit of good A in the future. In this case, wants I,II,III can be satisfied in the present, whereas only want I may be satisfied in the future. Suppose that good A is a perishable good, which cannot be stored and moved to the future, and the individual is offered to acquire one additional unit of good A .
At what time is this good preferred? Because wants of much lower intensity are satisfied in the present than will be satisfied in the future, the individual may prefer good A to be delivered in the future rather than in the present. From the present point of view of the individual, it might be preferable to satisfy want II in the future to gratify want IV in the present. ${ }^{101}$ In such a case, future good A will be preferred to present good A.
This simple example clearly illustrates a situation in which the given want is preferred to be satisfied in the present, regardless of the initial endowment, but at the same time a marginal future good is valued more than a marginal present good. The representative individual exhibits time preference in sense two (the Böhm-Bawerkian second ground is effective), but he fails to exhibit time preference in sense one (present good is not preferred to future good). The reason is that the first ground (even though in the opposite direction since present is better endowed) more than offsets ground number two. Specifically, in the foregoing example the marginal present good satisfies a want of much lower intensity compared with the marginal future good. As a result, the individual manifests negative time preference in sense one, even though he still possesses positive time preference in sense two.
Furthermore, this simple illustration easily fits the example of ice in winter and in summer. In the first place, man in winter has a large endowment of ice. At the same time, due to the typical weather conditions, ice in winter may satisfy only very low marginal needs. On the other hand, summer can be characterised as a season with much smaller endowment of ice. Moreover, due to high temperatures in the summertime, very urgent wants may be potentially satisfied with ice. Hence, owing to the very low endowment, the last unsatisfied want can be in very great need. In such a case, the future cube of ice can be preferred to the present one. Thus, the individual may be prepared to exchange much more present ice just for a small amount of future ice. There is no need to define cubes of ice in different seasons as different goods. It is still the same good, only the individual is differently endowed with the given good - ice.

[^60]It should be stressed that the second meaning of time preference requires that it is the same need or want to be compared at different times. Surprisingly, Mises did not stress this fact in "Human Action", although it is present in his "Critique of Böhm-Bawerk". Rothbard (2004), on the other hand, carefully used the term "given satisfaction". Hence, the Misesian fundamental statement should stress that the given(!) satisfaction is preferred sooner rather than later. However, if the individual now(!) expects that future needs will be only poorly gratified, higher valuation might be put on marginal future goods compared with marginal present goods.
Furthermore, Mises based his definition on the ceteris paribus assumption, i.e. other things must be equal. Yet, different endowment at different times seems to violate this condition. However, unequal (income) endowment across time is so widespread in the real world income usually increases over time - that Mises should have discussed specifically what he had in mind by the assumption - other things being the same. We will return to this topic in section 4.1.

As was already said, Murphy identified two concepts of time preference that can be, nevertheless, easily mixed together. The first one refers to a value premium of present goods over future goods, which Böhm-Bawerk explained with three famous causes. The second meaning of time preference refers only to the second cause - underestimation of future wants. Thus, Murphy (2003:66) in his interpretation of Böhm-Bawerk explicitly stated that the time preference in sense one exists due to the better provision of goods in the future (Böhm-Bawerkian first ground for interest), the time preference in sense two (BöhmBawerkian second ground for interest), and owing to the technical superiority of roundabout methods of production (Böhm-Bawerkian third ground for interest).

However, at this place, we will modify Murphy's approach. We will follow his understanding of the second meaning - underestimation of future wants, or in the Misesian theory, the preference for the satisfaction of the given want sooner rather than later. Yet, the first meaning of the time preference will be identified only with the first two causes.

The reason for this modification is that the first two causes are subjective and may be used to derive the saving curve on the loanable funds market. Moreover, both have a clear representation in the standard indifference curve model. Hence, from now on, the first meaning of the time preference will refer to the subjective valuation of present goods compared to future goods caused only by the first two grounds, not the third one. BöhmBawerk (1891:273) himself claimed that these two causes are cumulative, whereas the third cause is alternative, or independent of the other ones. As a result, the time preference in the first sense in our approach will depend on the relative provision of goods over time and on the time preference in the second sense.

The third cause associated with the superior productivity of roundabout (or longer) methods will be separated since it might be used as an explanation for a downward sloping investment curve. ${ }^{102}$ The natural rate of interest will then be determined by the interplay of the saving curve (the first two causes) and the investment curve (the third cause), or in Fisherian terms - by the time preference ${ }^{103}$ (in the first sense) and by productivity (or investment opportunity).

[^61]
### 2.3 GRAPHICAL REPRESENTATION OF TIME PREFERENCE

Many neoclassical economists paid much attention to the proper interpretation of the BöhmBawerkian theory. ${ }^{104}$ They used traditional neoclassical tools to pin down what Böhm-Bawerk had presumably in mind. Many of them also touched the problem that time preference might be defined in two different ways. Here we will build on their research.

In section 2.2, we introduced a representative agent having four descendent wants. Carl Menger (2007) persuasively demonstrated that the first unit of every good is always used for the satisfaction of the most urgent want, the second unit for the next most urgent want, etc. From this observation, he derived the famous law of diminishing marginal utility - every additional unit of the (same) consumption good has a lower marginal utility to the consumer as it satisfies less and less pressing needs.

There is no reason to reject the validity of the law of diminishing marginal utility also for future consumption goods. To keep things as simple as possible, we will model only one single consumption good, ${ }^{105}$ whose marginal utility is positive but diminishing both in the present and in the future. Consistently with the traditional Fisherian approach, future will be represented just by one period. ${ }^{106}$ In section 5, this assumption will be relaxed.

If both marginal utility in the present and in the future fall, and by keeping some other technical assumptions, ${ }^{107}$ the marginal rate of substitution (MRS) between future consumption goods $\mathrm{C}_{1}$ and present consumption goods $\mathrm{C}_{0}$ decreases with higher quantity of present goods and lower quantity of future goods. As will be shown later on, the MRS is represented by the ratio of marginal utilities, but great care must be taken in the definition of these particular marginal utilities. As is well known, the $\mathrm{MRS}_{\mathrm{C} 1, \mathrm{C} 0}$ defines how many units of future goods the consumer is willing to exchange for one unit of the present good while keeping a constant level of utility.

[^62]This concept is usually best illustrated by the indifference curve model, where the MRS is represented by the slope of the indifference curve at a particular point. As can be clearly seen in panel (a) of Figure No.1, the MRS falls along the indifference curve as present is still better and better provided for compared with the future.

Panel (b) transforms the ideas from the previous section into this neoclassical model. Point A (satisfaction of want I in the present by, using the example of Böhm-Bawerk (1891:146), one baker's roll and no satisfaction of want $I$ in the future by the same good; i.e. endowment $[1,0]$ ) is preferred to point B (satisfaction of want I in the future by one baker's roll and no satisfaction of want I in the present by the same good; i.e. endowment $[0,1]$ ). The higher preference of A over B is indicated by its position on a higher indifference curve. As a result, a marginal present baker's roll is preferred to a marginal future baker's roll. On the other hand, if the initial endowment is $[3,1]$ (point C ), then point E is preferred to point D because the satisfaction of wants I,II,III in the present and wants I and II in the future is preferred to satisfaction of wants I,II,III,IV in the present and want I in the future. In other words, endowment [3,2] is preferred to [4,1], which results in the fact that for the initial endowment $[3,1]$ (point C), the marginal future good is preferred to the marginal present good.


Figure No. 1, Diminishing MRS and the indifference curve representation of the Mengerian theory

This model can easily illustrate two causes of interest as defined by Böhm-Bawerk, and it may also show the key difference between the two meanings of time preference. Let us start with the idea of the first cause of interest - better provision in the future compared with the present. Consider an initial endowment $\mathrm{C}_{0}{ }^{\mathrm{A}}, \mathrm{C}_{1}{ }^{\mathrm{A}}$ in Figure No. 2. At this point, the income endowment in the future is much larger than in the present. Panel (a) of Figure No. 2 demonstrates that the consumption good added in the present is of higher subjective value (gives higher utility) than if the same good is provided in the future because point B lies on a higher indifference curve than point C .
Alternatively, panel (b) of Figure No. 2 shows that the slope of the tangent at point A (i.e. MRS at point A ) is greater than one in absolute value. It means that the consumer is willing to exchange more than one unit of the future good (e.g. 1.2 units) for one additional unit of the present good. The lower the relative endowment in the present, the higher is the willingness to
forgo future goods in exchange for present goods. On the other hand, at point D , the marginal present good is valued less than the future good because present is so abundantly endowed that only wants of very low intensity remain to be satisfied, whereas wants in the future are expected to be poorly gratified. At this point, a present apple is by no means preferred to a future apple; it is exactly the other way round. Here, the marginal rate of substitution is lower than one.


Figure No. 2, First cause of interest in the indifference curve diagram

The question that immediately springs to one's mind is as follows: what is the slope of the indifference curve at the 45 -degree line? At this line, present and future are equally endowed, so the first ground for interest is not effective. At any given point of this line, it is theoretically the same want at the margin that is waiting for satisfaction both in the present and in the future (e.g. feed the dog today or tomorrow). The Böhm-Bawerkian second cause (or the Misesian postulate that the given want is preferred to be satisfied sooner rather than later) requires that the slope of the indifference curve at the diagonal line is greater than one. This can be seen in panel (b) of Figure No. 3. At point E, the consumer is prepared to forgo more than one unit of future goods (e.g. 1.1) in order to get one additional unit of present goods, even though his endowment is the same in both periods. In other words, the MRS at point E is higher than one. ${ }^{108}$
Alternatively, panel (a) in Figure No. 3 demonstrates that the additional good is preferred in the present since point F lies on a higher indifference curve than point G . The movement from point $E$ by one unit either to point $F$ or to point $G$ explicitly uncovers the second meaning of time preference. In the preference relation, this idea may be represented as follows: $\mathrm{F}[\mathrm{x}+1, \mathrm{x}]$

[^63]$\succ \mathrm{G}[\mathrm{x}, \mathrm{x}+1]$. If both present and future are equally endowed with goods (e.g. apples, point $\mathrm{E}[\mathrm{x}, \mathrm{x}])$, a present apple is preferred to a future apple because in both periods, it is the same want at the margin that is waiting to be satisfied. And because the given want is preferred to be satisfied sooner rather than later, the additional apple will be preferred in the present. ${ }^{109}$


Figure No. 3, Second cause of interest in the indifference curve diagram

We can see that the general Misesian statement that a present apple is always preferred to a future apple makes sense only at the $45^{\circ}$ line, where the first cause is not effective, because endowment is the same in the present and in the future. At the $45^{\circ}$ line, the second cause can be isolated in its pure form. Unfortunately, the first cause cannot be separated in a similarly elegant way. Graphically, it may be only reflected as the difference between $\alpha$ and $\beta$ in panel (b) of Figure No. 2.

Thus, points A and D in Figure No. 2 contain both causes as the indifference curve is not symmetrical around the $45^{\circ}$ line due to the existence of the second cause. At point A in Figure No. 2, both causes operate in the same direction - a present good is preferred to a future good owing to a higher endowment in the future and owing to the "underestimation of future wants." On the other hand, at point D in Figure No. 2, they operate in the opposite direction.

[^64]Moreover, the first cause (here with the opposite sign) dominates the second cause. As a result, at point D , the marginal future good is preferred to the marginal present good.

Finally, somewhere between point E and D both causes offset each other (point H in panel (a) of Figure No. 4). Graphically, it can be found by constructing a perpendicular line to the $45^{\circ}$ line. At this hypothetical point, the MRS is equal to one (panel b). As can be seen in panel (a), the marginal present good is valued the same as the marginal future good because point I lies on the same indifference curve as point J. Obviously, the present endowment must be larger than the future endowment to offset the time preference in sense two. This particular difference in endowments will be discussed in more detail in the next section.


Figure No. 4, Zero time preference in sense one in the indifference curve diagram

The two meanings of time preference can be easily distinguished with this apparatus as well. The MRS at point A in Figure No. 2 represents the time preference in the first sense - a present good is preferred to a future good due to the cooperation of the two Böhm-Bawerkian causes. However, the MRS represents the first meaning of time preference at any point of the indifference curve as it accounts for the willingness to substitute present goods for future goods. Moreover, the MRS can fall below one (point D), so the time preference in the first sense can be negative - a future apple might be preferred to a present apple. In the wintertime, future ice (in summer) might be preferred to present ice (in winter). Murphy (2003:36) considered the first sense of time preference endogenous because it may be altered by a different flow of income over time - any point on the indifference curve might be achieved. As a result, time preference in sense one can take on any value - positive (point A), negative (point D), or zero (point H). ${ }^{110}$

On the other hand, the second meaning of time preference is represented by the slope of the indifference curve only at the diagonal line. As we have already seen, it is generally believed to be positive - the MRS at this point is higher than one. In this respect, the biggest

[^65]contribution of the Misesian theory is that it offered an axiomatic reason for why it should be so - the given satisfaction is always preferred sooner rather than later. According to Murphy (2003:36), the second meaning of time preference is exogenous as it stands outside the model, it operates regardless of other parameters or variables of the model. In the next section, we will see that the second meaning may be attributed to the subjective discount rate in standard models. However, we will also show that it can be endogenous.
According to our analysis, Mises was wrong in implying that if present satisfaction is preferred to future satisfaction, then present goods are always preferred to future goods. In our model, an ideal good is posited on both axes, and it is not transformed to some other good if it satisfies a different want at different times. Even if people exhibit positive time preference in sense two (the MRS is higher than one on the diagonal line), time preference in sense one is not guaranteed (the MRS may be below one, e.g. at point D ) because it also depends on the shape of the income stream (relative income endowment over time). As a result, people may always prefer the given want to be satisfied as soon as possible. Yet, future goods may be preferred to present goods.

It seems that Rothbard was well aware of the fact that future scarcity may overcome time preference:
The case of berries or of any other good is similar. If Crusoe decides to postpone consuming a portion of his stock of berries, this must mean that this portion will have a greater intensity of satisfaction if consumed later than now-enough greater, in fact, to overcome his time preference for the present. The reasons for such difference may be numerous, involving anticipated tastes and conditions of supply on that future date. (Rothbard 2004:70)

As the following passage demonstrates, Rothbard well understood that the law of diminishing marginal utility must eventually determine the optimal mix of present goods and future goods. However, instead of reconsidering his own position, he redefined the given good with respect to its time position. Moreover, Rothbard's inclusion of the marginal utility makes his analysis completely puzzling because within the given period, every additional berry must satisfy wants of lower intensity. Yet, in the intra-temporal analysis, the given good keeps its status; it is not redefined as a different good despite the fact that it satisfies a different want. In other words, berry in the future satisfying a more urgent want compared with the present berry is defined as a different good, whereas additional berry within the given period satisfying a less urgent want is not redefined as a different good, e.g. berry-at-time-to-satisfying-want-I and berry-at-time-to-satisfying-want-II, etc.:

At any rate, "berries-eaten-a-week-from-now" become a more highly valued good than "berries-eaten-now," and the number of berries that will be shifted from today's to next week's consumption will be determined by the behavior of the diminishing marginal utility of next week's berries (as the supply increases), the increasing marginal utility of today's berries (as the supply decreases), and the rate of time preference. (ibid.) ${ }^{111}$

The tools developed so far may also help us demonstrate inconsistencies that are present in Rothbard's numerical example which was designed to prove the existence of positive time

[^66]preference (Rothbard 2004:380). Rothbard envisioned a representative consumer whose intertemporal preferences are represented by a table in Figure No. 5. Moreover, Rothbard considered present and future money instead of goods. This is quite unfortunate because only goods can ultimately satisfy human wants. Without the knowledge of the purchasing power of money in each period, his schema is quite incomplete. More on this will be said in section 3.1.2. Thus, to be more consistent with our previous analysis, we will consider present and future apples instead of money.


Figure No. 5, Representation of intertemporal preferences of individual 1 in Rothbard (2004:380)

Rothbard's schema suggests that Mr. Smith values 11 units of future goods more than 10 units of present goods added to his present stock. He also values an additional 12 future units more than the first 10 units forgone in the present. This implies that if the market interest rate was $20 \%$, he would be prepared to forgo 10 units of present goods to obtain 12 units of future goods. If it increased to $30 \%$, he would be willing to give up an additional 10 units in the present because an additional 13 units in the future would be valued more.

The schema is interrupted at a critical point where the individual compares 10 present units with 10 future units. According to Rothbard, man can never value 10 future units more than 10 present units. Nonetheless, we will demonstrate with the help of the graphical apparatus developed above that such a conclusion is not accurate. Rothbard interrupted the schedule of intertemporal preferences too early.

The key problem of this schedule is that we do not know the initial intertemporal endowment (income stream) of the representative consumer. To keep things as simple as possible, let us suppose that his initial endowment is 100 units both in the present and in the future (point A in Figure No. 6). The Rothbard's schema in Figure No. 5 suggests that the endowment $\mathrm{A}[100,100]$ is preferred to $\mathrm{B}[90,111]$ because an additional 11 units in the future are valued less than 10 forgone present units. However, combination C[90,112] dominates A[100,100] for the opposite reason. As a result, A lies on a higher indifference curve than B, but at a lower indifference curve than C . If we focus on point C , it is perfectly clear that $\mathrm{D}[80,124]$ is valued less compared with C because from Mr. Smith's point of view, an additional 12 future units will not be enough to compensate the loss of the second dose of 10 present units.

However, combination $\mathrm{E}[80,125]$ is preferred to $\mathrm{C}[90,112]$ since the consumer is willing to accept 13 future units as a compensation for the second forgone dose of 10 present units. The same holds for $\mathrm{F}[80,126]$ and $\mathrm{G}[80,127]$.

If we focus on $\mathrm{E}[80,125]$, it is preferred to $\mathrm{H}[70,140]$ because 15 future units will not offset the loss of the third dose of 10 present units. Nevertheless, point I[70, 141] is more valuable than E as an additional 16 units obtained in the future are valued more than the loss of 10 present units. In a similar manner, we can imply that $\mathrm{J}[70,142]$ and $\mathrm{K}[70,143]$ are also preferred to $\mathrm{E}[80,125]$. By the same procedure, point $\mathrm{L}[60,159]$ is dominated by $\mathrm{I}[70,141]$, whereas M[60, 160] is preferred to I[70,141].


Figure No. 6 Reconstruction of Rothbard's schema, Mr. Smith I

So far, we have analysed all intertemporal combinations of present goods and future goods (or money) implied by the original Rothbard's schema. Figure No. 6 portrays all these critical points in a simple indifference curve model. As can be clearly seen, more and more future units must be added in order to persuade the consumer to give up marginal units of present goods. According to Rothbard, the consumer's time preference gradually increases.

It must be stressed that this statement is perfectly compatible with our previous analysis. However, the key question is which sense of time preference Rothbard was talking about. And the answer seems to be quite simple: As we are moving along a hypothetical indifference curve, it is the first sense. It is still more and more painful for the consumer to forgo additional present goods. So the fundamental reason for an increase in the time preference is the increasing marginal utility of present goods (as their supply decreases) and the diminishing marginal utility of future goods even though the latter phenomenon Rothbard neglected, as he himself admitted (2004:381, n.8).
Constructed in this way, Rothbard's concept of time preference is definitely endogenous since he was obviously talking about the increasing MRS as moving to the top left along the indifference curve, despite the fact that he presented this idea in a simple numerical schema and not in a usual neoclassical indifference curve language. ${ }^{112}$ His vision of time preference in this particular case was definitely of sense one. However, Rothbard's previous quotations reported above stressed the dominance of present satisfaction over future satisfaction. Hence, at this place, he referred to time preference in sense two. As a result, we may conclude that Rothbard confused two senses of time preference. His numerical schema does not exclude the possibility that present goods are valued less than future goods, as the following extension of his own numerical example will demonstrate.
From Rothbard's schema in Figure No. 5, it is obvious that 10 units added in the present are valued less than 11 units forgone in the future. Hence, $\mathrm{N}[110,89]$ is dominated by $\mathrm{A}[100,100]$. Similarly, $\mathrm{O}[120,78]$ is dominated by $\mathrm{N}[110,89]$. Nevertheless, $\mathrm{N}^{\prime}[110,90]$ is preferred to $\mathrm{A}[100,100]$ since 10 units added in the present are valued more than 10 units lost in the future. Similarly, $\mathrm{O}^{\prime}[120,80]$ is preferred to $\mathrm{N}^{\prime}[110,90]$, so $\mathrm{O}^{\prime}$ is preferred to A as well. Figure No. 7 displays these preferences.


Figure No. 7 Reconstruction of Rothbard's schema, Mr. Smith II

[^67]Rothbard's schema evidently stopped at point O or $\mathrm{O}^{\prime}$. However, consider a hypothetical point $\mathrm{P}[130,68]$ and its relation to $\mathrm{O}[120,78]$. The question is which of these is more preferred. Since present is very well endowed at O , one might suggest that P is dominated by O. This would require that the line " 3 rd added unit of 10 oz ." would lie below ( 10 oz . future) in Rothbard's schema. If the individual's endowment was $\mathrm{P}[130,68]$, the consumer would be willing to forgo 10 units of present goods to obtain 10 units of future goods (i.e. to move from point P to point O ). At this particular point, there would be no time preference (in sense one) and the Rothbardian (and Misesian) theory would collapse since present goods would possess no value premium over future goods.

Nonetheless, Rothbard could raise three objections against the hypothetical point P where his theoretical system does not work. First, as future is much less endowed with the particular good (or money), different wants (or more precisely more urgent wants) are satisfied with the given good (or money) in the future compared with the present, so we have to define this good (money) in the future as a different good (money). However, we have already discussed the absurdity of this argument that will become even more obvious if the analysis is carried out in terms of money.
Secondly, if people were endowed with point $P$, Rothbard could argue that nobody would be willing to exchange 10 present goods for 10 future goods. Because it is money (gold) that he employed in his example, i.e. a non-perishable good, the individual could simply transfer 10 present units to the future (he would move from point P to point O by a simple "intrapersonal"/ intertemporal transfer). However, here we do not discuss the optimum of the consumer that might be achieved by various ways, even by a simple transfer. We are just constructing his intertemporal preferences over the entire domain of all possible combinations of present and future consumption. Every single point must be taken into account, including those that would be never chosen by an acting man if some particular conditions were satisfied. ${ }^{113}$

Moreover, as we will see later, the assumption of a non-perishable good is of critical importance. It may easily happen that point P is the optimum, once the good deteriorates over time. However, at this place we only claim that there may exist a situation where the marginal present good is not preferred to marginal future good (point P).
And finally, Rothbard could argue that the inclusion of our last line is fundamentally wrong. The $3^{\text {rd }}$ added unit of 10 oz . must lie above 10 oz . in the future, not below, because:

It will be noticed that there is no listing for less than 10 ounces of future goods, to be compared with 10 ounces of present goods. The reason is that every man's time preference is positive, i.e., one ounce of present money will always be preferred to one ounce or less of future money. Therefore, there will never be any question of a zero or negative pure interest rate. (Rothbard 2004:380)

A man could not prefer 10 ounces or even less of future money to 10 ounces of present money (ibid.:386)

Apart from the fact that Rothbard employed interchangeably present and future goods in one sentence, and present and future money in the next sentence, which might obscure the analysis, as we will see in section 3.1, his statement would require that all n added units of 10 oz. in the present should be always preferred to 10 oz . in the future, regardless of the endowment or the operation of the law of diminishing marginal utility.

[^68]If we look at Figure No. 7, the difference between N and $\mathrm{N}^{\prime}$ is very small, and the time preference is very close to zero between these two points. At the same time, both points are posited near the symmetrical endowment $\mathrm{A}[100,100]$. It would be quite surprising if the time preference (in sense one) did not fall below zero by moving closer to the horizontal axis. It would require that the MRS should suddenly stop declining. This would mean that the law of diminishing marginal utility with regard to present and future goods must cease to operate at some point. So instead of a logical picture in Figure No. 8, in which the MRS gradually falls with higher present consumption, Rothbard's theory would either require to cut off part of the indifference curve at point $\mathrm{O}[120,78]$ or to fix its slope at the value above one from that point onwards (Figure No. 9).


Figure No. 8, Reconstruction of Rothbard's schema, Mr. Smith III


Figure No. 9, Reconstruction of Rothbard's schema if time preference (in sense one) never falls below 0 (i.e. if the MRS never falls below 1)

To reveal the absurdity of this statement from a different point of view, consider two hypothetical points - Z[200,0] and Y[190,10]. Since point Z implies death in the future, point Y is surely preferred to $\mathrm{Z} .{ }^{114}$ As a result, starting with endowment Z , man would willingly exchange 10 present goods for 10 units of future goods (and even for less than 10 units). As we can see, Rothbard critically confused two meanings of time preference. His numerical schema is a story about the diminishing MRS - the subjective exchange ratio between present goods and future goods. It increases with lower present consumption; however, the MRS can easily fall below one if the present is much better endowed compared with the future. Rothbard's statement that present goods are never valued less than future goods is therefore flawed, despite the fact that present satisfaction is always preferred to future satisfaction. ${ }^{115}$

| James Robinson |
| :---: |
|  |
| ......... 2nd unit of 10 oz . |
| .........................................(18 oz. future) |
|  |
| $\ldots . . . .1$ 1st unit of 10 oz . |
| ..........................................(16 oz. future) |
| ..........................................(15 oz. future) |
| $\ldots . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .(14 \mathrm{oz} . \mathrm{future)}$ |
| $\ldots . . . . .$. (1st added unit of 10 oz.$)$ |
|  |
|  |
|  |
| ..........................................(11 oz. future) |
| $\ldots . . . .$. (3rd added unit of 10 oz.$)$ |
|  |

Figure No. 10, Representation of intertemporal preferences of individual 2 in Rothbard (2004:381).

Rothbard also revealed preferences of the second man, Mr. Robinson, which are displayed in Figure No. 10. Let us assume that Robinson's initial endowment is also A[100,100]. By a similar procedure, we can deduce that $\mathrm{A}[100,100] \succ \mathrm{D}[90,116] \succ \mathrm{C}[90,115] \succ \mathrm{B}[90,114]$, but $\mathrm{E}[90,117] \succ \mathrm{A}[100,100]$. We also know that $\mathrm{E}[90,117] \succ \mathrm{F}[80,135]$, but $\mathrm{G}[80,136] \succ$ $\mathrm{E}[90,117]$. These relations may reconstruct the upper part of a hypothetical indifference curve.

To obtain the lower part, Rothbard's example implies that $\mathrm{A}[100,100] \succ \mathrm{H}[110,86]$, but $\mathrm{I}[110,87] \succ \mathrm{A}[100,100]$. Furthermore, $\mathrm{I}[110,87] \succ \mathrm{J}[120,75]$, but $\mathrm{K}[120,76] \succ \mathrm{I}[110,87]$. And finally, $\mathrm{K}[120,76] \succ \mathrm{L}[130,65]$, but $\mathrm{M}[130,66] \succ \mathrm{K}[120,76]$. From the last two relations, it is clear that if his endowment was $\mathrm{K}[120,76]$, he would not be willing to receive

[^69]additional 10 present goods for 11 future goods, but he would accept giving up only 10 future goods. As a result, his MRS in this region is below 1.1 and above 1. It was gradually falling from very high levels of points G,E, etc. (see Figure No. 11).

In the end, a critical point must arise at which the MRS falls to 1 . It might be, for example, point $\mathrm{N}[140,56]$, where Mr. Robinson is prepared to exchange 10 present goods for 10 future goods, $\mathrm{M}[130,66] \succ \mathrm{N}[140,56]$. At this point, his time preference (in sense one) is zero. Although it is attained for a more unequal endowment than for Mr. Smith (compare point $\mathrm{N}[140,56]$ of Mr. Robinson with point $\mathrm{P}[130,68]$ of Mr. Smith), it must definitely arise as well. The MRS cannot stop falling at some level, so even Mr. Robinson (or any other man) might value marginal present goods less than marginal future goods if future wants are expected to be poorly gratified.


Figure No. 11, Reconstruction of Rothbard's schema, Mr. Robinson
From the previous discussion, it should be perfectly clear that Mr. Robinson is more impatient than Mr. Smith. If they both have the same initial endowment, for example A[100,100], Mr. Smith is prepared to obtain just 12 units of future goods in exchange for 10 present units, whereas Mr. Robinson requires at least 17 units. From Figure No. 12 (or 8 and 11), it can be seen that the slope of the indifference curve at the $45^{\circ}$ line is higher for Mr. Robinson. Thus, his time preference (in sense two) exceeds that of Mr. Smith. In the next section, we will see that his subjective discount rate $(\rho)$ is higher than that of Mr. Smith.

However, the higher impatience of Mr. Robinson might be only spurious because Rothbard did not provide us with the information about the initial endowment of either man. Point A [100, 100], at which we started with the Rothbard's schema, was purely hypothetical. Hence, the higher impatience of Mr. Robinson could have stemmed from a different initial endowment compared with Mr. Smith. Suppose, for example, that his A point was [50,150], whereas that of Mr. Smith $[150,50]$. In that case, the higher patience of Mr. Smith would be caused by a decreasing shape of his income stream, and the impatience of Mr. Robinson would stem from a strong influence of the first Böhm-Bawerkian cause of interest because his future would be much better provided for. As a result, the hypothetical indifference curve of

Mr. Smith presented in Figure No. 12 should be steeper at the $45^{\circ}$ line, whereas that of Mr. Robinson would be flatter.


Figure No. 12, Time preference (in sense two) of Mr. Smith and Mr. Robinson

Although the initial endowment was chosen arbitrarily, increasing present consumption and decreasing future consumption should eventually reduce the MRS below one, regardless of the initial endowment. Due to the law of diminishing marginal utility, there must be a point on the indifference curve at which the time preference (in sense one) switches from positive to negative value and at which the marginal present good is subjectively valued less than the marginal future good.

### 2.4 MATHEMATICAL REPRESENTATION OF TIME PREFERENCE

Additional insight to this problem may be gained if we apply simple mathematical tools. The first attempt to mathematically model the problem of discounting was due to P. Samuelson (1937). Even though his approach was based on cardinal utility and other simplifying assumptions that we list below, its elegance is so attractive that it remained the workhorse of a majority of modern models.

Further research then extended this first approximation. Koopmans (1960) in a highly technical paper developed an ordinal approach, and very rigorous analysis of time preference may be found in other papers as well. See, for example, Lancaster (1963), Koopmans et al. (1964), and Fishburn and Rubinstein (1982). Nevertheless, equation (1), which is a discretetime version of the original model in Samuelson (1937), will give us enough intuition to reveal the problems discussed in previous sections.
Following Olson and Bailey (1981), and Loewenstein (1992), consider a mathematical model of a representative consumer who knows with certainty that he will live for T years:

$$
\begin{equation*}
U=\sum_{t=0}^{T} \frac{u\left(C_{t}\right)}{(1+\rho)^{t}}=u\left(C_{0}\right)+\frac{u\left(C_{1}\right)}{1+\rho}+\frac{u\left(C_{2}\right)}{(1+\rho)^{2}}+\ldots+\frac{u\left(C_{T}\right)}{(1+\rho)^{T}} \tag{1}
\end{equation*}
$$

$\mathrm{C}_{\mathrm{t}}$ denotes consumption (i.e. the quantity of final goods consumed) of the individual at time t . U represents the lifetime utility function, $\mathrm{u}(\cdot)$ is known as the instantaneous utility function (Strotz 1956). $u(\cdot)$ is assumed to be increasing and concave: $u^{\prime}>0, u^{\prime \prime}<0$, which means that (in every period) marginal utility of consumption is positive but diminishing.
The crucial parameter in this model is $\rho$ - the subjective discount rate. It should be greater than zero because it gives discount on future utilities. Hence, it represents the second cause of interest in the Böhm-Bawerkian work. This mathematical reasoning should not be alien even to the Misesian theory because if the present satisfaction is preferred to later satisfaction, then, from the present perspective, the utility of the given amount of goods consumed in the future is lower than the utility of the same amount consumed today. Rothbard (2004:63) himself wrote about discounting of future utilities, so equation (1) is just a reasonable and simple representation of this idea. Furthermore, the more distant the period of consumption, the higher the subjective discount. This phenomenon is reflected in the increasing exponent in the denominator of every term in expression (1). ${ }^{116}$
It should be obvious from the previous exposition that parameter $\rho$ represents the second sense of time preference. In this mathematical representation, the Misesian statement that the given satisfaction is preferred sooner rather than later may be translated as the preference for immediate utility over future utility (Frederick et al. 2002:352).

This discounted utility model is very elegant; however, let us list some of its simplifying assumptions and properties (Frederick et al. 2002:356-360, Ghez and Becker 1975:8-9):

1) Parameter $\rho$ is constant and exogenous, so the discount rate does not depend either on time or the level of consumption. ${ }^{117}$ The "exponential" discounting form also guarantees timeconsistent behaviour. ${ }^{118}$
2) The instantaneous utility at time $t$ depends solely on consumption in the given period. The lifetime utility function is additively separable. Hence, the marginal utility of consumption in the given period does not depend on consumption levels in other periods. It also implies that the MRS between time $t$ and $t+1$ depends only on consumption levels in these two periods.
3) The well-being at time $t$ does not depend on well-being in any other period. The model exhibits utility independence.
4) The instantaneous utility function is constant over time. As time passes, people do not change preferences, and their tastes are constant. ${ }^{119}$ In other words, the instantaneous utility from consumption in the given period does not depend on that particular period. ${ }^{120}$

[^70]5) People have perfect foresight, so they know with certainty their future consumption levels (Fisher 1930). After all, the main task of this paper is to clarify the existence of interest in the world without risk and uncertainty.

This model can be transformed into a two-period form, graphical representation of which was discussed in the previous section. Bailey and Olson (1981:4) considered an individual's choice only in period $t=0$ and $t=1$, holding the consumption pattern for all later periods fixed. However, the analysis might be generalized for any two periods. A more comprehensive model reflecting T periods or even an infinite number of periods will be developed in section 5.
To derive the MRS between time 0 and 1 , take the differential of (1) letting $\mathrm{dC}_{2}, \mathrm{dC}_{3}$ to $\mathrm{dC}_{\mathrm{T}}$ equal to zero.
$d U=u^{\prime}\left(C_{0}\right) d C_{0}+\frac{u^{\prime}\left(C_{1}\right) d C_{1}}{1+\rho}$
The individual is indifferent (i.e. his total utility is constant) by setting $\mathrm{dU}=0$. Hence (2) is transformed to:
$-\left.\frac{d C_{1}}{d C_{0}}\right|_{\text {Uconst }}=\frac{u^{\prime}\left(C_{0}\right)}{u^{\prime}\left(C_{1}\right)}(1+\rho)$
where the right hand side of the equation gives us the formula for the marginal rate of substitution (MRS). ${ }^{121}$ As was stated before, the MRS represents the slope of the indifference curve at every point. It defines the exchange ratio between present goods and future goods from the subjective point of view.

To make this point even clearer, consider the MRS at point A in Figure No. 2 presented in section 2.3. We already know that $\mathrm{MRS}^{\mathrm{A}}$ is definitely higher than one:

$$
\begin{equation*}
-\left.\frac{d C_{1}^{A}}{d C_{0}^{A}}\right|_{U c o n s t}=\frac{u^{\prime}\left(C_{0}^{A}\right)}{u^{\prime}\left(C_{1}^{A}\right)}(1+\rho)=M R S^{A} \tag{4}
\end{equation*}
$$

The first part of this formula $u^{\prime}\left(\mathrm{C}_{0}{ }^{\mathrm{A}}\right) / \mathrm{u}^{\prime}\left(\mathrm{C}_{1}{ }^{\mathrm{A}}\right)$ is greater than one at point A because future is better provided for than present and because the marginal utility of consumption is diminishing in every period. ${ }^{122}$ This ratio represents the first Böhm-Bawerkian cause for interest. The second part of this expression $(1+\rho)$ condensed the second Böhm-Bawerkian cause. It is also higher than one due to the fact that $\rho$ is always, by assumption, positive (at least implicitly in the writings of the Austrian authors). The entire term - MRS - includes both Böhm-Bawerkian causes, and, as we said before, it represents the time preference in the first sense.

The subjective discount rate $\rho$ reflects the time preference in the second sense, as can be easily seen from the numerical value of the MRS at point E in panel (b) of Figure No. 3:

[^71]$-\left.\frac{d C_{1}^{E}}{d C_{0}^{E}}\right|_{\text {Uconst }}=\frac{u^{\prime}\left(C_{0}^{E}\right)}{u^{\prime}\left(C_{1}^{E}\right)}(1+\rho)=(1+\rho)=M R S^{E}$

The MRS at point E is equal to $(1+\rho)$ because the consumption levels are the same in both periods, and the Böhm-Bawerkian first cause is not effective. At the diagonal line, the MRS is exclusively determined by $\rho$, i.e. by the preference for the given want to be satisfied earlier rather than later. In other words, at this point, the time preference in the first sense is purely determined by the time preference in the second sense. Thus, the preference for the marginal present good over the marginal future good is here effective only due to the preference for the given satisfaction to be gratified as soon as possible (due to the preference for present satisfaction over future satisfaction).
On the other hand, the MRS at point D in panel (b) of Figure No. 2 is lower than one because the ratio $u^{\prime}\left(C_{0}{ }^{\mathrm{D}}\right) / \mathrm{u}^{\prime}\left(\mathrm{C}_{1}{ }^{\mathrm{D}}\right)$ is so low that it more than offsets $(1+\rho)$. At this point, the time preference in the first sense is negative. The marginal future good is valued more than the marginal present good even though the time preference in the second sense is positive present utility is preferred to future utility because $\rho$ is positive.

With the help of this model, we can also find the relative size of consumption levels in the two periods that will lead to zero time preference in the first sense (i.e. MRS = 1). The graphical representation of this consumption flow can be found at point H in Figure No. 4. It is implicitly defined by the following formula:

$$
\begin{align*}
& -\left.\frac{d C_{1}^{H}}{d C_{0}^{H}}\right|_{\text {Uconst }}=M R S^{H}=1=\frac{u^{\prime}\left(C_{0}^{H}\right)}{u^{\prime}\left(C_{1}^{H}\right)}(1+\rho)  \tag{6a}\\
& \frac{u^{\prime}\left(C_{0}^{H}\right)}{u^{\prime}\left(C_{1}^{H}\right)}=\frac{1}{(1+\rho)} \tag{6b}
\end{align*}
$$

As can be clearly seen, $\mathrm{C}_{0}{ }^{\mathrm{H}}$ must be large enough compared with $\mathrm{C}_{1}{ }^{\mathrm{H}}$ to overcome the a priori positive time preference in sense two (i.e. $\rho>0$ ) and to depress time preference in sense one to zero (i.e. MRS $=1$ or alternatively MRS $-1=0$ ).
It is very tempting to call the first part of the MRS, $u^{\prime}\left(\mathrm{C}_{0}\right) / \mathrm{u}^{\prime}\left(\mathrm{C}_{1}\right)$, the ratio of marginal utilities. However, this is not so clear-cut due to the presence of $u^{\prime}\left(\mathrm{C}_{1}\right)$, which is a very problematic term. What does this term represent? It might be called the future marginal utility from the future perspective. However, the valuation of man can only take place in the present. Hence, it would be more in line with our verbal exposition to rewrite the MRS as:

$$
\begin{equation*}
M R S_{C 1, C 0}=\frac{u^{\prime}\left(C_{0}\right)}{\frac{u^{\prime}\left(C_{1}\right)}{(1+\rho)}} \tag{7}
\end{equation*}
$$

and define $\mathrm{u}^{\prime}\left(\mathrm{C}_{1}\right) /(1+\rho)$ as the future marginal utility from the present perspective. In this form, the individual can compare both marginal utilities - MU of $\mathrm{C}_{0}$ and of $\mathrm{C}_{1}$ - from the perspective of the same (i.e. present) period. As a result, the entire term of the MRS in (7) is to be defined as the ratio of marginal utilities, not just the part $u^{\prime}\left(\mathrm{C}_{0}\right) / \mathrm{u}^{\prime}\left(\mathrm{C}_{1}\right)$ in equation (3).
Alternatively, equation (7) can be defined in a slightly different form:

$$
\begin{equation*}
\varepsilon\left(C_{1}, C_{0}\right)=M R S_{C 1, C 0}-1=\frac{\frac{u^{\prime}\left(C_{0}\right)}{\frac{u^{\prime}\left(C_{1}\right)}{(1+\rho)}}-1}{} \tag{8a}
\end{equation*}
$$

$\varepsilon$ might be called the marginal rate of time preference. ${ }^{123}$ In our model, it explicitly represents the time preference in the first sense. It can be rearranged to:
$\varepsilon\left(C_{1}, C_{0}\right)=\frac{u^{\prime}\left(C_{0}\right)(1+\rho)-u^{\prime}\left(C_{1}\right)}{u^{\prime}\left(C_{1}\right)}$

$$
\begin{equation*}
\varepsilon\left(C_{1}, C_{0}\right)=\frac{\rho+\left[1-\frac{u^{\prime}\left(C_{1}\right)}{u^{\prime}\left(C_{0}\right)}\right]}{\frac{u^{\prime}\left(C_{1}\right)}{u^{\prime}\left(C_{0}\right)}} \tag{8c}
\end{equation*}
$$

Expression (8c) suggests that the time preference in the first sense $\varepsilon$ is not exogenous. It changes with the varying consumption flow over time. As can be seen, the higher the provision in the future relative to present, the higher the time preference in the first sense. It would coincide with the time preference in the second sense $\rho$ only if the provision of consumption goods was the same in both periods. However, it might be lower than $\rho$ if present was more abundant than future. Moreover, it could fall to zero or even below zero if present consumption was sufficiently large compared to expected future consumption.
As we can see, this simple mathematical model gives us the same results as the previous graphical apparatus. The MRS between present and future consumption (or MRS - $1=\varepsilon$, the marginal rate of time preference in the first sense) is not fixed, it is truly an endogenous concept that depends on the relative (income) endowment. However, even parameter $\rho$ (or more exactly $1+\rho$ ) may be defined as the marginal rate of substitution - not between consumptions, but between utilities (or satisfactions) in the two periods. By taking total differential of the lifetime utility function with respect to the instantaneous utility at time 0 and 1 (i.e. not with respect to consumption as before), we get: ${ }^{124}$

$$
\begin{align*}
& d U=d u_{0}+\frac{d u_{1}}{1+\rho}  \tag{9a}\\
& d U=0  \tag{9b}\\
& -\frac{d u_{1}}{d u_{0}}=1+\rho \tag{9c}
\end{align*}
$$

[^72]

Figure No. 13, $(1+\rho)$ as the MRS between instantaneous utilities

The marginal rate of substitution between instantaneous utilities is constant. It says how much utility in the future man will require if he sacrifices one unit of utility in the present. ${ }^{125}$ This rate depends on $\rho$, and its graphical representation is given in Figure No. 13.

At this point, it should be stressed that it is the consumption goods that are traded on the intertemporal market, not utilities (or satisfactions). Hence, the MRS between present and future consumption goods (the time preference in the first sense), not the MRS between utilities (the time preference in the second sense), is crucial for the determination of the interest rate on the intertemporal market. In Section 3, we will see that the natural rate of interest can take on any value (as the time preference in the first sense), it is not necessarily positive as Mises tried to show in his theoretical system.


Figure No. 14, Constant $\rho$ and constant MRS at the diagonal line

[^73]Let us further demonstrate in a simple figure what the assumption of constant $\rho$ implies for our graphical model. First of all, positive $\rho$ results in the fact that the slope of the indifference curve at the diagonal line is greater than one (see Figure No. 3). Furthermore, the higher the impatience of the individual (i.e. higher $\rho$ ), the higher the slope of the indifference curve at the diagonal line (compare Mr. Robinson and Mr. Smith in Figure No. 12). Conversely, if $\rho$ was hypothetically zero, the slope would be perpendicular to the $45^{\circ}$ line. Negative $\rho$ greater weight put on future utilities - would result in the slope lower than one at the $45^{\circ}$ line (Ghez and Becker 1975:9).

Second, the constancy of the subjective discount rate implies that the slope of every indifference curve at the diagonal line is the same (see Figure No. 14). Time preference in the second sense (i.e. the pure time preference) does not depend on the average level of income. We will relax this assumption (together with other assumptions made before) in the following paragraphs, and then we will discuss the effect of this change on our graphical apparatus.

Not only the slope of the indifference curve at the $45^{\circ}$ line but also the curvature of the entire indifference curve is of great importance in the theory of interest and in the discussion with the pure time preference theory. Panel (a) in Figure No. 15 represents an individual with very low discount rate (the MRS on the diagonal is only slightly above one) and with very high elasticity of substitution - indifference curve exhibits a very low curvature. An increase in consumption in the present and a decline in the future do not much alter the MRS (time preference in the first sense). Alternatively, it can be said that the reallocation of consumption across time does not significantly affect marginal utilities (the instantaneous utility function is close to linear as well). ${ }^{126}$


Figure No. 15, Consumer with high elasticity of substitution, and with low subjective discount rate (panel a) and with high subjective discount rate (panel b)

[^74]However, it is perfectly conceivable that preferences of the individual might be such that the subjective discount rate is very high, and the elasticity of substitution is considerable as well. This combination will result in a very steep (due to high $\rho$ ) and almost linear (due to very high elasticity of substitution) indifference curve (see Figure No. 15, panel b). ${ }^{127}$


Figure No. 16, Consumer with infinite elasticity of substitution - case of perfect substitutes

Both examples also suggest that if the elasticity of substitution rises beyond all limits, the MRS at any point (time preference in the first sense) is solely determined by the subjective discount rate (time preference in the second sense). In this extreme case of perfect substitutes, ${ }^{128}$ in which the elasticity of substitution goes to infinity, the MRS will be determined only by parameter $\rho$ not only at the diagonal line but also along the entire indifference curve (Figure No. 16). We can clearly see that indifference curves are linear and parallel. In this case and only in this case, the first sense of time preference will coincide with the second sense. Time preference would be solely determined by the Misesian postulate of the superiority of the earlier satisfaction, and no role would be played by the relative income endowment over time (income stream). An immediate question is whether Mises did not have this case in mind. ${ }^{129}$

On the other hand, assume that the elasticity of substitution is very low. In such a case, the marginal rate of substitution declines rapidly with a fall in the future endowment and an increase in the present endowment. Alternatively, this means that the marginal utility of consumption falls very quickly with higher consumption. In the Mengerian language, it is the case when the urgency of lower wants, which the given good is able to satisfy, is much lower

[^75]than the urgency of higher wants. It can be also said that the intensity of the given want abruptly falls with its gradual gratification.

In our simple example from the previous sections, this would mean that one's hunger is of much larger importance than feeding the dog. In such a case, the particular individual is rather reluctant to substitute consumption over time. For instance, to postpone the meal to the future (i.e. want "I" would not be gratified in the present), the individual would require to be compensated by gratifying at least four wants in the future, i.e. eating, feeding the dog, cat, and fish. In other words, with very low intertemporal elasticity of substitution in consumption, one would require to receive at least four baker's rolls in the future as a compensation for losing one present baker's roll. The instantaneous utility function is much curved in this case generating rather convex intertemporal indifference curves.


Figure No. 17, Consumer with low elasticity of substitution, and with low subjective discount rate (panel a) and high subjective discount rate (panel b).


Figure No. 18, Consumer with zero elasticity of substitution - case of perfect complements

Figure No. 17 represents such preferences. Panel (b) illustrates a person with a very high subjective discount rate and a very low elasticity of substitution. Hence, the curvature of the indifference curve is high as well as its slope at the $45^{\circ}$ line. Panel (a) is consistent with a very low discount rate and a very low elasticity of substitution. Surprisingly, both pictures are almost indistinguishable. In this situation, the time preference in the first sense is largely affected by the relative endowment, whereas the subjective discount rate (undervaluation of future wants - time preference in sense two) retains its role only at the diagonal line. Any deviation from the smooth income stream results in a considerable change in the MRS and consequently in a change in the relative subjective valuation of present goods as against future goods.

To take the other extreme, if the consumer has Leontief preferences, the subjective discount rate, i.e. the second Böhm-Bawerkian cause, loses its power (see Figure No. 18). In this hypothetical case, $\mathrm{C}_{1}$ and $\mathrm{C}_{0}$ are perfect complements, and the sole determinant of the time preference is the relative income endowment. In this particular example, a perfectly smoothed profile of the consumption stream is preferred by the consumer regardless of the interest rate (or the shape of the income stream).
So far, we have not specified any form of the instantaneous utility function. At this point, we will introduce a typical mathematical form that will enable us to separate the role of the subjective discount rate and the elasticity of substitution. This form, a CRRA utility function, is given in (10). Expression (11) is then the resulting lifetime utility form.
$u(C)=\frac{C^{1-\theta}}{1-\theta}$
$U=\frac{C_{0}^{1-\theta}}{1-\theta}+\frac{1}{1+\rho} \frac{C_{1}^{1-\theta}}{1-\theta}+\frac{1}{(1+\rho)^{2}} \frac{C_{2}^{1-\theta}}{1-\theta}+\ldots+\frac{1}{(1+\rho)^{T}} \frac{C_{T}^{1-\theta}}{1-\theta}$

To pin down the two phenomena we are interested in, let us utilize the two-period model. The resulting MRS for the CRRA utility function (see equation 3) is as follows: ${ }^{130}$
$M R S=\left(\frac{C_{1}}{C_{0}}\right)^{\theta}(1+\rho)$

It is obvious that $\theta$ determines the sensitivity of the MRS on the shape of the consumption path. It can be also shown that the elasticity of substitution $\sigma$ is equal to $1 / \theta$ (see Appendix 2). ${ }^{131}$ Hence, the higher $\theta$ is, the lower the elasticity of substitution. Perfect substitutes are represented by $\theta=0$ because MRS $=(1+\rho)=$ constant. Logarithmic utility function is consistent with $\theta=1 .^{132}$ In this particular interval (i.e. between 0 and 1 ), the sensitivity of the individual's optimum consumption stream to changes in external conditions is very high, and the impact of $\rho$ on the MRS is dominant.
Again, the idea of the MRS can be represented by the variable $\varepsilon$ :

[^76]$\varepsilon=\left(\frac{C_{1}}{C_{0}}\right)^{\theta}(1+\rho)-1$
As can be seen, the time preference in the first sense $\varepsilon$ is positively related to the time preference in sense two $\rho$, regardless of the relative provision of consumption over time $\mathrm{C}_{1} / \mathrm{C}_{0}$ and the elasticity of substitution $1 / \theta$. However, the impact of a change in the subjective discount rate on the (endogenous) marginal rate of time preference $\varepsilon$ is magnified if the consumption stream is increasing ( $\mathrm{C}_{1}>\mathrm{C}_{0}$ ) and if the elasticity of substitution is rather low (high $\theta$ ). On the other hand, if the consumption stream is decreasing ( $\mathrm{C}_{1}<\mathrm{C}_{0}$ ), exactly the opposite statement holds.
Furthermore, for the given $\theta, \varepsilon$ rises with a larger provision of future goods compared with the provision of present goods. In such a case, the first Böhm-Bawerkian cause for interest is becoming stronger and stronger. Expression (13) also implies that if future is better provided for than present $\left(\mathrm{C}_{1}>\mathrm{C}_{0}\right)$, the time preference in the first sense $\varepsilon$ is larger for a lower intertemporal elasticity of substitution (high $\theta$ ). On the other hand, if present is more abundant than future ( $\mathrm{C}_{1}<\mathrm{C}_{0}$ ), $\varepsilon$ is higher with a higher elasticity of intertemporal substitution (low $\theta$ ). This conclusion is documented by the slope of the indifference curve at various points in Figure No. 1_A2 in Appendix 2. As can be seen, the MRS is much larger for high $\theta$ if we consider the part of the indifference curve above the $45^{\circ}$ line (compare panel (a) and panel (b)). On the other hand, the slope is much lower in panel (b) than in panel (a) if the points being compared lie below the $45^{\circ}$ line.
Moreover, the first Böhm-Bawerkian cause for interest depends not only on the relative provision of final consumption goods ( $\mathrm{C}_{1} / \mathrm{C}_{0}$ ) but also on the intertemporal elasticity of substitution ( $1 / \theta$ ). This observation therefore extends the original theory of Böhm-Bawerk.

As can be seen, the foregoing analysis implies that the time preference in the first sense $\varepsilon$ depends on the time preference in the second sense $\rho$ (i.e. on the Böhm-Bawerkian second cause for interest), the relative provision of consumption goods over time $\left(\mathrm{C}_{1} / \mathrm{C}_{0}\right)$, and the intertemporal elasticity of substitution in consumption represented by ( $1 / \theta$ ), where the latter two account for the Böhm-Bawerkian first cause for interest.

Expression (13) may also help us to show that in the case of perfect substitutes, $\theta=0$, the time preference in the first sense is solely determined by the time preference in the second sense, i.e. $\varepsilon=\rho$. On the other hand, the case of perfect complements, $\theta \rightarrow \infty$, suggests that if the present consumption is larger than the future consumption ( $\mathrm{C}_{0}>\mathrm{C}_{1}$ ), the time preference in the first sense is zero, $\varepsilon=0$, regardless of the size of the time preference in the second sense $\rho$. The reason is that the only optimum consumption path is perfect consumption smoothing ( $\mathrm{C}_{0}=\mathrm{C}_{1}$ ). Thus, an excess of present consumption over future consumption makes the given surplus of present goods completely useless. As a result, the consumer is willing to dispose of the present goods for free. In other words, the consumer is unwilling to sacrifice any single unit of future goods to obtain one more unit of present goods. The MRS is zero in this interval (see Figure No. 18). The opposite case holds if the future endowment of consumption goods is larger than the present endowment $\left(\mathrm{C}_{1}>\mathrm{C}_{0}\right)$. In such a case, the time preference in the first sense is infinite, $\varepsilon \rightarrow \infty$, regardless of the size of $\rho$. The consumer is infinitely impatient, accepting any reduction in future goods to get one additional unit of present goods, the MRS is infinite.

The figures presented above might be identified with the following combinations of parameters:
a) Figure No. 15, panel (a): low $\rho$ and low $\theta$
b) Figure No. 15, panel (b): high $\rho$ and low $\theta$
c) Figure No. 16: $\theta=0$; MRS (and $\varepsilon$ ) depends solely on $\rho$
d) Figure No. 17, panel (a): low $\rho$ and high $\theta$
e) Figure No. 17, panel (b): high $\rho$ and high $\theta$
f) Figure No. 18: infinite $\theta$; no role of $\rho$

We will conclude this section with several modifications of the assumptions listed at the beginning. Fisher (1930) in his seminal work did not assume that the MRS at the diagonal line is always constant. He explicitly stated that the impatience typically falls with higher average income, which results in the fact that the slope of the indifference curve decreases along the $45^{\circ}$ line. Furthermore, he also claimed that the response of the MRS to changes in the relative income endowment is much greater for people with lower income. The economic explanation is that if the bare life of a person is endangered, he is not willing to forgo present consumption even in exchange for a very high increase in future consumption. As a result, his time preference rises sky high. Hence, an increase in income not only diminishes the person's impatience but it also raises the internal stability of his time preference (in the sense of MRS). We can say that higher income brings about tranquillity to his mind.

In modern terms, this means that the subjective discount rate diminishes and the elasticity of substitution increases with the growth in average income. The graphical apparatus we used above is affected in such a way that indifference curves closer to the origin have a very large slope at the diagonal line, and at the same time they are much more curved (see Figure No. 19). In mathematical terms, parameters $\rho$ and $\theta$ are no longer exogenous; both should decrease with the growth in average income (i.e. with higher position of the entire income stream). They might be endogenous.


Figure No. $19 \rho$ and $\theta$ decreasing with higher average income

This behaviour of the discount rate was widely discussed in the literature. See, for example, Ghez and Becker (1975), and Becker and Mulligan (1997). However, it may produce
unfortunate results in dynamic models of the Ramsey style (Blanchard and Fischer 1989, Chapter 2). If an increase in income raises patience of the individual, this should lead to the situation that the least impatient person (or dynasty in the infinite horizon) in the economy will ultimately own all the assets of the society. ${ }^{133}$ A model with more stable outcomes was developed by Epstein and Hynes (1983). They assumed that the subjective discount rate depends positively on the level of future utility (and average consumption).
Further modification of the modelling of the discount rate was offered by Becker and Mulligan (1997) who explored the second Böhm-Bawerkian cause for interest. They focused on the first explanation for the second cause - the lack of imagination leading to the underestimation of future wants. They developed a model where a rational individual can invest in better imagination or anticipation of future wants, and by this act he can reduce his subjective discount rate. They specifically defined and investigated a future oriented capital that reduces $\rho$ (and maybe $\theta$ as well).

Another interesting analysis within the neoclassical framework modifying the standard approach can be found in Ryder and Heal (1973), and Trostel and Taylor (2001). Ryder and Heal developed a model in which the instantaneous utility is negatively related to the past consumption. Here, preferences are inter-temporally dependent, and the model may generate a satiation point, first discussed in the dynamic intertemporal framework by F. Ramsey (1928).
Authors of the second article relaxed the assumption of stationary instantaneous utility function. In their opinion, this function might vary with age in the sense that the ability to enjoy consumption eventually deteriorates. This property gives another reason for discounting future utilities (since they are of lower intensity) even without an explicit presence of time preference (subjective discount rate). ${ }^{134}$

All modifications mentioned above can extend our simple model. However, for the discussion with the PTPT, the simple model thoroughly explained in the previous part is sufficient as it reflects all properties and ideas necessary for a thorough analysis of the pure time preference theory.

## 3. OBJECTIVE AND PRODUCTIVITY ELEMENT - CLOSING THE SYSTEM

As was stressed in the first part of this study, the pure time preference theorists deny that the productivity of capital should play any role in the explanation of the interest phenomenon. Mises (1996) frequently claimed that only time preference determines the originary (natural) interest. Any increase in the productivity of capital will result merely in temporary profits. After some time, the value difference between present goods and future goods will be reestablished at the previous level. Rothbard (2004:424) reasoned that if the physical productivity of capital goods increases, their market value will eventually rise, but the value difference between those particular capital goods and the final output of consumption goods will return to the previous level that is dictated solely by the time preference. It must be stressed again that the essence of time preference in the PTPT is the statement that the given satisfaction is always preferred sooner rather than later. As Mises put it:

[^77]Originary interest is the ratio of the value assigned to want-satisfaction in the immediate future and the value assigned to want-satisfaction in remote periods of the future. It manifests itself in the market economy in the discount of future goods as against present goods. It is a ratio of commodity prices, not a price in itself. (Mises 1996:526)
Thus, in the Misesian system, productivity of capital is not important for the explanation of the phenomenon of interest, which is always positive due to the a priori existence of positive time preference. However, in this section we will try to demonstrate that time preference does not possess such unique superiority as Mises believed. The analysis that follows directly builds on our findings from the previous sections.

Fisher (1930) in his classic work on interest presented an example in which the interest rate must be necessarily zero. His reasoning is so tempting on the one hand, and so explicitly at variance with the Misesian theory, on the other hand, that it must be analyzed in great detail at this point.

Fisher introduced a story of shipwrecked sailors left only with a given stock of non-perishable hard-tacks. These hard-tacks represent the only good consumed by sailors. Their stock cannot be increased, so the only problem these sailors face is as follows: what is the optimal allocation of hard-tacks over their lives?

The above assumption implies that future consumption of sailors is possible only if some hard-tacks are saved in the present. Saving cannot be used for productive investment purposes, hence one present non-consumed hard-tack can "produce" one hard-tack ready for consumption in the future provided that they do not deteriorate over time. In this environment, Fisher claimed that the only equilibrium exchange ratio between present and future hard-tacks is one. In other words, Fisher concluded that in this economy no interest on hard-tacks may emerge, which is in direct contradiction with the fundamental statement of the Misesian theory.

According to Fisher, if the interest on hard-tacks was positive, no sailor would be willing to borrow. Nobody would accept borrowing 10 hard-tacks today so as to return 11 hard-tacks in the future because the same increase in the present consumption could be made just by a reallocation of the sailor's own stock - simply 10 hardtacks would be consumed today instead of in the future. On the other hand, everybody would be willing to lend for the opposite reasons. This obvious imbalance should rapidly reduce the interest rate back to zero.
Fisher further concluded that the optimum MRS of every sailor must be so adjusted as to make his subjective intertemporal valuation of hard-tacks consistent with the objective reality. The optimum MRS must be one; every sailor must shape his consumption stream in such a way that at the margin, the present hard-tack will be valued the same as the future hard-tack. Hence, there will be no discount on future goods.

At this point, we will use the tools developed in the previous section to demonstrate that Fisher's reasoning seems to be superior to Mises's theory. Consider a representative sailor who lives just in two periods. ${ }^{135}$ This sailor prefers the given satisfaction earlier rather than later, so if the first need on his value scale is to eat, the second one is to feed his dog, the third one to feed his cat, etc., he will use the first hard-tack for eating now, the second hard-tack to feed his dog now, the third hard-tack to eat tomorrow, the fourth hard-tack to feed his cat today, the fifth hard-tack to feed his dog tomorrow, etc. This sailor perfectly meets the Misesian requirement of the superiority of present satisfaction. He exhibits typical time preference in sense two, and his subjective discount rate is thus surely positive.

[^78]Even though every sailor prefers present satisfaction to future satisfaction, and his time preference (in sense two) is positive, the interest rate in this economy must be necessarily zero. According to Mises, the combination of positive time preference and zero interest rate is absolutely unthinkable because on the unhampered market, the positive time preference must be always reflected in a positive rate of interest. ${ }^{136}$

It could be also argued that all resources would be depleted in the present if the interest was zero and the time preference was positive. However, we will see that this is not the case. Consumption may be (more or less) evenly spread across periods even if the subjective discount rate is positive and the interest rate is zero.
In Figure No. 20, we can see a graphical representation of preferences of a representative sailor. His preference for the present gratification of the given want is reflected in the slope of the indifference curve exceeding 1 at the diagonal line. In this figure, we also added his resource constraint. His stock of hard-tacks is depicted at point A. Because hard-tacks can be easily moved to the future, his resource constraint (or a very degenerate investment opportunity line or the production possibility frontier) is linear and with the slope 1 (in absolute value). At the same time, this resource constraint must perfectly coincide with his intertemporal budget constraint (IBC) because the market interest rate on this desert island (in terms of hard-tacks) is necessarily zero (see panel (a)).



Figure No. 20, Optimum of a shipwrecked sailor on a desert island. Hard-tack economy

The reason for zero interest is as follows: a positive interest rate would result in the excess of lending over borrowing, whereas a negative interest rate would lead to the opposite situation. The graphical proof of this statement is presented in panel (b). If the interest rate was positive, for example $10 \%$, no hard-tack from the entire fund of A would be retained in the stock for the future. For a $10 \%$ interest rate and a linear PPF with the slope 1 (in absolute value), point A would generate the maximum present value of assets, as can be seen by comparing this

[^79]point with point $\mathrm{D}_{0}$. If the number of $\left(\mathrm{A}-\mathrm{D}_{0}\right)$ hard-tacks were left in the stock, the present value of the sailor's assets (or the present value of the income stream $\mathrm{D}_{0}, \mathrm{D}_{1}$ ) would be lower.

Thus, assuming positive interest rate, all sailors ( N ) in this economy would not store any hardtack in their stocks. Each (non-consumed) piece would be offered on the intertemporal market for a $10 \%$ interest rate. Only this decision would maximize the present value of their assets. However, in such a case, the total demand for present hard-tacks ( $\mathrm{N} \mathrm{x} \mathrm{C}_{0}{ }^{2 *}$ ) would fall short of the total supply of present hard-tacks ( $\mathrm{N} x \mathrm{~A}$ ). Alternatively, it can be said that the net per capita supply of present hard-tacks $\left(\mathrm{A}-\mathrm{C}_{0}{ }^{2 *}\right)$ could not find the corresponding net demand. As a result, the interest rate must decrease to zero to equalize the demand and supply. It must decline to equilibrate the intertemporal market.

As can be seen in panel (a), only a zero interest rate will eliminate the excess of present hardtacks in the intertemporal market. The IBC will then perfectly coincide with the PPF. At the same time, the demand for present hard-tacks will be equal to the supply of present hard-tacks even though they will not be traded on the intertemporal market, as will be seen later on. At the individual level, $\mathrm{C}_{0}{ }^{*}$ hard-tacks will be consumed in the present, and ( $\mathrm{A}-\mathrm{C}_{0}{ }^{*}$ ) will be retained in the stock for future consumption $\mathrm{C}_{1}{ }^{*} .{ }^{137}$

Panel (a) clearly shows that the sailor's optimum cannot be at point A. This means that even in the situation of zero interest and positive time preference (in sense two), all hard-tacks are not consumed in the present. The optimum does not lie at point B either, where the consumption stream is perfectly smoothed. At this point, the sailor is prepared to forgo more future hard-tacks just to obtain one additional present hard-tack. The economic environment enables him to perform such reallocation because the objective ratio is lower - just one-forone. Hence, the reallocation of hard-tacks will continue up to the point where his MRS is equal to 1 (the same value as the marginal rate of transformation in this case). The optimum will be at point E at which the present hard-tack will be valued the same as the future hardtack.

Intuitively, the optimum must be very close to the point at which marginal utilities in both periods are not far off of each other. In this particular case, they are exactly equal (keeping the proper definition of future MU). Equations (14) to (16) support this conclusion:

$$
\begin{align*}
& \text { MRS }^{\text {optimum }}=1  \tag{14}\\
& M R S=\frac{u^{\prime}\left(C_{0}\right)}{\frac{u^{\prime}\left(C_{1}\right)}{(1+\rho)}}  \tag{15}\\
& u^{\prime}\left(C_{0}\right)=\frac{u^{\prime}\left(C_{1}\right)}{(1+\rho)} \tag{16}
\end{align*}
$$

Figure No. 21 demonstrates this process of equalization of marginal utilities across time. At point B , consumption is the same in both periods. Nevertheless, marginal utility from present consumption is higher than the discounted future marginal utility, ${ }^{138} u^{\prime}\left(\mathrm{C}_{0}{ }^{\mathrm{B}}\right)>\mathrm{u}^{\prime}\left(\mathrm{C}_{1}{ }^{\mathrm{B}}\right) /(1+\rho)$,

[^80]hence the MRS exceeds one. A reallocation of hard-tacks from the future to the present (i.e. leaving more hard-tacks in the present rather than in the future) leads to a net increase in total well-being, as can be seen by comparing areas F and G. Area F represents the gained utility, area $G$ the forgone utility. Optimum is found at point $E$ where no net gain can be obtained by further reallocation of consumption over time. Furthermore, at both points the budget constraint is satisfied because the total consumption of hard-tacks over the lifetime does not exceed the initial stock: $\mathrm{C}_{0}{ }^{\mathrm{B}}+\mathrm{C}_{1}{ }^{\mathrm{B}}=\mathrm{C}_{0}{ }^{\mathrm{E}}+\mathrm{C}_{1}{ }^{\mathrm{E}}=\mathrm{A}$.


Figure No. 21, Reallocation of consumption over time to achieve maximum utility

As can be seen, the argument that consumption goods will be entirely reallocated to present does not hold. The necessary break is performed by the tendency to equalize marginal utilities. ${ }^{139}$ A one-way shift of all goods to one particular period would radically reduce marginal utility in this period. In other words, needs of very low intensity would be satisfied in this period at the expense of needs in other periods. And this cannot be optimal.

Finding the exact ratio between present and future consumption requires a concrete form of the utility function. Assuming CRRA, the solution for the optimum is as follows (see equation (12):
$M R S=1=\left(\frac{C_{1}}{C_{0}}\right)^{\theta}(1+\rho)$
$\frac{C_{1}}{C_{0}}=\left(\frac{1}{1+\rho}\right)^{\frac{1}{\theta}}$

[^81]Equation (18) explicitly reveals the role of the subjective discount rate $\rho$ and also of $\theta$, a parameter reflecting the elasticity of substitution. An increase in the subjective discount rate (time preference in the second sense) raises present consumption at the expense of future consumption. A more impatient sailor would find his optimum closer to point A and further from point B in Figure No. 20. Parameter $\theta$ operates in the opposite direction. The lower the $\theta$, hence the higher the elasticity of substitution, the closer is the optimum of the sailor to point A, so more hard-tacks will be consumed in the present. Infinite elasticity of substitution $(\theta=0)$ will move the optimum to point A. All hardtacks will be consumed in the present only for such an extreme case. Panel (a) in Figure No. 22 illustrates this situation. A linear indifference curve having the slope $(1+\rho)$ is steeper than the intertemporal budget constraint (IBC) with the slope $(1+r)=1$. On the other hand, the case of perfect complements (infinite $\theta$ ) would result in the coincidence of E with point B . Optimum consumption would be perfectly smoothed regardless of the size of the subjective discount rate (panel b).


Figure No. 22, Optimum consumption path; the case of perfect substitutes (panel a) and perfect complements (panel b)

In our example, the market interest rate is not affected by sailors' subjective discount rates. Regardless of their impatience, the interest rate must be necessarily zero. The consumption path of each sailor will be so adjusted that his time preference in sense one will be depressed to zero. In other words, it will be shaped such that the present hard-tack will be valued the same as the future hard-tack, and the MRS will be one. The subjective discount rate (and the intertemporal elasticity of substitution in consumption) of each sailor will only determine his optimum shape of the consumption stream over time (see equation 18), i.e. the specific position of point E on the budget line. However, it must be stressed that in such a case, the budget line itself is determined by the objective phenomena - the initial stock of hard-tacks and their zero marginal productivity.

It is also obvious that in this hard-tack economy, if the subjective discount rate (time preference in the second sense) is positive, the optimum present consumption is larger than the optimum future consumption (except for the case of perfect complements). This is a direct
corollary of the fact that the interest rate is lower than the subjective discount rate (zero compared to a positive number). In such a case, the optimum path of consumption is decreasing. This will be proved generally below, after introducing the Euler equation. Especially in the continuous version of this model presented in section 5 with time horizon $T$ or even infinite horizon, we would say that the optimum growth rate of consumption is negative.
At this point, let us make a little digression and discuss the implications of a negative time preference in sense two, i.e. a situation where the given satisfaction is preferred later rather than earlier. ${ }^{140}$ Mises would reject such a possibility on the a-priori basis. According to Mises, the act of consumption would never occur (1996:484). However, we can demonstrate that such a conclusion is not correct. In our mathematical model, the negative time preference in sense two means that the subjective discount rate is negative. ${ }^{141}$ Recall that we consider a hard-tack economy in which the market interest rate is zero. According to our graphical and mathematical model (see equation 18), the optimum for negative $\rho$ would be somewhere to the top left of point B in Figure No. 20, and more would be consumed in the future. Nevertheless, it could not be at the end of the budget line at point M, as Mises would predict. Intuitively, if the given satisfaction is preferred in the future, the first hard-tack will be eaten in the future, and the second hard-tack may feed the dog in the future. However, due to the law of diminishing marginal utility, the third hard-tack might be eaten in the present because this present need is so pressing that it will more than offset the paradoxical tendency to satisfy the given want (in this case feeding the cat) in the future. Hence, Mises was not right in saying that if the given satisfaction is preferred in the future, the act of consumption will never occur. The law of diminishing marginal utility will offset the tendency to postpone consumption of every single good to the future. In the mind of our consumer, wants in the present would be so pressing and future desires (from the present perspective) would be so abundantly satisfied that some goods must be consumed in the present.

However, Mises's idea that the given satisfaction is always preferred in the present is still so attractive that it will be accepted here almost on the a-priori basis. In other words, unless otherwise indicated, we will assume that the subjective discount rate is positive.
Let us return to the hard-tack economy that is characterized by a zero interest rate and presumably by a positive time preference in sense two. Garrison (1979) objected that it is ridiculous to talk about the (zero) interest rate, intertemporal markets, intertemporal decisions, and even about human action as such in this highly stylized economy. After all, there is no intertemporal exchange at all; no hard-tacks will be traded among sailors. There will be no borrowers and no lenders.

It should be admitted that Garrison was perfectly right. There is clearly no intertemporal market in this example. However, this does not mean that zero interest rate is not the market equilibrium. The intertemporal exchange is eliminated because zero market interest rate gives the same return as storing hard-tacks in the stock. Nevertheless, an enormous number of intertemporal exchanges might exist even for this zero interest rate, although it would be quite irrational to trade one present hard-tack for one future hard-tack if the act of exchange is costly. ${ }^{142}$ On the other hand, even a tiny deviation of the market interest rate from the zero

[^82]equilibrium would provoke a creation of a true intertemporal market. However, a gigantic imbalance would be observed on this market. If the interest rate was positive, the market would be flooded by present hard-tacks offered in exchange for a higher amount of future hard-tacks. A negative interest rate would lead to the opposite tendency. As a result, only zero interest rate guarantees equilibrium in the intertemporal market even though this market as such will not exist. ${ }^{143}$

It is also inaccurate to claim that there is no human action. The sailors must decide upon the optimum allocation of hard-tacks over time. The formal dynamic model developed in section 5 will shed even more light on this problem. At this moment, we can conclude that zero market interest rate may exist, even if people prefer present satisfaction to future satisfaction, i.e. if their time preference in the Misesian meaning is positive. The hard-tack economy is also a very good example of a situation in which the time preference in the second sense exists $(\rho>0)$ - that is, all sailors prefer the given satisfaction to be gratified as soon as possible, whereas the time preference in the first sense is zero (MRS $=1$, or MRS $-1=\varepsilon=0$ ) present goods are valued the same as future goods. Both the subjective and the objective exchange ratio between a present hard-tack and a future hard-tack is one, where the former must have been adjusted to the latter.
Fisher (1930) in his work offered another example of the irrelevance of the time preference. In the second story, the saved goods exhibited positive rather than zero physical productivity. He envisioned a herd of sheep that, if saved and properly invested, would increase future output of sheep by a constant percent. According to Kirzner (1993), P. Samuelson speculated about a similar situation - each seed of rice, if not consumed, may (without any further costs) provide 1.1 seeds in the future.

Compared with the hard-tack economy, the marginal productivity of capital in the "rice economy" is positive. Capital has a form of the saved rice (or sheep), and it has a productive power to increase future output of consumption goods. Obviously, this example is elementary and highly stylized - both capital and consumption goods are represented by the same commodity. As such, it is a typical neoclassical one-good economy. However, even this example may provide us with interesting insights. ${ }^{144}$
Another simplification in this example is that the marginal product of capital does not diminish. It is constant, for example $10 \%$. Thus, every additional seed of rice invested provides a net return of ten percent, regardless of the number of seeds invested before. By the same reasoning as before, the only equilibrium interest rate (in terms of rice) in this economy is $10 \%$, as the following discussion clearly demonstrates.

The resource constraint (PPF) of each sailor can be represented by a linear line; in this particular case with the slope of 1.1 (in absolute value). Panel (b) in Figure No. 23 illustrates that if the market interest rate was lower than $10 \%$, for example $0 \%$, the intertemporal budget constraint (IBC) would be flatter than the PPF, and all present seeds of rice should be invested. By investing the entire stock, the producer would maximize the present value of his assets, as can be seen by comparing the present value of point $M$ with point $D$ at which only ( $A-D_{0}$ ) seeds of rice would be planted in the present. Thus, all sailors ( $N$ ) should invest their entire stock A. However, in such a case, the demand for present goods $\left(\mathrm{C}^{2{ }^{2 *}}+\mathrm{A}\right) \times \mathrm{N}$ would highly exceed the supply of present goods A x N. Alternatively, it can be said that the

[^83]aggregate investment $\mathrm{A} \times \mathrm{N}$ would fall short of aggregate saving $\left(\mathrm{A}-\mathrm{C}_{0}{ }^{2 *}\right) \times \mathrm{N}$. At the individual level, the per capita net demand for present goods $\mathrm{C}_{0}{ }^{2 *}$ could not find the corresponding net supply. As a result, the interest rate r should increase to the point at which the shortage of present goods is completely eliminated.

In the intertemporal equilibrium, the intertemporal budget constraint will coincide with the resource constraint, both having the slope of $(1+r)=1.1$. As can be seen in panel (a), the optimum MRS of every sailor must be 1.1 as well. Only at point E , the total demand for present goods due to consumption $\mathrm{Nx} \mathrm{C} 0^{2^{*}}$ and investment $\mathrm{N} \times\left(\mathrm{A}-\mathrm{C}_{0}{ }^{2^{*}}\right)$ will be equal to the total supply of present goods $\mathrm{N} \times \mathrm{A}$. Total investment will be equal to total saving, and all sailors will also achieve maximum utility.


Figure No. 23, Optimum of a shipwrecked sailor in the "rice" economy.
The optimum of each sailor can be found using the well-known condition that the interest rate r must be equal to the time preference in sense one, which is defined as $\varepsilon \equiv \operatorname{MRS}-1$ (see Appendix 4). ${ }^{145}$

MRS $=1+r$

[^84]\[

$$
\begin{align*}
& \frac{u^{\prime}\left(C_{0}\right)}{\frac{u^{\prime}\left(C_{1}\right)}{(1+\rho)}}=1+r  \tag{20}\\
& 1+\varepsilon=1+\mathrm{r} \\
& \varepsilon=\mathrm{r} \tag{21}
\end{align*}
$$
\]

In the rice economy, the marginal rate of time preference $\varepsilon$ in optimum must be $10 \%$. As can be seen from equation (20), the optimum allocation of consumption (of rice) over time depends on the (intertemporal) preferences of each sailor. Using the CRRA, the solution can be found by the same procedure as in (17) and (18).

$$
\begin{align*}
& M R S=1.1=\left(\frac{C_{1}}{C_{0}}\right)^{\theta}(1+\rho)  \tag{23}\\
& \frac{C_{1}}{C_{0}}=\left(\frac{1.1}{1+\rho}\right)^{1 / \theta} \tag{24}
\end{align*}
$$

or generally:

$$
\begin{equation*}
\frac{C_{1}}{C_{0}}=\left(\frac{1+r}{1+\rho}\right)^{1 / \theta} \tag{25}
\end{equation*}
$$

Equation (25) is the Euler equation for this particular problem. As can be seen from equation (24), the relative size of $\mathrm{C}_{1}$ and $\mathrm{C}_{0}$ can take on any value. If the subjective discount rate is lower than $10 \%$, future consumption will exceed present consumption, but not necessarily by $10 \%$. The exact value depends on $\rho$ and $\theta$. These two parameters determine the optimum combination of $\mathrm{C}_{1}$ and $\mathrm{C}_{0}$. Figure No. 23 represents a sailor who is relatively patient; the slope of the indifference curve at the diagonal line is lower than the slope of the budget constraint. This sailor willingly consumes more rice in the future.
We can conclude that if the interest rate exceeds the subjective discount rate, future consumption will exceed present consumption. Optimum path of consumption is increasing, or we may say that the optimum growth rate of consumption is positive. All statements are interchangeable. In Section 5, after introducing a model with an infinite number of periods, we will discuss this topic in more detail.
It is theoretically possible that there is a sailor with the subjective discount rate of exactly $10 \%$ (Figure No. 24). This particular sailor will perfectly smooth his consumption over time because the interest rate is equal to his subjective discount rate (see equation 25). However, even if all sailors were of this type, it would not be correct to claim that the interest rate in this economy is determined by their time preferences (in sense two). For any value of $\rho$, the market interest rate in this economy is solely determined by the productivity of capital. It must be $10 \%$, regardless of the impatience of sailors. ${ }^{146}$ Parameter $\rho$ will only affect the optimum path of consumption over time, not the interest rate.

[^85]

Figure No. 24, Optimum of a shipwrecked sailor in the "rice" economy; the case of perfect consumption smoothing ( $\mathrm{r}=\rho$ ).

Another interesting observation derived from this example is as follows. Suppose that the given satisfaction is not preferred as soon as possible, so the individual is indifferent about the moment of gratification of the given need. In mathematical terms, suppose that the subjective discount rate is zero. ${ }^{147}$ This assumption implies that the slope of the indifference curve at the diagonal line is one. However, the optimum of this sailor can be illustrated by a similar diagram as is shown in Figure No. 23. Mises (1996) claimed that positive interest rate and zero time preference will lead the consumer to postpone every unit of consumption goods to the future. In other words, everything should be saved. As can be seen, this is not accurate, at least in the 2-period model. The reason lies in the fact that the consumer's optimum is quite close to consumption smoothing because marginal utilities of consumption in each period should not be very far off of each other. In each period, wants of similar urgency are to be satisfied. Thus, the allocation of consumption goods over time will follow this requirement. ${ }^{148}$

[^86]If the subjective discount rate is zero (and the interest rate is $10 \%$ due to constant productivity of capital), the optimum path of consumption is implicitly defined by the following equation:
$\frac{u^{\prime}\left(C_{0}\right)}{u^{\prime}\left(C_{1}\right)}=1.1$

The optimum consumption path must be shaped such that the ratio of marginal utilities equals 1.1. Since the marginal utility decreases with consumption, $\mathrm{C}_{1}$ must exceed $\mathrm{C}_{0}$. For a logarithmic utility function (i.e. CRRA, $\theta=1$ ), $\mathrm{C}_{1}$ will exceed $\mathrm{C}_{0}$ by $10 \%$ (see equation 25). For a square root utility function (CRRA, $\theta=1 / 2$ ), this difference will increase to $21 \%$, whereas for $\theta=2$, it falls to $4.8 \%$. As can be seen, the higher the $\theta$, the flatter the optimum profile of the consumption path.

In the preceding paragraphs, we have demonstrated that the positive interest rate is perfectly consistent with zero time preference (in sense two). The example of rice economy also shows that there might exist time preference in sense one (MRS > 1; see equation 26) even if the time preference in sense two is absent $(\rho=0)$. Thus, people in this economy do not prefer the given satisfaction to be gratified as soon as possible, yet they do prefer the marginal present good over the marginal future good. Their time preference in the first sense is $\varepsilon=10 \%$. This endogenous meaning of time preference is perfectly adjusted to the ongoing rate of interest that is solely determined by the productivity of capital. ${ }^{149}$
To conclude, the previous two examples proved that the time preference in the Misesian sense (the superiority of present satisfaction) is neither a necessary nor a sufficient condition for the existence of interest. In example No. 1 - hard-tacks, we observed a zero market interest rate even if people preferred the given satisfaction to arrive as soon as possible ( $\rho$ was positive). In example No. 2 - constant marginal productivity of capital (rice), the interest rate was positive even in the economy where people were indifferent about the time at which the given want would be satisfied ( $\rho$ was zero).

Moreover, Mises's predictions about a complete postponement of all goods to the future did not hold either. The simultaneous existence of zero time preference in the second sense (i.e. the absence of the preference for the satisfaction of the given want sooner rather than later, $\rho=0$ ) and positive interest rate were consistent with the positive present consumption. The reason was that the time preference in the first sense (i.e. the relative valuation of present goods as against future goods, MRS $-1=\varepsilon$ ) was adjusted to the prevailing positive rate of interest $(\varepsilon=r>0)$. We have also shown that all goods would not be postponed to the future even if the time preference in the second sense was negative ( $\rho<0$ ). The law of diminishing marginal utility would act as a necessary break for such an outcome. Thus, the time preference in the first sense is always so adjusted that in the consumer's optimum, it is equal to the ongoing rate of interest, be it zero (hard-tack economy) or positive (rice economy).
At this place, Garrison (1979) and other authors writing in the PTPT tradition might object that even though rice in our example have a physical productivity, it does not necessarily exhibit value productivity. However, in the one-good model, value productivity coincides with physical productivity. Value is usually measured in terms of money or in terms of other commodity. In the single-good model, this distinction is immaterial.

[^87]Furthermore, assuming that more goods are always preferred to less, ${ }^{150}$ one present unit of rice must have a higher subjective value than one future unit of rice. The reason lies in the fact that the former turns into 1.1 units of the latter. Consequently, 1.1 units of future rice should be valued more than 1 unit of future rice. As a result, present rice must be valued more than future rice; there exists the phenomenon of interest.
Nevertheless, Garrison (1979:146) objected to this example that productivity cannot explain the interest phenomenon because it neglects valuations of acting man. He provided a mythical example of an island that is washing away by $30 \%$ per year. Using the same reasoning as with rice (or as with figs mentioned below), Garrison implied that the interest rate in terms of the island should be $-30 \%$, regardless of valuations of acting man or regardless of the existence of humankind as such.

Obviously, this is absurd. The interest is always a subjective phenomenon that depends on subjective valuations of acting people. There would be no interest in the case of an island that washes away because it is not an object of valuing minds. On the other hand, the interest phenomenon will exist in the rice economy since rice is a valuable good. It can be exchanged on the intertemporal market. If the valuing mind prefers more rice to less rice, the productivity phenomenon can solely determine the intertemporal price of rice regardless of the time preference; productivity may solely determine the interest rate. As a result, even though it is the subjective valuations of acting men that ultimately creates (intertemporal) value of goods, the productivity element might be its crucial determinant, as our simple rice example demonstrated, totally neglecting the influence of time preference (in sense two).
To put it in different words, if more goods are preferred to fewer goods, productivity might uniquely determine the interest rate without any role of the time preference. The subjective element is ensured by (intra-temporal) valuations of goods - by the law of diminishing (but positive) marginal utility. The time preference in the second sense, i.e. the preference for the gratification of the given need as soon as possible, is not necessary. Thus, it is the objective element of productivity (together with the preference of having more rather than less and together with the diminishing MU) ${ }^{151}$ that ultimately determines the interest rate in the "rice economy".
We will conclude this part with one theoretical possibility that is also unthinkable in the Misesian system - a negative originary (or natural) interest. Fisher's third famous example is about shipwrecked sailors furnished just by a perishable good - figs. Sailors are endowed with a given amount of figs that, however, deteriorate over time. Suppose that the rate of decay is $10 \%$ per year.

The solution of this problem is virtually the same as before. In a two-period model, even if no figs are consumed today, their stock will fall by ten percent in one year. In this economy, the only equilibrium exchange rate between present and future figs is 0.9 . In other words, the only equilibrium interest rate is $-10 \%$ (i.e. minus $10 \%$ ). If it was, for example, $0 \%$, everybody would be prepared to offer present figs just to obtain the same amount of figs in the future. On the other hand, no one (honest) would be willing to borrow 10 present figs to pay back the same amount in the future. A mere reallocation of his own figs to present (i.e. eating additional 10 figs today instead of in the future) would obviously represent a better option. Thus, the resulting surplus of the supply of present figs over their demand on this

[^88](momentary, as we already know) intertemporal market will necessarily reduce the interest rate to the equilibrium level of $-10 \%$. ${ }^{152}$

Hence, if only 9 out of 10 figs survive to the future, their subjective intertemporal valuation must follow this ratio. Figs will be allocated over time such that the optimum MRS of every sailor (regardless of his $\rho$ ) is 0.9 . Figure No. 25 illustrates a situation of one representative sailor. It is very similar to the hard-tack example; the only difference is that the interest rate is negative, and (for the same $\rho$ and $\theta$ ) the optimum path of consumption decreases more rapidly (see equation 25 ).


Figure No. 25, Optimum of a shipwrecked sailor on a desert island; the fig economy

We can conclude that in this particular example, a negative interest rate will emerge even if sailors exhibit positive time preference in sense two. ${ }^{153}$ Sailors have a negative time preference in sense one (MRS $-1=\varepsilon=-0.1$ ), whereas in sense two, it is positive ( $\rho>0$ ). In other words, sailors do prefer the given want to be satisfied as soon as possible, yet the subjective (and objective) exchange ratio between present goods and future goods is 0.9 . Every man in this economy is willing to exchange one present fig for nine tenths of a future fig. Thus, present goods are valued less than future goods. Moreover, the interest rate is determined by the objective element - by the rate of decay of that particular good.

As a result, this example reveals again the inaccuracy of the Misesian statement that:
Satisfaction of a want in the nearer future is, other things being equal, preferred to that in the farther distant future. Present goods are more valuable than future goods. (Mises 1996:483)

[^89]
### 3.1 PHYSICAL PRODUCTIVITY AND VALUE PRODUCTIVITY, REAL INTEREST RATE VERSUS NOMINAL INTEREST RATE

Modern proponents of the Austrian school and the PTPT seem to be at least partly aware of the objections raised in the previous section. ${ }^{154}$ They offer two fundamental statements against the approach presented so far. First, a single-good model with constant productivity explains the growth rate of output (or attainable real income stream in terms of this good) rather than the interest phenomenon. Secondly, following Böhm-Bawerk (1890) in his criticism of naïve productivity theories, PTPT theorists claim that the interest theory must explain value productivity, not only physical productivity. They accept that capital is productive in the sense that it produces goods. They also admit that it may produce more goods than is the amount originally invested, as in the example with rice. However, the key problem of the interest theory is why the value of output is eventually higher than the total value of inputs used in a time-consuming process of production. They ask why the competition does not eliminate this value difference between inputs (representing future consumption goods) and the eventual output of the resulting consumption goods. ${ }^{155}$ The interest theory must therefore explain why present goods are valued more than future goods of the same type and quality. ${ }^{156}$ In a singlegood model, both concepts (the value and physical productivity) coincide and, as a result, the answer to the fundamental problem raised is avoided or escaped.
R. Garrison in the following passage mentioned both problems:

Modern textbook writers have attempted to skirt this problem by using a one-good model. In all such models, questions of value, which may be affected by changes in the rate of interest, simply do not arise. Value productivity and physical productivity are indistinct; productivity is modelled as the rate of increase in the quantity of the good. The phenomenon of interest is being analogized once again to sheep that reproduce or to plants that grow. But, as Professor Rothbard often reminds us, the rate of interest is a ratio of values, not of quantities. This modelling technique unavoidably conflates growth rates with interest rates and fails thereby to shed any light on the phenomenon of interest. (Garrison 1988:170)
I. Kirzner raised similar objections to the one-good model:

One hundred units of 1987 rice exchange, in 1987, for 110 promised units of 1988 rice. With this trade repeated each year, the rice owner can consume 10 units of rice each year ("real interest income") without eroding the ("capital") base that yields this annual income. We shall attempt to show, however, that from the Fetter-Mises PTPT view, these demonstrations do nothing to advance understanding of the general phenomenon of interest. (Kirzner 1993:109)

Let us first deal with the objection that the single-good model explains the growth rate of output rather than the interest phenomenon. In the example with rice, the net marginal product of capital (i.e. of saved rice) is $10 \%$. In other words, one unit of rice invested this year can

[^90]produce 1.1 units of rice next year. However, this does not necessarily imply that output of rice in the next period will also be higher by $10 \%$. Recall the Euler equation (24) or (25). The eventual increase in the (demanded) output depends on the subjective discount rate $\rho$ and the intertemporal elasticity of substitution $1 / \theta$. Since the MPK is constant and the investment opportunity line is linear, the cost curve in this economy is a horizontal line at the given relative price between future goods and present goods (i.e. 1/1.1 in our case). The intertemporal demand (given by the Euler equation) determines only the optimal ratio of present and future output because a perfectly elastic supply curve can meet any demand for the given relative price.


Figure No. 26, Possible optima in the "rice" economy

Figure No. 26 graphs possible optima in this rice economy. Panel (a) reflects $\rho=0$ and $\theta=1$ (i.e. logarithmic utility function), panel (b) is for $\rho=0$ and $\theta=2$. Panel (c) displays $\rho=4 \%, \theta=1$, and panel (d) is for $\rho=4 \%$ and $\theta=2$. All combinations will be used further in the text to elucidate other confusions in the PTPT.
In all examples, the interest rate is $10 \%$, and the optimum $\varepsilon=$ MRS- 1 must be $10 \%$ as well, regardless of $\rho$ and $\theta$. However, only in panel (a), the growth rate of output (and consumption) is also $10 \%$. In this case, the indifference curves are only moderately curved (due to relatively low $\theta=1$ ), and their slope at the diagonal line is 1 (due to $\rho=0$ ). In panel (b), the slope at the $45^{\circ}$ line is also 1 , but due to the fact that the elasticity of substitution is lower and the indifference curves are therefore more curved, the optimum consumption (and output) growth is only $5 \%$. In diagram (c), the slope of the indifference curve exceeds one at the $45^{\circ}$ line (because $\rho>0$ ), and the optimum consumption growth is $6 \%$. And finally in (d), a relatively low elasticity of substitution depresses the optimum consumption (and output) growth to $3 \%$ for the same subjective discount rate as in panel (c).
As we can see, a $10 \%$ growth in output is not general, it holds only for a particular combination of parameters. Furthermore, the Euler equation (25) implies that the optimum growth in consumption (and output) is negatively related to $\rho$ (time preference in the second sense), and if $r>\rho$, it is positively related to the elasticity of substitution ( $1 / \theta$ ). As a result, the optimum of less patient people (for the given $\theta$ ) is closer to the horizontal axis and further from the vertical axis (compare panel (c) and (a), or (d) and (b)).

On the other hand, people with lower elasticity of substitution (for the same $\rho$ ) prefer a smoother path of their optimum consumption (compare panel (b) and (a), or (d) and (c)). Hence, their optimum point is closer to the $45^{\circ}$ line. If the interest rate is greater than the subjective discount rate, their present consumption is larger than that of people with higher intertemporal elasticity of substitution. The opposite conclusion would hold if $\mathrm{r}<\rho$.

The second objection of modern PTPT proponents is more fundamental - the theory of interest must explain the difference between the value of inputs (representing future consumption goods) and the eventual value of output of the resulting consumption goods. According to the pure time preference authors, their theory could fulfil this difficult task, whereas the productivity theory cannot.
Let us first show how the PTPT theorists would argue in the rice example against the productivity theory. Suppose that 100 present units of rice can produce 110 units of rice next year owing to constant productivity (Kirzner 1993:112). Assume also that people do not obey the fundamental Misesian assumption, so they do not prefer the given satisfaction earlier ( $\rho$ is zero). As a result, the emergence of interest, if any, must be only due to physical productivity. Suppose further that both the present price of rice and future price of rice are $\$ 20$. Hence, if 100 units of rice are invested today, the eventual rate of interest will be as follows: ${ }^{157}$
interest rate $=$ (value of future goods - value of present goods) $/$ value of present goods interest rate $=(20 \times 110-20 \times 100) / 20 \times 100=10 \%$

Böhm-Bawerk (1891) was eager to explain why the eventual value of output (\$20x110 units of rice $=\$ 2,200$ ) is higher than the total value of factors of production invested $(\$ 20 \times 100$ units of rice $=\$ 2,000$ ). Böhm-Bawerk claimed that the key problem in the interest theory is to explain this value productivity (\$200), not the physical productivity (additional 10 units of

[^91]rice). According to the PTPT, this value difference exists due to the time preference and the fact that the production process takes time. In the absence of time preference, competition should gradually eliminate this $\$ 200$ "profit opportunity".

We assumed at the start of this example that there is no time preference (in the second sense). Hence, Kirzner argued that the expected physical increase in output created by the investment of present rice must be sooner or later imputed to its present price. In the absence of time preference, the present price of rice must increase to $\$ 22$. In equilibrium, the interest (rate) is necessarily $0=(20 \times 110-22 \times 100)$. There is no value-difference to be explained. As a result, the pure physical productivity cannot induce value productivity, which may emerge only due to time preference.
One hundred units of 1987 rice are expected to ripen into 110 units of 1988 rice. Suppose that the "value" of the 100 units of 1987 rice has indeed risen to anticipate this physical growth. Then in terms of the interest problem (formulated at the outset of this paper) the perpetual annual rice consumption income so made possible does not present an example of interest. (Kirzner 1993:112)

Herbener added that if the time discount rate of people was $5 \%$, the market interest rate must be $5 \%$, not $10 \%$, as would be predicted by the productivity theory:
A 10 percent annual increase in the physical stock of a Crusonia plant, or a bushel of rice, or a flock of sheep does not dictate any particular time-discount or rate of interest, let alone exactly 10 percent. If the Crusonia plant, or bushel of rice, or flock of sheep were a tradable good in the monetary market and if the time discount rate was 5 percent, then the rate of interest earned from investing in any one of them would be 5 percent. (Herbener 2011:44)

The previous statements are so perplexing that their proper elucidation requires us to go very slowly. At first glance, Kirzner's reasoning seems to be quite persuasive. The absence of time preference should result in zero interest because $\$ 2,200$ invested today will turn into $\$ 2,200$ received next year. There is no value productivity without time preference. There is no interest without time preference. On the other hand, the Fisherian neoclassical approach from the previous sections seemed to be correct as well because the $10 \%$ interest rate was the only logical solution. However, both solutions cannot be correct simultaneously. Or can both approaches be somehow reconciled?
An elementary textbook approach to the interest problem is usually as follows: If Ann lends 10 apples to John, she requires 11 apples to be paid back next year because the present apple is (subjectively) valued more than the future apple. In other words, the interest problem is usually presented as the problem of the intertemporal exchange ratio between goods. And as it is, it is certainly correct. The entire microeconomic theory tries to explain exchange ratios of goods traded on the market - two apples can be exchanged for one orange, 10 present apples can be exchanged for 11 future apples, etc. The fundamental question for the economic science is why it is so. The veil of nominal variables is usually of secondary importance. Only the real variables matter.

The key question in this connection is: What is the interest rate Kirzner was talking about and what is the interest rate presented in the previous sections of this paper? It should be perfectly clear that Kirzner's value-difference argument is about the nominal interest rate. In his attack on the rice example, the nominal interest rate is zero - by investing $\$ 2,200$ now, one can receive $\$ 2,200$ next year. However, what is the real interest rate in this rice economy? If rice is the only good in the economy and the initial price level is 1 , then the next year price level
will fall to $0.91=(20 / 22)$. As a result, $\$ 2,200$ earned next year will have $10 \%(=1 / 0.91-1)$ higher purchasing power. Next year, one can buy $10 \%$ more rice than this year. The real interest rate in this economy is $10 \%$, which is exactly the figure predicted in the previous sections. ${ }^{158}$ Even though the nominal interest rate is $0 \%$, the $10 \%$ real interest rate is generated by a $10 \%$ fall in prices. Although there is no time preference (in sense two), the real interest rate is positive $10 \%$.
In our example, it is quite surprising that the time preference (in the second sense) is zero as well as the nominal interest rate. Is it just a coincidence? PTPT authors argued that if people discount future satisfaction, the interest on money should reflect this discount, and the nominal interest rate (interest rate earned on money) should be positive. Herbener wrote:

The exchange of present money for future money isolates pure time preference and permits the emergence of the rate of interest as the intertemporal exchange ratio of present money for future money. The exchange ratio between a present good and a future good is not the rate of interest, but is based on time value, and could either have a premium of the present or a premium of the future. (Herbener 2011:56)

Herbener explicitly argued that it is the exchange of present money as against future money that is at the centre of the theory of interest. As can be seen, he was not concerned in the purchasing power of money in any period because, according to him, the exchange ratio between present goods and future goods is not the thing the theory of interest should explain; it is not the rate of interest.

It seems that the entire controversy between the Austrian PTPT and the neoclassical theory is "only" about the definition of interest. The former school tries to explain the nominal interest rate - the value difference, the premium paid on present money as against future money. For the Austrian authors, the problem is why it is possible to receive $\$ 2,200$ in the future after investing $\$ 2,000$ in the present. The neoclassical school, on the other hand, attempts to explain the real interest rate - the exchange ratio between present goods and future goods. In this theory, the key question is why it is possible to receive 110 units of rice in the future just by investing 100 units today.
What approach is the more fundamental? We will argue that in the economic science, real phenomena are of primary importance. Nominal phenomena are just derived from real phenomena after the introduction of money into the given model. In the theory of interest, it is not important how much money one can receive by investing present money in an investment opportunity, in a (roundabout) process of production. The crucial question is how many future real consumption goods one can receive by forgoing present real consumption goods since only real consumption goods are ultimately capable of satisfying human wants (either in the present or in the future), not money. ${ }^{159}$

First of all, a careful reading of Böhm-Bawerk gives a first-round impression that he tried to explain why a lower amount of present goods is capable of purchasing a higher amount of future goods. And this is obviously a real concept (Murphy 2003:79). At the end of this section, we will investigate the critical words of Böhm-Bawerk in more detail. Secondly,

[^92]which value difference did the PTPT theorists have in mind? Was it the difference presented in our example, i.e. the total value of expenditures on present inputs ( $\$ 22 \times 100$ units of rice), compared with the total revenue from the future output ( $\$ 20 \times 110$ units of rice)? It is difficult to find evidence in the PTPT literature that would be against this interpretation.
However, Böhm-Bawerk himself talked about the value difference between present goods and future goods. If we consider the price of a good as representing value of that good, then the price of the present good (rice) is $\$ 22$ and the price of the future good (rice) is $\$ 20$. Even though there is no nominal interest rate, present goods are valued more than future goods since for 1 unit of present rice one gets 1.1 units of future rice. And this is precisely the phenomenon Böhm-Bawerk tried to explain. The sole knowledge of the nominal interest rate gives us no information about the relative value of present goods and future goods. It does not tell us how many units of future goods are required to be exchanged for one present good.

Let us now focus on the statement of Herbener from the previous quotation - "the exchange of present money for future money isolates pure time preference". With the help of the mathematical model developed in the previous sections, this peculiar statement might be analysed from a different perspective. However, if nominal variables should have any meaning, we must introduce money into our single-good economy.
So far, prices were selected more or less ad hoc. We just assumed along with Kirzner that competition should eradicate the initial value difference between output and expended inputs. Then we simply increased the present price from $\$ 20$ to $\$ 22$. However, we could have also decreased the future price from $\$ 20$ to $\$ 18.18$. Or any other change in prices could have been chosen just to obtain the relative price $\left(\mathrm{P}_{1} / \mathrm{P}_{0}\right)$ of $\$ 20 / \$ 22=\$ 18.18 / \$ 20=0.91$. Moreover, this relative price itself was ad hoc too, as will be proved in the following part.

In the first place, it should be stressed that the price structure of an economy is usually explained in the demand/supply framework. To determine the intertemporal price of rice (e.g. 20/22) in the example above, we have to explore the intertemporal structure of the demand and supply of rice. To solve this difficult problem, we will proceed as follows. First, we assume that the real interest rate is the primary phenomenon. The relative quantities produced in both periods then determine the intertemporal relative price of rice. And finally, the knowledge of the real interest rate and the relative price (i.e. the rate of price inflation, which was $-10 \%$ in our hypothetical example) may determine the nominal interest rate.
The real interest rate in our hypothetical rice economy is $10 \%$ even though some Austrian authors (see, for example, the words of Herbener or Kirzner presented above) would say that this does not represent the interest phenomenon at all. The optimum output of present and future consumption goods depends on the relative demand for these goods, which is in turn determined by the Euler equation. Consider first the set of parameters from panel (a) of Figure No. 26, $\rho=0 \%$ and $\theta=1$. In this case, there is no time preference in sense two, and the intertemporal elasticity of substitution in consumption is 1. PTPT authors would argue that productivity does not determine interest and the absence of time preference should result in zero interest in this economy. However, we already know that the only optimal marginal rate of substitution between present and future consumption is 1.1 - one present good must be exchanged for 1.1 units of future goods.
According to the Euler equation, the optimum ratio of future and present consumption for this particular set of parameters is $\mathrm{C}_{1} / \mathrm{C}_{0}=(1+0.1) /(1+0)=1.1$. As has been already said, the optimum future consumption should be higher by $10 \%$ compared with the present consumption. Due to the linear investment opportunity line (and the resulting horizontal intertemporal supply curve), the optimum ratio of output in the future and in the present is 1.1
as well provided that all consumers in this economy have the same time preference in sense two (in this case zero). Thus, future output of consumption goods will be higher by $10 \%$.

The most difficult task is to introduce money into this economy. There is only one good and virtually no intertemporal market. However, let us assume that sailors must use money to buy consumption goods both in the present and in the future. Consider some definite amount of money that is proportionately distributed among sailors at time 0 . It is then completely used to buy the total output of consumption goods at time 0 . In the second period, the same amount of money is given to the sailors via a lump sum transfer to purchase the next period output of consumption goods. Hence, it is assumed that the nominal amount of money is constant over time, and the monetary part of the economy may be described by the following equations:
$\mathrm{M}=\mathrm{P}_{0} \mathrm{C}_{0}$
$\mathrm{M}=\mathrm{P}_{1} \mathrm{C}_{1}$

Equations (27) and (28) represent a very simple equation of exchange with unitary velocity. However, they enable us to determine the equilibrium intertemporal price ratio of rice in this economy. The exogenous real interest rate of $10 \%$ (determined by the fixed marginal productivity of capital), exogenous subjective discount rate of $0 \%$, and exogenous $\theta=1$ will determine the optimum relative output of consumption goods $\mathrm{C}_{1} / \mathrm{C}_{0}=1.1$. Applying (27) and (28), it is easy to calculate that the equilibrium relative intertemporal price is 0.91 , which is the same number as in the "Kirzner's" example.
Let us now list the key characteristics of this economy - the real interest rate is $10 \%$, the rate of inflation is $-10 \%$ (or more exactly $-9.1 \%$ ), and the resulting nominal rate of interest is $\mathrm{i}=(1+\mathrm{r})(1+\pi)-1=1.1 \times 0.91-1=0 \%$. In our model, the nominal interest rate is the same as suggested by the PTPT authors. However, they would claim that since there is no value difference between expenditures on inputs and the eventual revenue from the final output, there is no phenomenon that deserves explanation, at least within the theory of interest.

Nonetheless, if the centre of the theory of interest is the real interest rate, then the analysis must continue. In this economy, 1 present unit of rice is exchanged for 1.1 units of future rice. The real interest rate is $10 \%$, and it is generated by the fact that the nominal interest rate is 0 and the rate of inflation is roughly $-10 \%$ (so there is almost a $10 \%$ price deflation).

Let us now experiment with the model in panel (c) of Figure No. 26 in which $\rho=4 \%$ and $\theta=1$. According to the Euler equation, the equilibrium relative output is 1.06 . As a result, the rate of price inflation in this economy is $-6 \%$ (i.e. fall in prices by $6 \%$ ). Because the real interest rate is $10 \%$, the resulting nominal rate of interest is $+4 \%$. Compared with the previous example ( $\rho=0 \%$ ), there exists a value difference between present inputs and future output. By investing $\$ 2,000$ today, one will earn $\$ 2,080$ in the future. However, in real terms, no actual change has emerged. Suppose that $\mathrm{P}_{0}$ is $\$ 20$. $\mathrm{P}_{1}$ must be $\$ 18.8$ (i.e. $6 \%$ lower). Investment of 100 units of present rice $(\$ 2,000 / \$ 20)$ will earn 110 units of rice in the future $(\$ 2,080 / \$ 18.8)$, which gives a $10 \%$ real return - the same as before. Hence, regardless of the size of the subjective discount rate, the real rate of interest must be $10 \%$.

The nominal interest rate of $4 \%$ in panel (c) is, interestingly enough, the same number as would be predicted by the PTPT. "The exchange of present money for future money isolates pure time preference" because the value difference between output and input is exactly the same as the subjective discount rate. Thus, at first glance, the verbal exposition of the PTPT authors and their economic intuition seem to be accurate, as far as the nominal interest rate is
concerned - the value difference is determined by the pure time preference (i.e. by the subjective discount rate in our model). ${ }^{160}$

However, as we will see below, this is just an artificial outcome of our model. In panel (b) and (d), in which the growth rate of output is depressed due to the lower elasticity of substitution, this relationship disappears. In panel (b) with $\rho=0$ (no time preference) and $\theta=2$, the PTPT would suggest that there should be no interest. However, the optimum growth rate of output is about $5 \%$, the fall in prices is about $5 \%$ as well, and the resulting nominal interest rate is the sum of the real interest rate of $10 \%$ and the rate of inflation of $-5 \%$, which results in $\mathrm{i}=+5 \%$, not $0 \%$. At this place, the prediction of our model differs from the PTPT. By investing $\$ 2,000$ in rice today, one can get $\$ 2,100$ in the future. There is a positive value difference even for zero time preference.

And finally, in panel (d) with positive time preference (in the second sense) of $4 \%$ and $\theta=2$, the optimum growth in output is only $3 \%$. For constant money, prices must fall by $3 \%$. The resulting nominal rate of interest in this economy is $7 \%$, which is much higher than the $4 \%$ that would be suggested by the PTPT. Thus, the investment in present rice for $\$ 2,000$ earns $\$ 2,000 \times 1.07=\$ 2,140$ in the future, not $2,000 \times 1.04=\$ 2,080$, as the PTPT would predict.

As we can see, predictions of the PTPT are in line with our approach only for unitary elasticity of substitution. Thus, the pure rate of time preference (i.e. the subjective discount rate) is equal to the nominal interest rate only for $\theta=1$. Other values of $\theta$ give results that are at odds with the Misesian theory. It might be interesting to compare this observation with our previous analysis. In section 2.4, we concluded that Mises's theory was accurate only for an infinite elasticity of substitution $(\theta=0)$. However, in that case we discussed the exchange ratio between present goods and future goods, not the value difference between output and expended inputs. Hence, our previous discussion considered the real rate of interest. In the present example, we explore the nominal interest rate. Thus, our analysis suggests that the pure time preference theory is correct with regard to the real interest rate only for $\theta=0$. On the other hand, if the core of the interest problem is the value difference (i.e. the nominal interest), then the PTPT is accurate for $\theta=1$.
Let us now derive the exact formula for the nominal interest rate in our simple model. From equations (27) and (28), it can be seen that:

$$
\begin{equation*}
\mathrm{P}_{1} / \mathrm{P}_{0} \equiv(1+\pi)=\mathrm{C}_{0} / \mathrm{C}_{1} \tag{29}
\end{equation*}
$$

Using the Euler equation (25), equation (29) yields:

$$
\begin{equation*}
(1+\pi)=\left(\frac{1+\rho}{1+r}\right)^{1 / \theta} \tag{30}
\end{equation*}
$$

If we define the nominal interest rate $i$ in accordance with the Fisher theory:

$$
\begin{equation*}
(1+i)=(1+r)(1+\pi) \tag{31}
\end{equation*}
$$

[^93]We can conclude that the nominal interest rate in our model is:

$$
\begin{align*}
& (1+i)=(1+r)\left(\frac{1+\rho}{1+r}\right)^{1 / \theta}  \tag{32}\\
& (1+i)=(1+r)^{\frac{\theta-1}{\theta}}(1+\rho)^{1 / \theta} \tag{33}
\end{align*}
$$

Equation (33) provides us with the key insights about the determinants of the nominal interest rate in this economy. The nominal interest rate is equal to the subjective discount rate $\rho$ only if $\theta$ is equal to 1 (i.e. for logarithmic utility function). Only for a unitary intertemporal elasticity of substitution in consumption, the predictions of this model coincide with the Austrian approach. ${ }^{161}$ As a result, the PTPT can be considered as a special case of a more general theory. For a lower elasticity of substitution (e.g. $\theta=2$ ), the equilibrium nominal interest rate is higher than suggested by the PTPT, whereas for a higher elasticity of substitution $(\theta<1)$, the nominal interest rate is lower than the subjective discount rate. ${ }^{162}$

Furthermore, it is even possible to set parameters to achieve a negative nominal interest rate. Consider, for example, $\rho=4 \%$ and $\theta=0.5$. This combination would lead to $\mathrm{i}=-1.7 \%$. As a result, this would mean that the investment of $\$ 2,000$ today would result only in the total revenue of $\$ 1,966$ next year. However, nobody would be willing to invest money in this case because its pure hoarding would earn a higher return ( 0 instead of $-1.7 \%$ ). This money hoarding should result in a fall in prices of the factors of production (the price of present rice ready for investment in our example), which would push the nominal interest, i.e. the value difference between output and expended inputs, upwards. However, the real interest rate, the growth in output and the resulting price deflation are ultimately determined by the structural parameters of our model, so the future price of rice must consequently fall by the same percentage as before. Hence, a fall in the present price would not raise the nominal rate of interest above zero because the future price would fall accordingly due to the price deflation that is dictated in the model by the values of exogenous parameters. As can be seen, the existence of (constant) money and nominal variables may upset the attainment of the intertemporal equilibrium that would emerge in a pure barter economy. ${ }^{163}$

Thus, it is usually believed that the money rate of interest cannot fall below zero (neglecting costs to store money and other costs) due to the zero lower bound imposed on the nominal interest rate. If we inspect such an economy in more detail, we will see that for the given set of parameters, the growth rate in output of consumption goods of $11.9 \%$ exceeds the real interest rate ( $10 \%$ ), which is a sign of dynamic inefficiency in usual growth models. It is wellknown that such a state of an economy may lead to curious results (which the negative nominal interest rate certainly is). Nevertheless, more on this will be said in section 3.1.3.

[^94]The PTPT authors strongly oppose the possibility of a negative (or zero) rate of interest. If our interpretation of this theory is correct, the difference between the value of output and the value of invested capital represents the nominal interest. And this interest cannot (in normal times and under normal conditions) fall below zero. Section 3.1.3 will discuss this topic in more detail.

In the section analyzing sailors with deteriorating goods (figs), we found out that the only equilibrium rate of interest, and now we may add - the real rate of interest - was negative. This outcome was at odds with the Austrian PTPT that claims that it is impossible for the interest rate to decrease below 0 . However, there was no escape from the conclusion about the negative rate of interest. In the present extension of the model that includes money, the negative real interest rate can be easily achieved by zero nominal interest rate and $10 \%$ price inflation. ${ }^{164}$ This combination is reached with $\rho=0$ and $\theta=1$ (and MPK $=r=-10 \%$ ). Notice that it is exactly the same set of parameters as in panel (a) of Figure No. 26. The only difference is that in the present example, the real interest rate is $-10 \%$, not $+10 \% .^{165}$

Now, if we reinterpret the PTPT as a theory of the nominal interest rate, the negative real interest rate in the "fig economy" should not be so troubling for the PTPT authors as before. Even more, for some Austrian authors, the fact that 10 present figs are exchanged for only 9 future figs does not represent the phenomenon of interest at all.
In the previous section, we did not say anything about the money rate of interest in the fig economy. With the help of equation (33), it is quite easy to reach a positive nominal interest rate simultaneously with a negative real rate of interest. Consider just any case for which the inflation rate exceeds the negative of the real rate of interest (i.e. $\pi>-$ r). And because the inflation rate in our model is the negative of the growth rate in output of consumption goods (i.e. $\left.\pi=-\left(\mathrm{Q}_{1}-\mathrm{Q}_{0}\right) / \mathrm{Q}_{0}\right)$, positive nominal rate of interest is obtained for any real interest rate (e.g. $-10 \%$ ) which exceeds the growth rate of consumption (e.g. $-13 \%$ ). ${ }^{166}$ To achieve this combination, set $\theta=1$ and $\rho=4 \%$ (and MPK $=r=-10 \%$ ). The nominal interest rate in the fig economy will be $+4 \%$ even though the real interest is $-10 \%$. One present dollar in this economy will be exchanged for 1.04 future dollars. The exchange ratio between present and future money will then perfectly reflect the pure time preference even though one present good will be exchanged for only 0.9 units of future goods. Thus, the PTPT would hold even in the fig economy - present money must be exchanged for more units of future money. However, as we concluded before, it is the real interest rate that should be of primary importance in the economic science, i.e. the exchange ratio between present goods and future goods (in this example between present figs and future figs), not the money rate of interest.

Furthermore, the apparent equality between the subjective discount rate ( $\rho=4 \%$ ) and the nominal interest rate ( $\mathrm{i}=4 \%$ ) is a direct outcome of the logarithmic utility function $(\theta=1)$. For a lower elasticity of substitution, $\theta=2$ (and for $\rho=4 \%$ and $r=-10 \%$ ), the equilibrium nominal interest rate is $-3 \%$ (i.e. less than zero). For a higher elasticity of substitution, $\theta=0.5$, the resulting nominal interest rate is $+20 \%$. As was indicated before, if $r<\rho$, the nominal interest rate is positively related to higher elasticity of substitution. Furthermore, assuming zero time preference in sense two (i.e. $\rho=0 \%$ ) and negative real interest rate $\mathrm{r}=-10 \%$,

[^95]positive nominal interest rate is achieved only for a high elasticity of substitution $(\theta<1)$. In such a case, large negative difference between the real interest rate and the subjective discount rate will motivate people to choose a sharply decreasing shape of the optimum consumption stream (see the Euler equation 25 ). Hence, present consumption will be very high compared to future consumption. Since the intertemporal supply of goods can easily meet this demand structure due to the assumption of constant productivity, the time shape of the output stream will follow consumption. Thus, consumption and output of future goods will be much lower than that of present goods, which will result in a high inflation rate ( $\pi=23.4 \%$, if $\mathrm{r}=-10 \%, \rho$ $=0 \%$ and $\theta=0.5$ ). As a result, the nominal interest rate will rise to $11 \%$. As can be seen, negative real interest rate is consistent with positive nominal interest rate and zero time preference (in sense two) if the elasticity of substitution is high enough $(\theta<1)$.

To conclude, the foregoing analysis suggested that the PTPT approach might be partly saved from the critique of the neoclassical school if it was reinterpreted as a theory of the nominal interest. However, in such a way, it lost its potential to be accepted as a general equilibrium theory because economists usually accept only economic theories elucidating the determination of real phenomena.

### 3.1.2 BÖHM-BAWERK ON REAL INTEREST OR NOMINAL INTEREST?

At the end of this section, we will investigate the work of the founder of the Austrian theory of interest - Eugen von Böhm-Bawerk - as regards the fact whether he defined the problem of interest in terms of real interest or nominal interest. If the phenomenon of interest is defined as a problem of the exchange of present goods for future goods, it is definitely a real approach. If it is defined as a value difference between the total value of invested inputs and the value of the resulting output, it is rather a nominal approach.
It will be shown that Böhm-Bawerk did not provide a precise and consistent definition of the problem of interest. This might have led to confusions of various authors presented in the foregoing sections. In the first place, we will present Böhm-Bawerk's fundamental statements about the problem of interest that indicated a real approach, i.e. the exchange of present goods for future goods. For easier orientation in the further discussion, each quotation will be marked with a number.
(1) PRESENT goods are, as a rule, worth more than future goods of like kind and number. (Böhm-Bawerk 1891:237)
(2) We arrive thus at a proposition which is a fundamental one in our inquiry: As a rule present goods have a higher subjective value than future goods of like kind and number. And since the resultant of subjective valuations determines objective exchange value, present goods, as a rule, have a higher exchange value and price than future goods of like kind and number. (ibid.:248)
(3) A loan is nothing else than a real and true exchange of present goods for future goods; indeed, it is the simplest conceivable phenomenal form, and, to some extent, the ideal and type of such an exchange. (ibid.:285)
(4) The means of production, and their result,-the finished product towards which the buyer is looking in purchasing them,-are future commodities, and the price is measured and paid in (more valuable) present goods. That, in this case, the greater number of less valuable future goods is purchased by a smaller number of more valuable present goods, is not "cheap buying, " (ibid.:301)
(5) Knowing now that the undertaker buys the future commodity, "Means of Production," for a smaller number of pieces of present goods than the number of pieces which will compose their future product, we ask, How does he come by his profit? The answer is very simple. From his "cheap" purchase, indeed, he does not get any result; for, estimated by its present value, the commodity is dear. (ibid.:301)

However, the following parts expose rather the nominal point of view - the value difference between output and invested inputs (or capital).
(6) The theorist, then, who professes to explain interest must explain the emergence of Surplus Value. The problem, more exactly stated, will therefore run thus: Why is the gross return to capital invariably of more value than the portions of capital consumed in its attainment? Or, in other words, Why is there a constant difference in value between the capital expended and its return? (Böhm-Bawerk 1890:116)
(7) I grant at once that capital actually possesses the physical productivity ascribed to itthat is to say, by its assistance more goods can actually be produced than without it. I will also grant-although here the connection is not quite so binding-that the greater amount of goods produced by the help of capital has more value than the smaller amount of goods produced without its help. But there is not one single feature in the whole circumstances to indicate that this greater amount of goods must be worth more than the capital consumed in its production,-and it is this phenomenon of surplus value we have to explain. (ibid.:138)
(8) Interest is a surplus, a remainder left when product of capital is the minuend and value of consumed capital is the subtrahend. The productive power of capital may find its result in increasing the minuend. But so far as that goes it cannot increase the minuend without at the same time increasing the subtrahend in the same proportion. For the productive power is undeniably the ground and measure of the value of the capital in which it resides. If with a particular form of capital one can produce nothing, that form of capital is worth nothing. If one can produce little with it, it is worth little; if one can produce much with it, it is worth much, and so on;-always increasing in value as the value that can be produced by its help increases; i.e. as the value of its product increases. And so, however great the productive power of capital may be, and however greatly it may increase the minuend, yet so far as it does so, the subtrahend is increased in the same proportion, and there is no remainder, no surplus of value. (ibid.:179)

It should be stressed that the passages suggesting nominal approach can be found in Capital and Interest, i.e. in the book that analysed theories of interest prior to Böhm-Bawerk. As can be seen from quotation (7), Böhm-Bawerk criticised especially naïve productivity theories since they confused value productivity with physical productivity. In other words, capital is able to produce more goods and services; however, this does not immediately imply that the value of the eventual output is higher than the total value of inputs invested. Thus, it is quite understandable that Böhm-Bawerk focused in this particular case on the value difference the nominal interest - rather than on the exchange ratio between present goods and future goods. The latter, i.e. the real approach, can be found in the second book - Positive Theory (quotations 1 to 5), which was designed to explain this value difference and to present his own theory. It can be said that the premium in the intertemporal exchange on behalf of the present goods was fundamental for him for the explanation of the "surplus value" - the problem posed in the first book. In modern terms, Böhm-Bawerk presented a theory of real interest in Positive Theory to elucidate the problem of nominal interest raised in Capital and Interest.

Frank Fetter (1928) identified Böhm-Bawerk's interest theory primarily as a problem of the exchange of present goods for future goods. From modern authors, Murphy (2003) held the same position as he placed Böhm-Bawerk's theory among real theories of interest. However, several passages in the Positive Theory are quite inconclusive in a sense that it is not clear whether they refer to the nominal approach or to the real approach.
(9) Interest, then, comes, in the most direct way, from the difference in value between present and future goods. (Böhm-Bawerk 1891:286)
(10) They buy goods of remoter rank, such as raw materials, tools, machines, the use of land, and, above all, labour, and, by the various processes of production, transform them into goods of first rank, finished products ready for consumption. In doing so they obtainindependently of compensation for their own personal co-operation in the work of production as leaders of industry, head-workers, etc.-a gain approximately proportioned to the amount of capital invested in their business. This gain is called by some "Natural Interest on Capital" or "Profit," and, by others, "Surplus Value" (ibid.:299)
(11) It is during the progress of production that the future commodity ripens gradually into the present commodity, and grows at the same time to the full value of the present commodity. (ibid. 301)
(12) In short, as time passes it cancels the causes by reason of which the then future commodity suffered a shrinkage of value, and brings it up to the full value of the present good. The increment of value is the profit of capital. (ibid.:302)

The most disturbing is the reference to the increase in value in (11) and (12). It is not clear whether Böhm-Bawerk had in mind the exchange value between present goods and future goods as in (2) or whether he referred to an increase in nominal value. However, even in (2) he continued with the word "price" ("present goods have higher price than future goods") immediately after the word "exchange value" (between present goods and future goods). It is not clear whether the word "price" refers to only one good, be it present or future, or to the total value of present goods and future goods as the latter gradually ripens into the former.

At first glance, the previous analysis might seem a mere game with words since the exchange value, price, and the value difference must refer to the same thing. However, let us present three examples that could shed more light on the problems presented so far. They clearly demonstrate that the issues raised do not represent only terminological inconsistencies, but rather fundamental questions in the theory of interest.
Suppose that production process No. 1 uses factors of production bought at time 0 for $\$ 1,000$, and after one year it will produce final consumption goods for $\$ 1,100$ (i.e. in period 1). Suppose further that the price of a representative present (i.e. at time 0 ) consumption good is $\$ 10$ and that it remains the same till time 1 . This production process exhibits typical value productivity. The value difference between inputs and final output is $\$ 100$. At the same time, it also exhibits physical productivity. The owner of 100 units of present consumption goods may hire factors of production for $\$ 1,000$ at time 0 . Their employment in the given production process will result in the output of 110 units of consumption goods in the future. If all processes in this economy were like this one, the nominal rate of interest must be $10 \%$ ( $\$ 100 / \$ 1000$ ). The factors of production represent future consumption goods, and their value gradually rises as their product matures to the value of $\$ 1,100$. The same value of $10 \%$ should also reach the real rate of interest because the ownership of 100 present consumption goods might be exchanged for 110 future consumption goods (if present goods are exchanged for present factors of production, and these are then employed in the production process No. 1).

However, one point is not clear in this example - what is the price of the future good? According to quotation (2), it should be lower than the price of the present good. To keep consistent with Böhm-Bawerk's exposition, the price of the future good cannot be considered as $\$ 10$, but rather as the discounted price of $\$ 10$, i.e. $\$ 10 / 1.1=\$ 9.1$. At this time, we set aside how the given nominal interest rate, the real interest rate, and prices in periods 0 and 1 were determined. Nevertheless, this example should not present any problem either for BöhmBawerk, or for the PTPT, or even for productivity theorists. ${ }^{167}$ According to the PTPT, the value difference in our example is undoubtedly caused by the pure time preference, whereas for Böhm-Bawerk by all three causes of interest. And finally, the productivity theorists might claim that the value difference emerged only due to the productive powers of capital.
Now, let us consider production process No. 2 that is characterized by the following set of information: Factors of production are bought for $\$ 1,000$, and the eventual output is sold for $\$ 1,100$ one year later. Hence, this production process exhibits a value productivity of $\$ 100$, and the nominal rate of interest is $10 \%$. However, suppose that the price of the representative present consumption good is $\$ 10$, whereas the price of the same good in the future is $\$ 11$, i.e. $10 \%$ more. This production process exhibits no physical productivity, as the ownership of 100 present goods ( $\$ 1,000 / \$ 10$ ) may (after the exchange for the appropriate factors of production and their employment in the proper production process) "produce" only 100 units of future consumption goods $(\$ 1,100 / \$ 11)$. The real interest rate is $0 \%$ because one present good is exchanged for one future good. The price of the future good in the Böhm-Bawerkian sense is $\$ 11 / 1.1=\$ 10$. Hence, it is not valued less than the present good. This particular example once again demonstrates that the value productivity might exist even if the present goods are not preferred to future goods. In this economy, the investment of money yields a $10 \%$ interest; however, present goods are not exchanged for a higher amount of future goods. This example is particularly disturbing for the Böhm-Bawerkian exposition since the "surplus value" can not be explained by the premium possessed by the present goods over future goods. The set of parameters in our model that may produce such a curious result will be presented below.
And finally, consider production process No. 3. Suppose that the employment of factors of production in period 0 for $\$ 1,000$ will create output in period 1 for $\$ 1,000$. Thus, there is no value productivity, no nominal interest. There is no "surplus value" to be explained. However, if the price of a representative present good (i.e. in period 0 ) is $\$ 10$ and the price of the same good in period 1 is only $\$ 9.1$, production process No. 3 clearly exhibits a physical productivity of $10 \%$. The endowment of 100 units of present goods ( $\$ 1,000 / \$ 10$ ) may produce (after the exchange for present factors of production and their employment in the given production process) 110 units of future consumption goods ( $\$ 1,000 / \$ 9.1$ ). In this economy, the real rate of interest is $10 \%$. Present goods are exchanged for future goods with premium in the sense that one present good might be exchanged for 1.1 units of future goods. ${ }^{168}$
It can be clearly seen that these examples resemble the ones from the previous section, namely the rice economy before and after the adjustment in prices, and the hard-tack economy. Furthermore, an explicit inclusion of the factors of production was not necessary since the logical reasoning would not be harmed by their complete omission. ${ }^{169}$ Thus, we can

[^96]directly compare $\mathrm{P}_{0} \mathrm{Q}_{0}$ and $\mathrm{P}_{1} \mathrm{Q}_{1}$, where $\mathrm{P}_{0}$ and $\mathrm{P}_{1}$ are prices of the representative consumption good at time 0 and 1 , and $\mathrm{Q}_{0}$ and $\mathrm{Q}_{1}$ stand for their respective quantities. PQ then represents total expenditure in the given period.
Even though example No. 1 presented no problem for the PTPT, examples No. 2 and 3 clearly demonstrated its inaccuracy. In problem No. $1, \mathrm{P}_{1} \mathrm{Q}_{1}>\mathrm{P}_{0} \mathrm{Q}_{0}$, so the value difference between present factors of production and their eventual output emerged. However, in example No. 2, there existed a positive value productivity $\left(\mathrm{P}_{1} \mathrm{Q}_{1}>\mathrm{P}_{0} \mathrm{Q}_{0}\right)$, but present goods could not be exchanged for a larger amount of future goods. Hence, they had no value premium in the Böhm-Bawerkian sense. In example No. 3, we encountered no value productivity, $\mathrm{P}_{1} \mathrm{Q}_{1}=$ $\mathrm{P}_{0} \mathrm{Q}_{0}$, nevertheless, one present good could have been exchanged for a higher amount of future goods.

One might argue that all three examples were purely hypothetical and arbitrary or could not represent a stable situation. However, the simple model from the previous section is consistent with all three possibilities. Outcomes in example No. 1 are generated for a constant marginal productivity of $10 \%$, subjective discount rate of $10 \%$, unitary elasticity of substitution, and constant money supply. In such an economy, the real rate of interest must be $10 \%$ (due to MPK $=10 \%$, not due to $\rho=10 \%$ ). According to the Euler equation (25), output in this economy is constant over time because $r=\rho$. If money is constant, prices must be stable over time as well. Hence, the nominal rate of interest, representing the value difference between output and input, is $10 \%$.

The second example might be represented by a hard-tack economy with MPK $=0 \%=r$, the subjective discount rate of $10 \%$, unitary elasticity of substitution, and constant money. According to (25), the growth rate in consumable output will be $-10 \%$, so for the constant money supply, prices must go up by $10 \% .^{170}$ This will generate a positive nominal rate of interest (value difference) for the given zero real rate of interest.

And finally, example No. 3 could represent a rice economy with MPK $=10 \%=r$, zero time preference $(\rho=0 \%), \theta=1$, and constant money. Output in this economy must grow by $10 \%$, and prices will fall by about $10 \%$ as well. Real rate of interest will be $10 \%$, and the nominal rate should stabilize at $0 \%$.

As can be seen, the existence or non-existence of the surplus value (i.e. the positive value difference between factors of production and the eventual output) can be totally independent of the exchange ratio between present goods and future goods. Thus, we can argue exactly in the opposite way compared with the PTPT. It is not the value difference that is to be explained by the interest theory because it is a completely empty concept. The fundamental problem in the theory of interest is the exchange ratio between present goods and future goods. Even though in normal times it is most probably positive on behalf of present goods (rice economy), positive exchange ratio on behalf of future goods (fig economy) cannot be ruled out, as was demonstrated in the foregoing parts of our investigation.
Thus, for the modern theory of interest, Böhm-Bawerk did only part of the job. Firstly, the primary problem in theory of interest is not to explain the difference between the value of output and the value of expended inputs as it is a mere nominal phenomenon. Secondly,

[^97]present goods are NOT, as a rule, worth more than future goods of like kind and number. ${ }^{171}$ However, the three causes presented by Böhm-Bawerk are of the utmost importance in the theory of interest even though fundamental questions of this theory as such lie in a slightly different field.

Yet, several arguments of the pure time preference theory might be raised against our reasoning. First, the value difference emerged only in the examples with positive discounting of future utilities $(\rho>0)$. However, we explicitly demonstrated that this outcome might arise only for a specific curvature of the utility function $(\theta=1)$. In the next section, we will see that the positive value difference may emerge even for zero time preference (or recall panel (b) in Figure No. 26 and the discussion in the relevant section).
Secondly, one may argue that it is the nominal interest (i.e. the interest on money) that is of primary importance and that must be explained by the interest theory in the first place. The real rate of interest (exchange ratio between present goods and future goods) does not represent the phenomenon of interest at all, or it is just of secondary importance. In addition, the evolution of prices only blurs the true phenomenon - Mises himself envisioned the change in prices in his theory of interest under the name of price premium (Mises 1996:541ff).

Let us deal with the arguments above step by step. First, Mises's reasoning about the price component in the interest rate is indistinguishable from the usual Fisherian theory and the Fisher effect. Nominal interest rate must be adjusted according to the expected change in prices. However, this does not explain the emergence of the interest rate "corrected for inflation".

Secondly, the primacy of the nominal approach would mean that $\mathrm{P}_{1} \mathrm{Q}_{1}>\mathrm{P}_{0} \mathrm{Q}_{0}$ is the principal phenomenon to be explained. Nonetheless, this would keep the ratio of prices and quantities, and the real rate of interest completely undetermined. Any combination would be possible, even a positive premium on future goods if the inflation rate is rapid enough compared with the nominal rate of interest. In such a case, present goods would not be preferred to future goods, which would be at odds with the Misesian maxim. Thus, the PTPT does not offer any explanation of how $\mathrm{P}_{1}, \mathrm{P}_{0}, \mathrm{Q}_{0}$, and $\mathrm{Q}_{1}$, or at least their relative ratios might be determined. Without this knowledge, it is impossible to find the exchange value between present goods and future goods.
We tried to demonstrate that the chain of logical reasoning must always start from the real rate of interest, i.e. from the exchange ratio between present goods and future goods. In the Fisherian examples, it was determined by exogenous productivity ( $0 \%$ for hard-tacks, $10 \%$ for rice, and $-10 \%$ for figs). Time preference (in the second sense, i.e. $\rho$, and also the elasticity of substitution $1 / \theta$ ) only affected the optimal flow of consumption and therefore output. This enabled us to determine the evolution of prices for the given behaviour of the money supply. The nominal interest, i.e. the value difference between present expenditures and future revenues, emerged at the end of this reasoning.
At the end of this section, we will extend our analysis by two topics. First, we will relax the assumption of constant productivity. And second, we will determine the combination of parameters that will generate positive value productivity (i.e. positive nominal rate of interest).

[^98]In the three Fisherian examples, it was quite disturbing that the real rate of interest was determined from the outside - by constant productivity of capital. As we will demonstrate in the next section, if the marginal productivity of capital is diminishing, the real rate of interest is co-determined by the productivity of capital and the time preference. A typical picture of such an economy can be seen in Figure No. 27. The interest rate is determined both by the curvature of the investment opportunity line (given by the marginal productivity of capital) and the curvature of the indifference curve (determined by $\rho$ and $\theta$ ). What is the proper chain of reasoning in this case?


Figure No. 27, Time preference and the marginal productivity of capital determining the real rate of interest and the relative output

The marginal product of capital and the time preference simultaneously determine the real rate of interest (slope of the tangent line at the optimum point E ) and, as can be seen in the picture, consumable output in both periods as well. For the given money supply, this will decide the evolution of prices, i.e. the inflation rate. And finally, the nominal rate of interest is given as the sum of the real rate of interest and the inflation rate. Thus, the productivity of capital and the time preference have not only a decisive effect on the real rate of interest, but for the given evolution of the money supply, they also affect the eventual "surplus value" (the nominal interest) due to their impact on the relative size of output in both periods and therefore prices. Hence, our reasoning must always start from real phenomena (time preference and productivity) determining the real rate of interest and end up with considerations about nominal phenomena (surplus value, nominal interest, i.e. the value difference). The opposite direction is not possible as it leaves the entire system undetermined.
So far, we have glorified the real approach as a true and the only consistent way to understand the problem of interest. However, serious problems arise in this connection as well. A straightforward calculation of the real rate of interest in our examples was critically dependent on the assumption of a single-good economy. If the number of goods is extended to n , the precise calculation of the real rate of interest is impossible.

First of all, what would be the real rate of interest if two present apples and three present oranges were exchanged for three future apples and four future oranges? Precise calculation could be done only if weights of these particular goods in the consumption bundle did not change. In other words, goods must be consumed in fixed proportions in every period (Hayek 1941:220). If the composition of the consumption bundle changed, and this is most probably the rule rather than exception, the calculation of the real rate of interest would require, first, the knowledge of the nominal rate of interest, and secondly, the construction of a general price index, which is always, however, more or less arbitrary.
Hence, in the n-good economy, the only interest rate that can be calculated is that on money. The real rate of interest is impossible to find unless some price index is constructed, the creation of which is, however, arbitrary in one way or another. The only real rates of interest that have any sense are the "own rates" of interest of various goods. ${ }^{172}$ These should be consistent with the intertemporal MRS of each good. Furthermore, the intertemporal movements in prices of each good will then guarantee that the money rate of interest is the same for every single good in the economy.
In the n-good economy, huge problems also arise as regards the deep structural parameter $\rho$. The time preference with respect to various goods might be dependent on the overall composition of the consumption basket. ${ }^{173}$ Only the assumption of homogeneity leaves the subjective discount of future independent of the particular structure of the consumption basket in each period (Lancaster 1963). However, homogeneity seems to be too strong an assumption to hold in the real world. Hence, a change in prices of goods will surely affect the optimum composition of the consumption basket, which in turn modifies the time preference and consequently the (real) rate of interest. Thus, via this channel the real interest rate might be affected by changes in the intra-temporal relative prices. Moreover, if the varying average income modifies the structure of the optimum consumption bundle, the time preference ( $\rho$ ) might be affected in unexpected directions. As a result, new and formerly unthought-of connections will immediately occur; connections that could not arise in a single-good model.
As can be seen, the introduction of $n$ goods into our model will complicate the analysis to such an extent that, at this point, it is impossible to deepen the investigation beyond some superficial notes presented above. However, the main message of the previous sections should be preserved. In the interest theory, real phenomena are of primary importance even though it might be impossible to determine the precise value of the real rate of interest. Nominal variables (i.e. value differences between total expenditures) are derived from real phenomena after the introduction of money into the theoretical system. Real variables and relationships (e.g. relative prices) can never be determined from the sole knowledge of nominal variables.

### 3.1.3 REAL INTEREST, NOMINAL INTEREST AND DYNAMIC (IN)EFFICIENCY

As a final note, let us briefly mention one interesting aspect of the theory presented above. First, we have clearly demonstrated that the real rate of interest might be negative, i.e. a higher amount of present goods might be exchanged for a lower amount of future goods. However, as far as the nominal rate of interest was concerned, the discussion was rather inconclusive.

[^99]It is generally believed that the nominal rate of interest cannot fall below zero. In other words, the value of output may not fall short of the total value of expended inputs. However, if it is costly to preserve money, even a negative rate of nominal interest might be accepted by rational agents. Yet, let us keep the zero bound on nominal rate as a barrier for its further decline. In other words, let us analyse the combinations of parameters in our model that will always generate a positive value difference between output and expended inputs.
Recall equation (31) determining the nominal rate of interest. We know that $i=r+\pi$. If the money supply is constant, $\pi=-$ growth rate of output. Thus, $\mathrm{i}<0 \%$ if $\mathrm{r}<-\pi$, hence if $\mathrm{r}<$ growth rate of output. As a result, for the constant money supply, the negative value difference between output and inputs may emerge if the real rate of interest is lower than the growth rate of (consumable) output. Interestingly, this is exactly a situation of a dynamically inefficient economy. ${ }^{174}$ The fall in prices is then too fast compared with the real rate of interest, so the equilibrium nominal interest rate should decline below zero. In this particular case, only monetary expansion is able to increase (or to partly offset the decline in) the price level in the next period and consequently to raise the nominal interest rate above zero. As a result, in a dynamically inefficient economy, constant money supply might be a barrier to establish an intertemporal equilibrium.
At this point, recall equation (33) for the nominal interest rate in our model. It can be easily shown that the positive nominal rate of interest is guaranteed if $(1+\rho)>(1+r)^{1-\theta}$. Thus, if $\theta=1$, the subjective discount rate must be positive. Furthermore, the higher the elasticity of substitution ( $1 / \theta$ ), the higher the subjective discount of future is required to drive up the nominal interest above zero. The reason lies in the fact that higher $\rho$ decreases the growth rate of output in our model (and reduces price deflation), as can be clearly seen in Figure No. 26.

Figure No. 1_A5 in Appendix 5 presents combinations of $\rho$ and $\theta$ that will generate various values of the nominal rate of interest. We assume constant money supply and a $10 \%$ real rate of interest, which is determined from the outside of the model (e.g. by constant MPK of rice). As can be seen, negative nominal rate of interest is possible only for $\rho$ that is lower than $r$. Furthermore, the lower the $\rho$, the wider the range of $\theta$ for which this may happen. At the same time, positive nominal rate of interest (i.e. positive surplus value) is generated even for zero time preference if $\theta$ is large enough. Similarly, a negative value difference between output and inputs might emerge even for a positive time preference (but lower than $r$ ) if $\theta$ is low enough.
Figures No. 2-5_A5 clearly show that the dynamic inefficiency (higher growth rate of output than the real rate of interest) generates negative nominal interest rate. It can be seen that this may happen for low $\rho$ and $\theta$. If $r=\rho, \theta$ plays no role. Furthermore, if $\rho$ exceeds $r$, negative nominal interest on money cannot emerge.

Let us conclude this section with Figure No. 6_A5 that displays various combinations of $\theta$ and $\rho$ for which the nominal rate of interest is at least zero for different values of the real interest rate. Assuming that $\rho$ cannot be negative, the higher the exogenous productivity of capital (and hence $r$ ), for the given elasticity of substitution ( $1 / \theta$ ), the higher the impatience (higher $\rho$ ) must be to guarantee a positive value difference between output and inputs (i.e. the positive nominal interest).

[^100]
## 4. NATURAL RATE OF INTEREST UNDER VARIOUS THEORETICAL ENVIRONMENTS

In this part, we will relax the assumption of constant marginal productivity of capital. The exogenous flow of real income will be introduced as well. As we will see, much richer conclusions might be drawn by these two extensions. However, all will demonstrate that the PTPT is just a special case of a more general theory.
In this section, we will use again the two-period model. This decision has one big advantage and one major drawback. The good thing is that all conclusions from this section can be directly compared with those from previous sections. However, one important aspect of the Austrian theory of capital and interest will be lost.
The core of the Austrian capital theory is that roundabout methods of production are more productive. ${ }^{175}$ In other words, (wisely chosen) time extension of the production process should lead to a higher output. However, the increments in the output are not proportional because it is generally believed that this process exhibits decreasing marginal (physical) productivity. The presence of time preference together with decreasing marginal productivity will eventually bring the process of the time extension of production to a halt. Although an infinite time extension of production could possibly create an infinite output, this option is never chosen by rational agents. Never-ending postponement of present consumption is impossible due to time preference. ${ }^{176}$
However, the two-period model does not allow us to analyse this aspect of the theory of capital and interest. With regard to time dimension, only two options are available for the investment of the factors of production. They can either be invested in processes that provide consumption goods immediately or in a relatively short period of time (i.e. in period 0 ), or they can be invested in longer processes that will take one period (Hayek 1941). The longer processes will then create consumption goods in period 1. Consistently with the assumption stated above, the given number of factors of production will produce more if they are invested for one period compared with their immediate use. At the same time, the longer the time for which the factors of production are tied up, the higher the eventual output, even though the marginal increments gradually diminish. However, in the two-period model, the number of periods cannot be extended. In other words, capital cannot be enlarged in height (Wicksell 1977). As a result, in the two-period model decreasing marginal productivity of capital cannot be reflected in the time dimension.
Nonetheless, we would like to introduce some kind of diminishing marginal productivity of capital even in the two-period model. Thus, suppose that every additional unit of input invested in a longer process (i.e. in the process that takes one period) increases future output, but at a decreasing rate. As more factors of production are directed to the longer process, the output of consumption goods in the next period increases. Yet, these increments are still lower and lower. As a result, in the two-period model decreasing marginal productivity can be reflected only in the breadth dimension of capital.
Obviously, with more units of input invested in a longer process, the average investment time of the entire stock of inputs (the average period of production in the Böhm-Bawerkian

[^101]system) increases (Hayek 1941). Nevertheless, the maximum time for which one single unit of input can be invested is just one period. Longer time extension is not possible.

Greater fruitfulness of longer methods and their diminishing marginal productivity can be illustrated in a simple Fisherian diagram (Figure No. 28). One possible interpretation of this schema is as follows: Point A can be considered as the initial endowment of present goods that might be transformed into future goods if properly invested. The first forgone present good may produce five units of future goods. However, this physical productivity gradually falls since the second present good may produce just three future goods, etc. We can imagine an interval (from point D to the left) in which the production process is so technically inferior and unfortunate that the marginal output of future goods falls short of the number of marginal present goods invested. In this case, the slope of the investment opportunity line is lower than one (in absolute value).


Figure No. 28, Greater fruitfulness of longer methods and their diminishing marginal productivity

However, our exposition would be more in line with the Austrian theory of capital if we interpreted point A as follows: This point represents the maximum amount of present consumption goods that can be produced if all factors of production are used only in direct methods of production. ${ }^{177}$ The investment of factors of production in a longer process decreases present output; however, the decline in present output is more than compensated by an increase in output of future consumption goods (point B). The fruitfulness of this reallocation of resources is, nevertheless, limited since the marginal increments of future output gradually fall. Illustration of the varying time extension of production would require

[^102](at least) a three-dimensional graph. Although this apparatus can be found in Hayek (1941), it will not be used in the analysis that follows.

Even this simplified version can provide us with key insights. As has been demonstrated before, the subjective exchange ratio between present goods and future goods depend on the MRS, i.e. on the time preference in sense one. However, the MRS is an endogenous variable as it depends on the time shape of the income stream. The indifference curve alone cannot determine the equilibrium interest rate. To close the model, one more curve (or equation) is required. One more relationship is necessary to determine what particular point on the indifference curve will be chosen. In the examples with shipwrecked sailors, the system was closed owing to the assumption of constant productivity. In this section, though, we assume diminishing marginal productivity of the invested capital.
By varying the amount of inputs invested in longer processes, the intertemporal flow of income may be changed. Since this flow affects the optimum MRS, the (real) natural rate of interest is co-determined by the marginal rate of substitution (subjective element) and the marginal productivity of capital (objective element).
Hayek (1941) in his magnum opus on capital thoroughly explained that the relative impact of time preference and productivity in determining the natural rate of interest depends on the relative curvature of the indifference curve compared with the investment opportunity curve. In the subsequent passages, we will extend his approach.


Figure No. 29 Equilibria for various time preference and marginal productivity schedules

Even though the model presented is quite simple, it enables us to analyze various aspects in the theory of interest. Panel (a) in Figure No. 29 portrays a situation in which the subjective discount rate (time preference in sense two) is relatively high and the productivity of inputs invested in a longer process rapidly falls. As a result, the slope of the indifference curve at the $45^{\circ}$ line is higher than the slope of the opportunity line. The optimum point lies to the right of the diagonal line. Compare this optimum with panel (b), representing a relatively patient individual (the subjective discount rate and hence the slope of the indifference curve at the diagonal is quite low) and moderately diminishing marginal productivity of capital. In panel (a), the optimal time shape of consumption is decreasing, whereas in panel (b) it is increasing. However, as can be seen in the picture, it is quite difficult to say in what panel the real rate of
interest is lower. In the first figure, the interest rate is pushed down by a rapidly diminishing marginal productivity, while a relatively high rate of impatience drives it up. In panel (b), exactly the opposite statement holds.

At this point, we can discuss in more detail why we departed from Murphy's approach to the time preference in sense one and why we separated the third reason for interest from the previous two. In our reasoning, the first two causes - the relative provision of goods over time and the time preference in sense two (preference for the given satisfaction to be gratified as soon as possible) - influence the slope of the indifference curve (the MRS) and hence the time preference in the first sense. They represent the subjective element in the theory of interest. On the other hand, the third reason is embodied in the investment opportunity line; it represents the objective or productivity element. All three causes together - the marginal rate of substitution (MRS) and the marginal rate of product transformation (MRPT) - codetermine the natural rate of interest.

Murphy suggested that all three reasons determine the time preference in sense one - the exchange ratio between present goods and future goods. However, it is more convenient to attribute the phenomenon of the time preference in sense one only to the subjective part of the model (the MRS and the saving function) and separate the third reason for the productivity element (the MRPT and the investment function). Time preference in sense one in our reasoning represents the subjective(!) exchange ratio between present goods and future goods, i.e. any MRS along the indifference curve. The eventual objective exchange ratio between present goods and future goods (but also the equilibrium MRS) then depends and is codetermined by the productivity element of the model. It is the point where the slope of the indifference curve (determined by the two causes) and the slope of the investment opportunity line (determined by the third cause) coincide. It can be said that Murphy's reasoning on time preference in sense one is about the point of general equilibrium (one particular and optimal point on the indifference curve). However, our approach to the time preference in sense one contemplates any point on the indifference curve and it seems to be more in line with standard neoclassical theory. The third reason is then required to find the eventual equilibrium, i.e. one particular point of optimum on the indifference curve.
The foregoing analysis allows us to show a very simple relationship between the Fisherian diagram and the loanable funds market. Both models are crucial in the discussion about the underlying factors of the natural rate of interest. Suppose that the investment opportunity line is close to linear (Figure No. 30, panel a). Consider an increase in the subjective discount rate (from $\rho_{1}$ to $\rho_{2}$ ), which might be represented by an increase in the slope of the indifference curve at the diagonal line. The equilibrium of this economy moves from point $\mathrm{E}_{1}$ to point $\mathrm{E}_{2}$. Note that the equilibrium natural rate of interest is not much affected. Slowly decreasing marginal productivity is reflected in a very flat investment curve presented in panel (b). An increase in the time preference (in sense two) leads to a shift of the saving curve to the left. In the end, the equilibrium quantity of invested capital declines leaving the natural rate of interest almost unaffected. In this particular situation, the key factor of the natural rate of interest is the physical productivity of capital, not the time preference. ${ }^{178}$

[^103]

Figure No. 30 Increase in time preference (in sense two) and the impact on the natural rate of interest if the marginal productivity schedule diminishes slowly.


Figure No. 31 Increase in productivity and the impact on the natural rate of interest if the elasticity of substitution is very high (low $\theta$ )

Figure No. 31 illustrates the opposite situation. The subjective intertemporal elasticity of substitution in consumption is very high ( $\theta$ is close to zero), which results in almost linear
indifference curves. An exogenous increase in the marginal productivity of capital (e.g. due to a positive technological shock) shifts the opportunity line outwards and the investment curve to the right. The amount of capital invested grows; however, the natural rate of interest is almost the same as before because the saving curve is close to linear. In this particular situation, the natural rate of interest is solely determined by the time preference (in sense two). As we can see again, the pure time preference theory is a special case in a more general theory of interest. It is valid just for a very high intertemporal elasticity of substitution in consumption.


Figure No. 32 Increase in productivity and the impact on the natural rate of interest if the elasticity of substitution is low (higher $\theta$ )

If the elasticity of substitution was lower (higher $\theta$ ), an increase in productivity would have a significant impact on the interest rate (Figure No. 32). Notice that the indifference curves are much more curved and the saving curve is much less elastic. It is theoretically possible that if the elasticity of substitution is low enough $(\theta>1)$, and the preferred path of the consumption stream is therefore rather close to perfect consumption smoothing ( $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ are close to $45^{\circ}$ line), the resulting saving curve might be downward sloping. Figure No. 33 illustrates this peculiar situation. In this case, the natural rate of interest is mainly determined by the productivity of capital. However, after the increase in productivity the resulting amount of invested capital paradoxically falls. ${ }^{179}$

The craziest situation may occur for an extremely low elasticity of substitution ( $\theta \gg 1$ ). A sudden increase in productivity should lead to a fall in the equilibrium interest rate because the saving curve exhibits not only a decreasing shape but it is even flatter than the investment curve (Figure No. 34) However, in this case the price mechanism most probably does not work as the natural rate of interest has a tendency to move away from the new equilibrium. The reason lies in the fact that after the shock, the investment (demand) exceeds saving

[^104](supply). As a result, the initial imbalance tends to expand over time and the equilibrium lower interest rate can never be achieved. In more complicated dynamic models (e.g. in the Diamond model), such a low elasticity of substitution may lead to sunspot equilibria and selffulfilling prophecies.


Figure No. 33 Increase in productivity and the impact on the natural rate of interest if the saving curve is decreasing $(\theta>1)$


Figure No. 34 Increase in productivity and the impact on the natural rate of interest if the saving curve is decreasing and more elastic ( $\theta \gg 1$ ) than the investment curve; the case of multiple equilibria

Fisher (1930) in his magnum opus envisioned a situation in which the simultaneous existence of very patient people and very poor investment opportunities results in the negative natural rate of interest. Figure No. 35 clearly shows that only the interest rate less than zero equilibrates investment and saving in this case. However, it must be also assumed that it is impossible to store present goods to the future, so the linear part with slope one (dashed line from point B in panel (a) is not effective, and the actual resource constraint is thus represented by the entire concave opportunity curve.


Figure No. 35 Negative natural rate of interest

Note that the natural rate of interest may be negative even if the economy is populated by "Misesian" people who prefer the given goal to be achieved as soon as possible ( $\rho>0$ ). As can be seen, this a priori positive time preference in sense two is represented by the slope of the indifference curve exceeding one at the $45^{\circ}$ line. However, time preference in sense one $\varepsilon=$ (MRS -1 ) is negative, and together with the diminishing marginal productivity of capital, they co-determine the (negative in this case) natural rate of interest. Once again, we arrived at a theoretical possibility that is unthinkable in the pure time preference theory.

At the same time, it must be clearly understood that the zero lower bound on the interest rate is not binding in this economy (as might be suggested by the FG distance on the horizontal axis in panel $\mathbf{b}$ ). The zero bound is a problem of the nominal rate of interest, not the real rate of interest studied here. As can be seen in panel (a), the present output of consumption goods is larger than future output, and the negative real rate of natural interest might be easily generated by a high rate of price inflation (see equations 29 and 31) if this rate exceeds the given positive (or zero) level of the nominal rate of interest.
So far, we assumed that the economy is populated by identical agents. Another interpretation could be that we displayed a situation of a typical (or average) consumer. Thus, we were not concerned with a possibility that at the individual level, saving need not equal investment. In other words, for the given market rate of interest, tangents to the indifference curves,
representing the highest attainable utility, and to the opportunity lines, representing maximum present value of the income stream, lied at an identical point.


Figure No. 36 An individual in an economy with heterogeneous agents

However, if agents were not identical, the tangencies for the given market rate of interest might be placed at different points. Consider a situation of one particular individual in Figure No. 36. If the (income) endowment of this individual was at point $\mathrm{A},{ }^{180}$ and no investment opportunities were available, for the given market rate of interest $\mathrm{r}_{\mathrm{E}}$, the optimum of this consumer would be at point $\mathrm{E}_{0}$. This consumer would be a saver with the optimum amount of saving DA. Yet, if we allow him to engage in investment activity, he might use the opportunity of a higher physical return on every dose of investment that exceeds the market real rate of interest $\mathrm{r}_{\mathrm{E}}$. The investment activity is profitable until the marginal rate of return (the slope of the investment opportunity line minus one) is greater than the real rate of interest $r_{\mathrm{E}}$. He is motivated to increase the investment activity as long as this condition is met. The optimum point is at F , where the real rate of return is equal to the real rate of interest. At this point, the present value of his income stream is maximised (Fisher 1930:223), and the difference between his returns (BF) over costs of investment (original investment plus interest, i.e. BG) is the greatest possible (i.e. FG) (Stigler 1987:316). As can be seen, the optimum amount of investment is AB.
With regard to consumption, higher income allows him to consume more in both periods compared with the original situation. His new optimum is at point $\mathrm{E}_{1}$. He is definitely better

[^105]off as the new optimum lies at a higher indifference curve. His present consumption rises from 0D to 0C, so his saving is reduced from AD to AC. Now, saving is too low to finance his optimum investment AB. Hence, he becomes a debtor - his optimum borrowing is represented by $(A B-A C)=B C$.
If all agents were like this one, the presented situation would not be sustainable since it is impossible for everyone to be a debtor. The equilibrium market real rate of interest must go up to equalise saving and investment. Only if agents were heterogeneous, the presented optimum would be stable. However, this would require that there were agents with an excess of saving over investment. These agents must be more patient and/or they must have less favourable range of investment opportunities compared with the agent considered here. Nevertheless, further discussion about the heterogeneity of agents will be postponed to the next section.

So far, we demonstrated that the natural rate of interest is determined by the time preference and the marginal productivity of capital. Yet, Olson and Bailey listed the following set of determinants of the (positive) rate of interest:
(1) diminishing marginal utility plus profitable investment opportunities with non-decreasing income (and zero time preference); (2) diminishing marginal utility with increasing endowment income (and zero time preference); (3) positive time preference with constant income or with profitable investment opportunities (and non-increasing marginal utility). (Olson and Bailey 1981:8)

The last part of this section deals with an exogenously given flow of real income. However, before we explore the impact of a fixed flow of real income on the interest rate, let us discuss objections of the PTPT authors against the conclusion presented above - namely that the increase in productivity should raise the natural rate of interest. The strongest opposition against this prediction can be found in Rothbard (2004), where he extended the original Mises's objection.

According to Rothbard, if the invention (unexpectedly) increases productivity of capital goods and hence the resulting output of consumption goods, the impact on the interest rate is only temporary. Higher revenues from the extra output should only lead to temporary profits for the users of capital. Sooner or later, the expanded output of consumption goods must reduce prices of these goods and/or higher profitability of capital goods must be reflected and imputed in higher prices of capital goods. ${ }^{181}$ This process will continue up to the point in which no profits are left to be reaped. At this point, the difference in value between output and inputs falls back to the level dictated by the pure time preference.

Our response to Rothbard's objection must be separated into several parts. First, if the growth in productivity of capital arises in the economy with constant returns to capital, the real rate of interest can never fall back. Suppose, for example, that in the rice economy the net return to capital suddenly rises from $10 \%$ to $15 \%$. It is not necessary to repeat the arguments from the previous section that there can be no equilibrium real rate of interest other than $15 \%$. In the loanable funds model, constant marginal productivity is reflected by a horizontal investment curve that solely determines the real rate of interest. ${ }^{182}$ After the exogenous productivity shock, it is shifted upwards to a new level of $15 \%$ (see Figure No. 37).

[^106]Furthermore, Rothbard's argument deals with the behaviour of the value difference between output and invested inputs. As we have already seen, this value difference is associated with nominal interest and says nothing about the market exchange ratio between present goods and future goods. An increase in productivity in the rice economy should move the optimum exchange ratio between present rice and future rice to 1.15 . However, nominal rate of interest might reach any value depending on the size of $\rho$ and $\theta$.


Figure No. 37 Increase in productivity in the economy with constant MPK

Recall again the examples in Figure No. 26, and equation (33). Assuming r $=15 \%, \rho=0 \%$ and $\theta=1$, the nominal rate of interest in panel (a) stays the same at the level of $0 \%$. However, output growth will increase to $15 \%$ and prices will fall by $15 \%$. Similarly, in panel c ( $\rho=4 \%$ ) the value difference between present and future rice will remain at the same level of $\$ 80$, so the nominal rate of interest will be $4 \%$ as before. As can be seen, Rothbard's prediction about the return of the value difference back to the level dictated by the pure time preference ( $\rho$ ) holds at least for $\theta=1$. Yet, the real rate of interest is undoubtedly affected by the increase in productivity. For the given rate of nominal interest, higher real interest rate is ensured by a sharper decline in prices.
As we already know, the equality between the nominal interest rate and the pure time preference $\rho$ was valid only for a unitary intertemporal elasticity of substitution. Yet, the nominal interest rate in panels (b) and (d) in Figure No. 26 does not fully reflect the subjective discount rate. A rise in productivity in panel (b) will result not only in a permanent increase in the real rate of interest to $15 \%$, but the nominal interest rate should grow to $7.24 \%$ from $4.88 \%$. ${ }^{183}$ The additional decline in prices will be only $2.36 \%$, since output will grow over time by only this extra percentage. Thus, after the increase in productivity, investment in present rice for $\$ 2,000$ will result in the net nominal interest of $7.24 \% \times \$ 2,000=\$ 145$, which

[^107]is more than the initial nominal interest of $\$ 100$. Since prices in panel (b) do not fall fast enough, the increase in physical productivity may result also in the growth in value productivity. A similar result is obtained in panel (d), in which the nominal rate of interest increases from $7 \%$ to $9.36 \%$.

Hence, according to our two period model, Rothbard's prediction about the return of the rate of interest back to its initial level is not correct. With constant MPK, the increase in productivity will affect not only the real rate of interest but (except for $\theta=1$ ) it will also influence the nominal interest rate, i.e. the difference between the value of expended inputs and the value of the resulting output.

A similar outcome will be reached, if we relax the assumption of constant MPK. Figures No. $31-33$, which are built on diminishing MPK, clearly show that the real rate of interest must be affected by higher productivity. The impact on the nominal rate of interest then depends on other parameters of the model (e.g. $\theta$ ). However, a return of nominal interest to exactly the same level is not very plausible.

Nevertheless, Rothbard's reasoning seems to be directed at more dynamic environment than analysed here with the help of a simple two-period model. Thus, if this basic model (of diminishing MPK) is extended to more periods (even to infinity in the limit case), different outcomes might be reached compared with a simple two-period model. As will be seen in section 5.1, in the infinite horizon model the impact of productivity on the real (and nominal) rate of interest critically depends on the permanence and nature of the productivity shock. The return of the interest rate back is possible, although the mechanism is different from Rothbard's reasoning, and this return will certainly take much more time.

At the end of this section, let us mention opinions of other PTPT authors about the impact of an increase in productivity of capital on the rate of interest. Frank Fetter (1928), for example, thought that the productivity growth should lead to a better provision of present goods. As a result, the interest rate must, according to this early PTPT theorist, fall rather than rise.

However, the objection to this idea is rather simple. There is no reason to expect that the productivity growth will not affect the provision of future goods in the positive direction as well (Pellengahr 1996). Thus, if the relative provision of present and future remains the same, Fetter's argument is not tenable.
And finally, Garrison (1979) believed that since the productivity growth brings about higher average income, it may decrease time preference. This argument was thoroughly discussed by Fisher (1930). As we will show in section 5.1, this outcome is possible to occur. However, the initial increase in the real rate of interest is inevitable and, moreover, higher real rate of interest should last for a relatively long period of time.

### 4.1 EXOGENOUS FLOW OF REAL INCOME, HETEROGENEOUS AGENTS AND THE NATURAL RATE OF INTEREST

The exogenously given flow of income can be understood as an extremely degenerated form of the investment opportunity line (Figure No. 38). Hayek (1941) argued that this might be the case if all factors of production consist of permanent (and non-renewable) resources. Hayek assumed that these resources provide a definite flow of goods and services. However, the service at the given point of time cannot be moved to any other period. Since the resources are permanent, factors of production here considered do not represent capital, but only land or labour. To keep the analysis as simple as possible, we will again contemplate the two-period model.

Fisher (1907:185ff), in this connection, envisioned a hard-tack economy in which doses of hard-tacks are provided at definite time moments. In such a case, sailors are not endowed with the entire stock at the beginning, as was assumed in section 3, but they face the "slowness of Nature", that "will give rise to a rate of interest" due to "man's impatience to exploit her".


Figure No. 38 Natural rate of interest and a constant flow of income

The exogenous flow of income can take any shape. The resulting natural rate of interest might be found either mathematically or graphically. We will start with a graphical approach. The mathematical solution is shown in Appendix 2 B. Consider an economy populated by people with an identical discount rate, who earn the same real income every period. This situation is closely related to the vision of Mises about the evenly rotating economy because all processes in this model are repeated every period in the same way and the same level of income is earned every period. It can be easily shown that the only equilibrium rate of interest is $r_{E}=\rho$ (see Figure No. 38). Any real interest rate lower than $\mathrm{r}_{\mathrm{E}}$ (e.g. $\mathrm{r}_{1}$ ) will result in the excess of demand for present goods over their available supply (see $\mathrm{C}_{0}{ }^{*}-\mathrm{Y}$ in panel b). Interest rate higher than $r_{E}$ will have the opposite effects. Excesses of supply or demand will eventually move the rate of interest back to the level of $\rho$. Notice that the optimum consumption stream of each individual will be perfectly smoothed (because $r=\rho$, see equation 25 in section 3 ) regardless of the elasticity of substitution $1 / \theta$. Since all agents are identical, there will be no individual saving or borrowing, because every agent will exactly consume his or her income endowment in that particular period. As a result, there will be no intertemporal market (recall the discussion with Garrison in section 3), even though the real interest rate must exist to guarantee equilibrium. Any deviation of the real rate of interest from $\rho$ will immediately create this market, however, it will be characterised either by a surplus of present goods or their deficit. Thus, in this case the only general intertemporal equilibrium is zero individual saving, non-existence of the intertemporal market, and a positive real rate of interest at the level of $\rho$. The mathematical solution is provided in Appendix 2 B, section A, equations 18 $22 .{ }^{184}$

[^108]In this particular economy, the Böhm-Bawerkian first reason does not operate, because the income endowment is the same in both periods. The third reason (higher productivity of roundabout methods) is not effective either, since there is no capital in the economy; it is a pure endowment economy. As we can see, in this economy (and virtually only in this economy), the natural rate of interest is solely determined by the time preference (in sense two). Here, not only the phenomenon of interest as such but also the particular size of the rate of interest exists only due to the fact that people prefer present satisfaction to future satisfaction $(\rho>0) .{ }^{185}$ Only in this economy, the PTPT is correct. Mises had maybe this model of pure constant endowment economy in mind when he envisioned the ERE and the dominance of the time preference in the interest theory.


Figure No. 39 Increasing (a) and decreasing (b) income stream and the corresponding natural rate of interest

Now we relax the assumption of a constant flow of income. Misesian economists would argue that the key assumption of the ERE is then violated. Nevertheless, varying income stream is so pervasive in the real economy that it must be studied here as well. Consider an economy with an increasing flow of income (Figure No. 39, panel (a)), and an economy with a sharply decreasing flow of income (panel b). In panel (a), only a positive natural rate of interest is consistent with the intertemporal equilibrium. Both Böhm-Bawerkian grounds for interest are effective because people are better provided for in the future and because they discount future utilities $(\rho>0)$. The natural rate of interest should be higher than the subjective discount rate (and hence higher than the natural rate of interest shown in panel (a) of Figure No. 38) due to the presence of the first ground. Thus, higher marginal valuation of present goods is also supported by their relative scarcity. Since $r>\rho$, the optimum time shape of consumption is increasing (see the Euler equation 25 in section 3, or equation 6 in Appendix 2 B).

[^109]On the other hand, in panel (b) a negative real rate of natural interest is the only level that will equilibrate demand and supply of present (and future) goods. Panel (b) once again represents an economy where people exhibit time preference in sense two ( $\rho>0$ ), i.e. they prefer the given want to be gratified as soon as possible, but not in sense one (MRS $<1, \varepsilon<0$ ), i.e. they do not prefer present goods to future goods (exactly the opposite is true). Consequently, the natural rate of interest in this economy is negative. The reason lies in the fact that the first ground for interest works in the opposite direction and it is stronger than the second ground ( $\rho>0$ ). The mathematical solution is provided in Appendix 2 B, section C, equations 27-31. As can be seen from equation (31), future income must be lower by at least $\rho \%$ compared with present income to depress the natural rate of interest below zero. In other words, future income must be sufficiently low compared with the present income in order to induce a premium of future goods over present goods. Very low future income will gratify wants of very high urgency, and if the representative present good cannot be moved to the future as we assumed at the beginning, the eagerness to postpone goods (but not the given satisfaction) to the future must decrease the natural rate of interest below zero. Since $r<\rho$, the optimum time shape of consumption is decreasing (see the Euler equation 25 in section 3, or equation 6 in Appendix 2 B).
If all individuals are identical, individual saving will be zero in both panels of Figure No. 39. If their income streams differ (even though being of the same time shape, i.e. increasing or decreasing) people in one group might become debtors and the other creditors. This conclusion is especially interesting for generally decreasing incomes. Individuals with a very sharp decline in the income stream will become creditors even for a negative real rate of interest (see Figure No. 40). They will lend more present apples in exchange for a lower amount of future apples (the loan of $\mathrm{Y}_{0}{ }^{\mathrm{B}}-\mathrm{C}_{0}{ }^{\mathrm{B}^{*}}=\mathrm{C}_{0}{ }^{\mathrm{A}^{*}}-\mathrm{Y}_{0}{ }^{\mathrm{A}}$ is lower than the repayment $\mathrm{C}_{1}{ }^{\mathrm{B}^{*}}-$ $\left.Y_{1}{ }^{\mathrm{B}}=\mathrm{Y}_{1}{ }^{\mathrm{A}}-\mathrm{C}_{1}{ }^{\mathrm{A}^{*}}\right)$. Mathematical discussion is provided in Appendix 2 B , section C .


Figure No. 40 Creditors might exist even if the natural rate of interest is negative (panel b)

Furthermore, for the given $\rho$ (and $\theta$ ) in the economy, a specific flow of income can be determined that will lead to zero natural rate of interest. This particular flow was discussed in section 2.4. In Appendix 2 B, section C, it is explicitly defined for logarithmic utility function
as $\mathrm{Y}_{1}=\mathrm{Y}_{0} /(1+\rho)$. Figure No. 4 illustrates how this income stream might be found. It lies at the perpendicular line to the $45^{\circ}$ line that exactly touches the highest possible indifference curve. Obviously, for impatient people ( $\rho>0$ ), this particular income stream must be decreasing $\left(\mathrm{Y}_{1}<\mathrm{Y}_{0}\right)$. In other words, present must be better provided for than future.
As we can see, even this simple Fisherian model may answer crucial questions in the theory of interest as well as fundamental questions about the optimum intertemporal consumption behaviour of people. First, it is quite easy to find an equilibrium size of the natural rate of interest. As was demonstrated above, it can be positive, zero, or negative regardless of the positivity of the subjective discount rate. Its negative value, i.e. the preference for marginal future goods over the marginal present goods, which is at variance with the pure time preference theory, is caused by a sharply diminishing time shape of the aggregate income in the economy. Hence, if people expect a reduction in their well-being in the future (e.g. due to an expected future stringency of economic conditions at the beginning of a very long recession), the natural rate of interest might fall below zero.

The optimum path of consumption can be also easily determined as it depends on the difference between the real rate of interest and the subjective discount rate. Surprisingly, it does not depend on the particular time shape of income. Thus, at the individual level a decreasing time shape of the optimum consumption stream might be consistent with an increasing time shape of income and vice versa. To see this, move the income endowment $\mathrm{A}^{\mathrm{A}}$ in Figure No. 40 along the budget line closer to the vertical axis such that this point will eventually lie to the left of the $45^{\circ}$ line. In such a case, $\mathrm{Y}_{1}{ }^{\mathrm{A}} / \mathrm{Y}_{0}{ }^{\mathrm{A}}>1$, but $\mathrm{C}_{1}{ }^{\mathrm{A}^{*}} / \mathrm{C}_{0}{ }^{\mathrm{A}^{*}}<1$ (see also Appendix 2 B, figures in simulation 1).

And finally, if flows of income vary across individuals that have an identical subjective discount rate, we may decide whether the particular individual will become a debtor or a creditor for any size of the natural rate of interest in the economy. Debtors (e.g. individuals A in Figure No. 40) are characterized by the fact that the growth rate of their income stream $\left(\mathrm{Y}_{1}{ }^{\mathrm{A}} / \mathrm{Y}_{0}{ }^{\mathrm{A}}-1\right)$ is higher than the growth rate of the income stream of creditors $\mathrm{Y}_{1}{ }^{\mathrm{B}} / \mathrm{Y}_{0}{ }^{\mathrm{B}}-1$ (panel b).
An interesting implication of the last point is that the net borrowing/lending position of each individual does not depend on the absolute average size of income, but rather at its time shape. Thus, individuals with very high present income might be debtors if they expect even higher income in the future. Their high demand for present goods might be satisfied by savings of relatively poor people having low present income and expecting its sharp decline over time. Obviously, the number of small savers must be large enough to meet the demand of important borrowers. Furthermore, according to equation (29) in Appendix 2 B, a very high future income of large borrowers may drive the real interest rate up, which will act as another brake that will ensure equilibrium in the intertemporal market. The irrelevance of the size of the income stems from the fact that all individuals have the same subjective discount rate and that their optimum consumption flow depends only on the difference between the real interest rate and the subjective discount rate. ${ }^{186}$ Hence, if the growth rate in income of the individual exceeds the difference between r and $\rho$ (for logarithmic utility), he will become a debtor because his consumption will grow at a lower rate than income. ${ }^{187}$ However, at the aggregate

[^110]level, the growth rate in income must be equal to $(r-\rho)$ as is perfectly clear from equation 30 in Appendix 2 B. The equilibrium rate of interest will adjust to ensure this condition.

As can be seen, the simple economy presented here is characterized by the fact that the growth rate in aggregate income is always lower than the real rate of interest provided that the subjective discount rate is positive. Thus, from the point of view of standard growth theories, if people prefer the given satisfaction to be delivered as soon as possible (i.e. $\rho>0$ ), this economy is always dynamically efficient. If we recall implications from section 3.1.3, for a constant money supply, the nominal rate of interest in this economy must be positive.
In the following part, we will relax the assumption that the subjective discount rate is identical for all individuals in the economy. Suppose that there are two groups of people with different rates of time preference in sense two. Group A has the discount rate of $\rho_{A}$, group B of $\rho_{B}$. Suppose that the first group is more patient than the second group, hence $\rho_{A}<\rho_{\mathrm{B}}$. Obviously, this assumption oversimplifies the real world because not only the subjective discount rate differs among individual people, but it is changing on the individual basis as well. However, as we will see, even this simplification may provide us with key insights.



Figure No. 41 More patient (a) and less patient (b) agents and the corresponding natural rate of interest.

Panels (a) and (b) in Figure No. 41 display equilibrium of an economy populated by two groups of people with different $\rho$, but (for simplicity) with a constant flow of income. If there was no intertemporal market, each group would consume its income in every period (point $\mathrm{A}_{1}$ and $\mathrm{A}_{2}$ ). The creation of the intertemporal market will make everybody better off because the optimum of a representative individual of each group $\left(\mathrm{E}_{1}\right.$ and $\left.\mathrm{E}_{2}\right)$ lies on a higher indifference curve. The interest rate must adjust such that positive saving of group A (more patient people) is exactly the same as negative saving of group $B$ (less patient people) - $\left(\mathrm{Y}-\mathrm{C}_{0} \mathrm{~A}^{\mathrm{A}^{*}}\right)=\left(\mathrm{C}_{0}{ }^{\mathrm{B}^{*}}-\right.$ Y). The equilibrium rate of interest will be between $\rho_{A}$ and $\rho_{B}$, and its precise value can be found in Appendix 2 B, in sections D and E. As is obvious from the mathematical analysis, the natural rate of interest does not depend on the level of income, if it is constant and

[^111]identical for all individuals (section D). However, if the size of the constant income stream varies across individuals, the real rate of interest is affected in the sense that the size of income of the particular agent gives relative weight to the subjective discount rate of this agent in determining the size of the real interest rate. Nonetheless, the limits on the natural rate are determined by $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$, and no level of income can push the interest rate outside these limits. And finally, it is obvious that the less patient agents will be characterized by a decreasing shape of their consumption flow because their subjective discount rate is higher than the interest rate, whereas the more patient individuals will consume relatively more in the future since their rate of time preference (in sense two) is lower than the rate of interest.

The foregoing approach might be generalized for $n$ possible values of $\rho$. The natural rate of interest is then so adjusted that the aggregate level of saving is zero. At the same time, the equilibrium of such an economy is characterised by the fact that every individual is maximizing his or her lifetime utility at the point where his or her $\mathrm{MRS}_{\mathrm{i}}$ is equal to $\left(1+\mathrm{r}_{\mathrm{E}}\right)$, or alternatively where the marginal rate of time preference - $\varepsilon_{\mathrm{i}} \equiv\left(1+\rho_{\mathrm{i}}\right) \mathrm{u}^{\prime}\left(\mathrm{C}_{0_{\mathrm{i}}}\right) \mathrm{u}^{\prime}\left(\mathrm{C}_{1 \mathrm{i}}\right)-1$ - is equal to the equilibrium natural rate of interest $\mathrm{r}_{\mathrm{E}}$. Furthermore, the optimum flow of consumption of each individual can be easily determined as well as the fact whether the individual is a lender, a borrower or does not enter the intertemporal market at all.

In the next discussion, we will relax the assumption that the time shape of the flow of income is the same for all people. Figure No. 42 illustrates a situation where the income stream of individual A is decreasing $\left(\mathrm{Y}_{0}{ }^{\mathrm{A}}>\mathrm{Y}_{1}{ }^{\mathrm{A}}\right)$ and that of individual B is increasing $\left(\mathrm{Y}_{0}{ }^{\mathrm{B}}<\mathrm{Y}_{1}{ }^{\mathrm{B}}\right)$. The real rate of interest must be so adjusted that the aggregate saving is zero. Compared with the example in Figure No. 41, the equilibrium real interest rate might be even negative if the income stream of one group is decreasing sharply enough. The mathematical solution can be found in the benchmark example of Appendix 2 B , equations 1-16.


Figure No. 42 Equalisation of the rates of time preference (in sense one) with the real interest rate and also among individuals, regardless of the time shape of their income stream and the size of their subjective discount rate.

In the example in Figure No. 42, the patience of individual A (due to low $\rho_{\mathrm{A}}$ ) is supported by a falling income over time, whereas high impatience of individual B (due to high $\rho_{\mathrm{B}}$ ) is enhanced by a rising income stream. Thus, at the income endowment point of individual A (point $A^{A}$ ) the MRS is very low, whereas in case of individual $B$ (point $A^{B}$ ) it is very high (see the dashed lines at $\mathrm{A}^{\mathrm{A}}$ and $\mathrm{A}^{\mathrm{B}}$ ). We can say that the individual A has a very low time preference in sense one (it might be even negative if MRS $<1$ at point $\mathrm{A}^{\mathrm{A}}$ ), and the individual $B$ has a very high time preference in sense one at point $A^{B}$. However, as we already know, the time preference in sense one (i.e. the subjective exchange ratio between present goods and future goods represented by the MRS) is an endogenous concept, and it eventually depends on the optimum point that is posited at the highest possible indifference curve that touches the budget line. In market equilibrium, the natural rate of interest is so adjusted that positive saving of patient individuals $\mathrm{A}\left(\mathrm{Y}_{0}{ }^{\mathrm{A}}-\mathrm{C}_{0}{ }^{\mathrm{A}^{*}}\right)$ is perfectly offset by negative saving of impatient individuals $\mathrm{B}\left(\mathrm{C}_{0}{ }^{\mathrm{B}^{*}}-\mathrm{Y}_{0}{ }^{\mathrm{B}}\right)$. Moreover, at the optimum (i.e. lifetime utility maximizing) points of both groups and in the eventual market intertemporal equilibrium, the rates of time preference (in sense one), or the rates of impatience as I. Fisher would call it, must be the same for all individuals and they must be equal to the equilibrium real rate of interest (i.e. $\operatorname{MRS}_{\mathrm{A}}-1 \equiv \varepsilon_{\mathrm{A}}=\mathrm{r}_{\mathrm{E}}=\varepsilon_{\mathrm{B}} \equiv \mathrm{MRS}_{\mathrm{B}}-1$ ). Thus, the market process leads to the equalization of time preferences (in sense one) of various individuals regardless of their subjective discount rates (i.e time preference in sense two) and the time shape and size of their income streams. The coordinating mechanism is due to the adjustment of the real rate of interest that ultimately guarantees that the objective exchange ratio between present goods and future goods is perfectly in accordance with the subjective exchange ratio of each individual.

Both Mises (1996) and Rothbard (2004) wrote about the eventual equalization of the rates of time preference among various individuals. It is quite difficult to imagine a different interpretation than the adjustment of the individuals' MRS. However, since the MRS can take on any value, greater weight might be put on future goods compared with present goods. Thus, the theory of Mises and Rothbard assuming a priori positive time preference (in sense one), i.e. a priori positive premium on the part of present goods, cannot be correct.
As a final note, let us discuss the optimum time shape of consumption of the individuals from the previous example. The answer is not so clear cut as before as it depends on the eventual size of the real interest rate $r$. The problem is that compared with Figure No. 41, the real interest rate is not bound by the interval determined by individual subjective discount rates ( $\rho_{\mathrm{A}}, \rho_{\mathrm{B}}$ ), because a non-constant time shape of the income streams can move it to any level (see equation 16 in Appendix 2 B). Thus, $r$ might be higher than $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$, it can be lower than both levels, or it can be in between (as in Figure No. 42). As a result, the optimum flow of consumption of each individual can take any time shape. A wide variety of possible outcomes are presented in simulations in Appendix 2 B.
As can be seen in simulations in Appendix 2 B, almost any combination is possible. The most noteworthy observations are as follows: An increasing time shape of income and a higher subjective discount rate lead to a borrowing position. If they operate against each other, the eventual position depends on their relative strength. ${ }^{188}$ Next, increasing income streams raise the equilibrium interest rate above both subjective discount rates, which results in the fact that the time shapes of consumption flows are also increasing (see simulation 6 in Appendix 2 B). Decreasing time shapes of income would imply a decline in the interest rate below both subjective discount rates. This would lead in turn to a decreasing time shape of consumption.

[^112]The natural rate of interest might even fall below zero if the general decline in income is sharp enough. And finally, the natural rate of interest might be stabilised between the discount rates of various individuals if the income streams are of the opposite time shapes or if they are constant over time. A perfectly smoothed consumption stream of an individual might be reached if the market interest rate is equal to his subjective discount rate, which seems to be an exception rather than a rule. Consumption will not be smoothed even if both income streams are perfectly smoothed provided that the subjective discount rates differ (see Figure No. 41 above and Appendix 2 B, simulation 5). In such a case, the consumption stream of a more patient individual will be increasing, whereas that of the less patient individual will be decreasing since the equilibrium interest rate will be stabilized in the interval between $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$.

However, two combinations of consumption streams of A and B are impossible. Because $\rho_{\mathrm{A}}<$ $\rho_{\mathrm{B}}$, if the consumption stream of A is decreasing $\left(\mathrm{r}<\rho_{\mathrm{A}}\right)$, that of $B$ cannot be increasing (since $\mathrm{r}>\rho_{\mathrm{B}}$ is inconsistent with $\rho_{\mathrm{A}}<\rho_{\mathrm{B}}$ and $\mathrm{r}<\rho_{\mathrm{A}}$ ). And conversely, if the consumption stream of $B$ is increasing ( $r>\rho_{B}$ ), that of A cannot be decreasing (since $r<\rho_{A}$ is inconsistent with $\rho_{A}<$ $\rho_{B}$ and $\rho_{B}<r$ ).
It should be stressed that the most common situation is probably an increasing time shape of income in general, positive real rate of interest that exceeds both the average growth rate in income and $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$, and a borrowing position of individuals with higher $\rho$ (i.e. B with $\rho_{\mathrm{B}}$ ) and a lending position of individuals with lower $\rho$ (i.e. A with $\rho_{\mathrm{A}}$ ). ${ }^{189}$ This situation is portrayed in Appendix 2 B, simulation 6.

As can be seen, heterogeneity of agents (both as regards income streams and subjective discount rates) leads to the creation of the intertemporal market in which borrowers and lenders exchange present goods for future goods. As is shown in Appendix 2 B, simulation 8, the intertemporal market will exist even if the natural rate of interest is zero (or negative, see equation 17 in Appendix 2 B). Thus, we have constructed another theoretical model that is at odds with Garrison's critique of the neoclassical theory. In our model, all individuals prefer the given want to be satisfied as soon as possible (both $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$ are greater than zero), so the key Misesian maxim is not violated. However, the natural rate of interest is zero, i.e. present goods are not preferred to future goods. Yet, a vivid intertemporal market has been created (i.e. some people lend and others borrow) even without the existence of interest. Hence, the critique of R. Garrison might be easily overcome.

In actual world, people differ in subjective discount rates, utility functions, and shapes of their income streams. We separated each factor of the natural rate of interest to analyze its specific impact. However, the main message of our analysis is clear - the natural rate of interest is a complicated function of the subjective discount rates of individuals ( $\rho_{\mathrm{i}}$ ), the shapes of their income streams ( $\mathrm{Y}_{0 \mathrm{i}}, \mathrm{Y}_{1 \mathrm{i}}$ ), and the shape of their utility functions $\left(\theta_{\mathrm{i}}, \ldots\right)$. We also demonstrated that the natural rate of interest may take on any value. The most plausible is its positive value because people prefer present satisfaction to future satisfaction ( $\rho_{\mathrm{i}}>0$ ). However, if the (expected) income stream of a considerable part of population is decreasing, it may fall below zero. As a result, the statement of the Misesian PTPT that the natural rate of interest is solely determined by the pure time preference holds only under very special circumstances.

[^113]Further extensions of the analysis of the natural rate of interest are also possible. We can combine the previous two sections - man can face investment opportunities, and he may also have an exogenous income endowment in both periods. Furthermore, if the leisure time enters the utility function, the income stream is no longer exogenous, and the key parameters of the utility function would determine not only the shape and position of the indifference curves but also the position of the income endowment. And finally, the assumption of a single-good economy might be relaxed and an effort (maybe futile) to find the natural rate of interest for an n -good economy might be carried out.
This section will be concluded with the first extension. Utility function that includes leisure (or labour effort) will be postponed to Appendix 3 B. The third extension, an n-good economy, was only briefly mentioned in the previous section since examining the natural rate of interest in such an economy would require much deeper investigation and much more time, space, and even intellectual and technical abilities than the author of the present study seem to be endowed with.


Figure No. 43 Natural rate of interest in the economy with income/labour (and land) endowment in both periods and with investment opportunities.

Figure No. 43 portrays an economy with investment opportunities between present and future and also with the income endowment in the second period. This may represent an economy with pure labour (and land) available in both periods in which labour in the first period can be used in a longer production process, whereas labour in the second period might be employed only in the short process. ${ }^{190}$ The natural rate of interest in this economy is co-determined by

[^114]subjective factors ( $\rho, \theta$ ) and by the productivity of capital (point E), or by the shape of the income stream and subjective factors for a different set of parameters (point A, not shown as the optimum in the diagram). In the latter case, the entire supply of present labour would be used only in short production processes, whereas in the former case present labour is employed also in longer processes, which reduces present output on behalf of future output. Figure No. 43 represents an economy with identical individuals. However, if agents differ in $\rho$ (and $\theta$ ) and in their investment opportunities, the individual investment need not equal individual saving. As a result, man can borrow from more patient agents to make even higher investment than in Figure No. 43 and fill the lack of saving by a loan from the others. This situation might be represented by Figure No. 36 if the extreme income endowment A is moved along the budget line from the horizontal axis closer to the vertical axis. However, the beginning of the investment opportunity line will stay at the horizontal axis (Stigler 1987:316). The optimum consumption, investment, saving, and loan can be easily described by a similar system of points as in Figure No. 36.


Figure No. 44 Storable income endowment that is positive in both periods. Natural rate of interest determined by the time preference in sense one (i.e. MRS)

It can be also assumed that the income endowment is easily storable (Figure No. 44, panel a), or it may have a constant productive power as in panel (b). The natural rate of interest then depends on the specific shape of the income stream. In Figure 44a, the natural rate of interest $r_{E}$ is determined by the subjective discount rate $\rho$ and the shape of the income stream (i.e. by the time preference in sense one - MRS), not by productivity. If $r$ was lower than $\mathrm{r}_{\mathrm{E}}$ (e.g. $\mathrm{r}=$ $0 \%$ ), the excess of borrowing would immediately emerge. This will in turn drive up the interest rate back to $r_{\mathrm{E}}$. A similar analysis holds for panel b. However, in panel (b) there is a higher chance that the natural rate of interest will be determined by constant marginal productivity of capital because the insignificance of productivity requires much sharper increase in the income endowment over time (i.e. $\mathrm{Y}_{1} \gg \mathrm{Y}_{0}$ ). It means that the amount of present original factors of production that may be used in a longer process (that exhibits constant and positive marginal productivity) must be quite small.


Figure No. 45 Storable income endowment that is positive in both periods. Case of zero interest

Yet, in Figure No. 45 the natural rate of interest is definitely zero (i.e. it is determined by productivity) due to the fact that the income stream is strongly decreasing over time and the good in question is storable. Again, even though people exhibit positive time preference in sense two $(\rho>0)$, the natural rate of interest (and time preference in sense one) is zero (MRS $-1 \equiv \varepsilon=\mathrm{r}=0$ ). Saving takes place in this economy ( $\mathrm{Y}_{0}-\mathrm{C}_{0}{ }^{*}$ ), which will not be, however, traded in the intertemporal market. It will take the form of a simple storage of non-perishable goods held to the poorly endowed future. Notice that the negative (real) natural interest rate could only emerge if the representative good was perishable (the slope of the linear line was below 1 ) and if it was in short supply in the future $\left(\mathrm{Y}_{1} \ll \mathrm{Y}_{0}\right)$.

## 5. DYNAMICS AND THE INFINITE HORIZON

In this part, we will relax the assumption of a two-period life. First, we will assume that the representative individual lives for T-periods. Next, we let T go to infinity. And finally, we will explore the behaviour of an economy in continuous time rather than in discrete time. The extensions made in this section will shed some light on the problems of the interest theory that are obscured in a two-period model or that cannot emerge in this model at all. On the other hand, the ideas developed here are much more difficult or sometimes virtually impossible to plot in a two-dimensional diagram.
We will start with an optimal allocation of consumption of a representative individual who lives for T periods. Recall the lifetime utility function represented by equation (1). To find the optimum consumption path of an individual, we have to add his intertemporal budget constraint (IBC). If his lifetime is T, a usual form of the IBC might be represented by (Olson and Bailey 1981:9): ${ }^{191}$

[^115]\[

$$
\begin{align*}
& C_{0}+\frac{1}{1+r} C_{1}+\frac{1}{(1+r)^{2}} C_{2}+\frac{1}{(1+r)^{3}} C_{3}+\ldots+\frac{1}{(1+r)^{T}} C_{T}= \\
& =Y_{0}+\frac{1}{1+r} Y_{1}+\frac{1}{(1+r)^{2}} Y_{2}+\frac{1}{(1+r)^{3}} Y_{3}+\ldots+\frac{1}{(1+r)^{T}} Y_{T} \tag{34}
\end{align*}
$$
\]

$\mathrm{Y}_{\mathrm{t}}$ denotes real income at time $t$. Real interest rate $r$ is assumed to be constant over time. For varying interest rate across time, (34) is modified to (35).

$$
\begin{align*}
& \mathrm{C}_{0}+\frac{\mathrm{C}_{1}}{\left(1+\mathrm{r}_{1}\right)}+\frac{\mathrm{C}_{2}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right)}+\ldots+\frac{\mathrm{C}_{T}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots\left(1+\mathrm{r}_{\mathrm{T}}\right)}= \\
& =\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{\left(1+\mathrm{r}_{1}\right)}+\frac{\mathrm{Y}_{2}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right)}+\ldots+\frac{\mathrm{Y}_{T}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots\left(1+\mathrm{r}_{\mathrm{T}}\right)} \tag{35}
\end{align*}
$$

Setting up a simple Lagrangian for (1) and (34), one can show (see Appendix 4 B) that FOCs of this problem lead to the following Euler equation for any time $t$ and $t+1$ :
$\frac{u^{\prime}\left(C_{t+1}\right)}{u^{\prime}\left(C_{t}\right)}=\frac{1+\rho}{1+r}$

Equation (36) implicitly defines the optimum path of consumption of a representative individual, and it should be familiar from the two-period model. ${ }^{192}$ Alternatively, the solution of this optimization problem can be expressed as:

$$
\begin{equation*}
\frac{u^{\prime}\left(C_{t}\right)}{u^{\prime}\left(C_{0}\right)}=\left(\frac{1+\rho}{1+r}\right)^{t} \tag{37}
\end{equation*}
$$

Equation (37) allows us to reconsider the Misesian statement that for zero time preference (in sense two, i.e. $\rho=0$ ) and a positive interest rate, an acting man will postpone his consumption to an indefinite future. If we extend the time horizon to infinity and set $\rho=0$, we get (Olson and Bailey 1981:12):
$\lim _{T \rightarrow \infty} \frac{u^{\prime}\left(C_{T}\right)}{u^{\prime}\left(C_{0}\right)}=\lim _{T \rightarrow \infty} \frac{1}{(1+r)^{T}}=0$

Equation (38) suggests that for a positive rate of interest and zero subjective discount rate, the ratio of the marginal utility from consumption in infinity and from consumption today is zero in optimum. This can be either achieved by an infinite marginal utility in the present or zero marginal utility in infinity. If the utility function satisfies usual Inada conditions, infinite marginal utility is achieved by zero consumption, and conversely zero marginal utility is obtained by infinite consumption. Alternatively, if we allow for a subsistence level $\mathrm{C}_{\text {min }}$, the infinite $M U$ is reached at this level. Correspondingly, a satiation level $\mathrm{C}_{\text {max }}$ would lead to zero MU.

[^116]Equation (38) therefore requires that in optimum, the present consumption must be depressed to a negligible level provided that the budget constraint (34) (or better the budget constraint A5_13 in Appendix 5 B) does not allow for an infinite consumption in infinity (Olson and Bailey 1981:12). As a result, all income should be postponed to infinity because the compounding of interest in infinite horizon may expand consumption beyond all limits. This outcome is so attractive that every unit of present consumption should be postponed to this remote future. ${ }^{193}$ Hence, it seems that the Misesian analysis should hold in the infinite horizon model since positive rate of interest is inconsistent with zero time preference (in sense two). The two-period model has therefore hidden this important outcome, and our critique of Mises was inaccurate.

Moreover, a positive interest rate and zero subjective discount rate cannot create long run equilibrium in the production part of the economy if capital exhibits diminishing returns. The never-ending postponement of all consumption implies that people save (almost?) entire income. Huge saving and immense accumulation should then extend the capital stock beyond all limits. This process would eventually depress the marginal product of capital to zero along with the interest rate. In the end, the real interest rate and the subjective discount rate must coincide at the zero level, and the accumulation of capital stops.
However, in real world we usually observe a positive real interest rate. At the same time, we do not witness a radical curtailment of present consumption. For Olson and Bailey (1981), this is an explicit evidence for the existence of positive time preference ( $\rho>0$ ). As a result, their approach leads to similar conclusions as made by L. von Mises. In equilibrium, the interest rate must be equal to the time preference, otherwise all consumption will be postponed to an indefinite future. ${ }^{194}$

The foregoing analysis will become even more obvious, if we introduce a particular form of the utility function. Consider, for example, the CRRA form. According to (25), equation (36) can be represented as:

$$
\begin{equation*}
\frac{C_{t}}{C_{t+1}}=\left(\frac{1+\rho}{1+r}\right)^{1 / \theta} \tag{39}
\end{equation*}
$$

and equation (37) as
$\frac{C_{0}}{C_{t}}=\left(\frac{1+\rho}{1+r}\right)^{t / \theta}$
If we expand the time horizon to infinity and set $\rho=0$, (40) yields:

[^117]\[

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{C_{0}}{C_{T}}=\lim _{T \rightarrow \infty} \frac{1}{(1+r)^{T / \theta}}=0 \tag{41}
\end{equation*}
$$

\]

If the time preference was zero and the real interest rate was positive, our individual would be in optimum either with zero present consumption or with infinite consumption in the infinite future. As a result, it seems that the only stable outcome of this analysis is that the real interest rate is perfectly equal to the subjective discount rate $(r=\rho)$ and the optimum consumption stream is smoothed over time ( $\mathrm{C}_{0}=\mathrm{C}_{1}=\ldots \mathrm{C}_{\mathrm{T}}=\ldots$ ).

However, in actual world we observe an increasing shape of the consumption profile. In other words, consumption is not constant as it grows across time at some definite rate that seems to be quite stable over very long periods of time. It can be shown that in standard growth models with labour-augmenting technological progress, consumption (per worker) may grow at the same rate as income (per worker). And this rate is equal to the rate of technological progress $g$ (Acemoglu 2011).
As a result, the Euler equation ( 40 or 41 ) may be consistent (owing to the technological progress) with an infinite consumption in infinity even if present consumption is not depressed to a negligible level. Suppose that the growth rate of consumption (and the technological progress) is $g=2 \%$. For a logarithmic utility function $(\theta=1)$, this implies that the difference between the real interest rate and the subjective discount rate is roughly $2 \%$ as well.

It can be shown (using 39) that the optimum growth rate of consumption must approximately obey the following expression:
$\frac{\Delta C_{t+1}}{C_{t}}=\frac{r-\rho}{\theta}$
Thus, if consumption grows at the rate of technological progress $g$, and if this rate is lower than the real interest rate $r$ (which is required for a dynamically efficient economy), ${ }^{195}$ equation (42) (or 39) might be satisfied even for a positive real interest rate and zero time preference $(\rho=0)$ if $\theta$ is big enough. This is what Olson and Bailey (1981:19) called a "drastically diminishing marginal utility". We already know that high $\theta$ is equivalent to a low elasticity of substitution and a highly curved utility function. For example, if $g=2 \%, r=4 \%$, and $\rho=0 \%$, the only $\theta$ consistent with equation (42) is equal to $2 .{ }^{196}$
We demonstrated that the positive interest rate is consistent with zero time preference (in sense two) and non-zero present consumption even in the infinite-time-horizon model. However, the model requires an exogenous growth in income endowment and sufficiently convex indifference curves (high $\theta$ ). Misesian economists would probably argue that an increasing labour income endowment violates the key assumption of the ERE. We may reply again that such a shape of the income stream is dominant in modern economies, so our model might accurately represent the actual world. ${ }^{197}$

The Misesian argument about the equality of the interest rate and the time preference ( $\rho$ ) does not hold if the time shape of income is increasing. To display this situation graphically, consider a time profile of income in the two-period model (Figure No. 39, panel a) which is replicated every period. In the infinite horizon model, $\varepsilon \equiv$ MRS -1 and $\mathrm{r}_{\mathrm{E}}$ may be positive even

[^118]for zero time preference in sense two. Even if people do not prefer the given satisfaction to be delivered as soon as possible, the real interest rate may be positive, and all consumption will not be postponed to the infinite future.

We may add that the economic reasoning for high $\theta$, and a positive difference between $r$ and $\rho$, and $r$ and $g$ runs as follows. If the income increases over time and people have high $\theta$ (i.e. low intertemporal elasticity of substitution), their preferred profile of consumption is rather smoothed. As a result, they prefer their high future income to be moved closer to present. They do not save very much, which increases the real interest rate both above the subjective discount rate $\rho$ and above the rate of technological progress $g$.

This model can be also used to give credit to Mises's critique of J. A. Schumpeter. As is well known, Schumpeter (1961) claimed that in a stationary economy the interest phenomenon should disappear. Mises (1996), on the contrary, believed that people a priori prefer the given satisfaction to be achieved as soon as possible, and positive time preference must exist even in a stationary economy (ERE in his system). This implies that positive interest can never be eliminated. If the government or the banking system artificially depressed the real interest rate to zero, a gradual consumption of capital should emerge. Our model gives similar predictions. According to (42), zero real interest rate and a positive subjective discount rate favour present consumption over future consumption. As a result, the optimum profile of consumption is decreasing. People consume today at the expense of future, and the capital stock must gradually fall. The artificially depressed interest rate should progressively exhaust the capital stock in the economy. The corresponding profiles of optimum consumption for various $\theta$ are displayed in Figure No. 49 below. In this particular respect, Mises was perfectly right. ${ }^{198}$

In the two-period model, we demonstrated that the natural rate of interest may be negative either for an initial endowment that deteriorates over time (Figure No. 25) or for a sufficiently decreasing flow of income (Figure No. 39, panel b). By a similar argument, we have shown that the natural rate of interest could be zero. However, Fetter (1928), Mises (1996), and Rothbard (2004) applied the infinite-horizon approach to deny the possibility of zero (or even negative) rate of interest. Consider a piece of land that provides an infinite flow of services. The present price of land in equilibrium should be equal to the discounted sum of the flow of these services. To keep the analysis in real terms, if the given piece of land provides an eternal real income of 100 apples every year, its market price should be:

$$
\begin{equation*}
\mathrm{PV}=100+\frac{100}{1+r}+\frac{100}{(1+r)^{2}}+\ldots=100 \frac{1}{1-\frac{1}{1+r}}=100 \frac{1+r}{r} \tag{43}
\end{equation*}
$$

where $r$ is the real rate of interest prevailing in the economy. Fetter and other PTPT authors claimed that for the zero rate of interest, the market price of this piece of land (and any other perpetuity) should be infinite. This approach therefore provides an indirect proof that the interest rate can never (permanently) fall to zero (or even below zero). ${ }^{199}$

However, let us now demonstrate that there may exist an important gap in this reasoning. The key problem is that the PTPT authors separated the analysis of the time shape of the flow of income on the one hand, and the equilibrium level of the rate of interest on the other. Fisher (1930) stressed that it is the time shape of the income flow that is of crucial importance in the interest theory. At this point, we will follow Fisher's ideas.

[^119]As we have seen, a negative (or zero) rate of interest might be generated only for a decreasing flow of income provided that the subjective discount rate is positive. Hence, the flow of constant perpetual income of 100 apples can never generate zero or a negative rate of interest. In this particular respect, the PTPT authors were right.
However, suppose that the given piece of land provides a perpetual flow of income that is falling at some definite rate $g$ (e.g. $g=-4 \%$ ). This means that the present output of apples is 100 , the next year output of apples is 96 , etc. The present price of land is then calculated as:

$$
\begin{equation*}
\mathrm{PV}=100+\frac{100(1+g)}{1+r}+\frac{100(1+g)^{2}}{(1+r)^{2}}+\ldots=100 \frac{1}{1-\frac{1+g}{1+r}}=100 \frac{1+r}{r-g} \tag{44}
\end{equation*}
$$

The sum of this infinite series converges if the interest rate exceeds $g .{ }^{200}$ Assume that the interest rate is zero, equation (44) then yields that the price of land is $100 / 0.04=2,500$. To make these calculations consistent with the Euler equation (42), consider $\rho=4 \%$ and $\theta=1$ (because $-4 \%=(0-4 \%) / 1) .{ }^{201}$ Hence, a finite price of land is guaranteed if $\mathrm{g}<\mathrm{r}$.
Thus, it can be argued that the approach of the PTPT is not valid. Even for positive time preference ( $\rho>0$ ), the interest rate might be zero (or even negative), and the value of perpetual land will not expand beyond all limits. The primary phenomenon is the flow of income. It influences not only the natural rate of interest, but also the definite price of the given asset generating this particular flow of income. In our example, a decreasing time shape of the income stream depressed the rate of interest to zero even though the time preference (in sense two) was positive, and it also implied a finite value of the given piece of land. It explicitly confirmed Fisher's statement that the analysis of the income stream can never be separated from the theory of the rate of interest, otherwise we obtain incomplete and erroneous results as Fetter, Mises and Rothbard.

### 5.1 DYNAMICS IN CONTINUOUS TIME AND THE NATURAL RATE OF INTEREST

In the last section of this paper, we will generalize our findings obtained so far by introducing a continuous time model. The analysis of either finite or infinite horizon is most elegant and rigorous in the continuous time approach. We will see that many ideas of Böhm-Bawerk are reflected in modern dynamic analysis. Moreover, the dynamic analysis with continuous time will demonstrate again that the pure time preference theory is at least incomplete.
A representative consumer facing dynamic intertemporal decisions can be described by the following life-time utility function (Samuelson 1937):
$\int_{0}^{T} e^{-\rho \mathrm{t}} u(C(t)) d t$
Equation (45) is a continuous-time version of our discrete-time model. It satisfies all usual neoclassical (and Austrian) assumptions. People prefer present satisfaction to future satisfaction, hence $\rho>0$. Marginal utility is positive and declines with higher consumption -

[^120]$\mathrm{u}^{\prime}(\mathrm{C})>0$ and $\mathrm{u}^{\prime \prime}(\mathrm{C})<0$ for all C. Positive first derivative for all levels of consumption guarantees that more is always preferred to less.

This simple model allows us to extend the analysis of many topics mentioned in the previous parts. Let us start with the Fisherian sailors shipwrecked with a definite amount of hard-tacks. A simple dynamic analysis can easily demonstrate that Fisher's predictions about the optimal allocation of hard-tacks over time were imprecise.

Fisher (1930) concluded that the interest rate in the hard-tack economy must be necessarily zero. However, Fisher (ibid.:188) offered the following figures that display possible optimum consumption paths (Figure No. 46).


Figure No. 46 Optimum paths of consumption of Fisher's shipwrecked sailors in a hard-tack economy; copied from Fisher (1930:188)

Let us now demonstrate that none of these are optimal. First, set up a dynamic optimization problem:
(39) $\max U=\int_{0}^{T} e^{-\rho \mathrm{t}} u(C(t)) d t$
s.t. $\int_{0}^{T} C(t) d t \leq K$

A representative sailor lives for $T$ periods. His initial endowment of hard-tacks is $K$. It cannot be extended either by labour effort, by exogenous transfer payments, or by productive investment. The only problem that sailors face is the optimal allocation of the initial endowment over time. Fisher explicitly demonstrated that the interest rate in this economy must be zero. The solution of this problem is given in Appendix 6 along with several technical comments. At this point, we just report the final solution for the CRRA utility function:
$C(t)=\frac{\rho}{\theta} \frac{e^{\rho(\mathrm{T}-\mathrm{t}) \theta}}{e^{(\rho / \theta) \mathrm{T}}-1} K$


Figure No. 47 Optimum path of consumption in a hard-tack economy for different time horizons. $\theta=1, \rho=0.05$.

Figure No. 47 depicts two optimal paths for different T. As can be seen, a longer life requires lower consumption in every period. Furthermore, Figure No. 48 displays that more patient sailors (lower $\rho$ ) have a flatter profile of the optimum consumption. And finally, Figure No. 49 demonstrates that for the given $\rho$, lower elasticity of substitution (higher $\theta$ ) results in a smoother optimum consumption path. ${ }^{202}$ None of these figures resembles the original pictures offered by Fisher. However, he could not have used modern modelling techniques. ${ }^{203}$

[^121]

Figure No. 48 Optimum path of consumption in a hard-tack economy for different subjective discount rates; $\theta=1, \mathrm{~T}=60$.


Figure No. 49 Optimum consumption profile for zero interest rate ( $\mathrm{r}=0 \%$ ) and positive subjective discount rate ( $\rho=5 \%$ ), $\mathrm{T}=60$.

[^122]As can be seen, constant consumption of hard-tacks over time in not optimal. The reason is the existence of positive subjective discount rate. Its presence requires that the un-discounted future marginal utility must be higher than present marginal utility. And this can be achieved only with lower consumption in the future (see Appendix 6 for technical details). Thus, the preference for present satisfaction over future satisfaction leads to a downward sloping profile of the optimum consumption path. However, even if the interest rate is zero, all hard-tacks are not consumed in the present. The reason lies in the diminishing marginal utility of consumption. This law requires that levels of consumption in two consecutive periods (whose distance is infinitely small in the continuous model) are very close to each other. Hence, we do not observe any dramatic jumps in consumption levels, but a smooth (though decreasing) path.

As a result, the existence of positive time preference and zero rate of interest do not result in a complete exhaustion of resources in the present as Mises might argue. The necessary break is performed by the law of diminishing marginal utility. This law is in turn derived from the idea that the given good is able to satisfy only wants of lower and lower intensity as its consumption rises in the given period. Hence, it cannot be optimal to move all goods to one period (be it present or future) and leave the needs in the rest of the life unsatisfied. ${ }^{204} \mathrm{As}$ can be seen, we derived a similar conclusion in the continuous-time model for lifetime $T$ as we observed in a simple two-period model.


Figure No. 50 Optimum consumption path if the subsistence level is achieved within the planning horizon.

A more complicated dynamics would be achieved if there was some minimum subsistence level of hard-tacks required every period till time T. Panel (a) of Figure No. 50 displays the optimum path of consumption for this possibility, which is also compared with a path at which no subsistence level is required (dashed curve), but which is constrained by the same amount of initial hard-tacks. An extreme version and the most unfortunate one would be if the

[^123]initial stock was not big enough to preserve life till time T (panel b). However, in this case there are not many economic decisions to analyse.

The previous example was based on constant marginal productivity of capital, being zero in case of hard-tacks. Now, we extend the analysis from section 4 where we considered an investment opportunity curve and diminishing marginal productivity of capital. The more advanced model of this section will allow us to make several extensions. First, the continuous time version narrows the distance between two periods to an infinitely small lapse of time. Second, there is an infinite number of periods rather than only two. However, the Austrian idea that the time extension of the production process provides higher output is not present here either. Decreasing marginal productivity of capital will again take place only in capital's breadth, not in its height. Nevertheless, many interesting insights might be found in this model as well.

An infinite-horizon continuous time version of the simple Fisherian model from section 4 closely resembles the Ramsey-Cass-Koopmans model, which is developed in Appendix 7. 205 Its building blocks consist of a representative consumer/dynasty maximizing lifetime utility, ${ }^{206}$ the law of motion of capital, and the production function with convential properties especially with the diminishing marginal productivity of capital. Its solution is derived in Appendix 7. In this section, we just use the fact that for a positive labour-augmenting technological progress growing at the rate of $g$, the steady state dynamic general equilibrium requires: ${ }^{207}$
MPK $-\delta=\rho+\theta g$

The interest rate in the economy must be so adjusted that its steady state value r* guarantees condition (48), thus MPK $-\delta=r^{*}=\rho+\theta$ g. Surprisingly, all three terms in (48) might be associated with one of the three causes of interest in Böhm-Bawerk's theoretical framework. Positive and diminishing MPK indicates the idea that roundabout (i.e. capital using) methods of production give higher output; yet, the increments of output are gradually decreasing. This term represents the productivity element - it is the third Böhm-Bawerkian cause for interest. The subjective discount rate $\rho$, as was discussed before, stands for the undervaluation of future wants; it is the second cause of interest. And finally, the first cause is hidden in the term $\theta \mathrm{g}$ even though at first sight, this might not be obvious. In the steady state (on the balanced growth path) of this model, the income per person grows at the rate of technological progress $g$. So the income endowment of each individual grows at this particular rate. Every individual is wealthier every subsequent period. This gives present goods additional premium over future goods in the minds of people because future is better provided for than present. Böhm-Bawerk identified this phenomenon as the first cause for interest.
Equation (48), expressed as MPK $-\delta=\mathrm{r}^{*}=\rho+\theta \mathrm{g}$, might be used to demonstrate that a zero rate of interest is possible even for a positive subjective discount rate, and conversely, a positive rate of interest can be generated even for a zero subjective discount rate. The first possibility may emerge if the income endowment falls at a sufficiently rapid pace. Consider zero population growth (ZPG), logarithmic utility function $(\theta=1), \rho=4 \%$, and $g=-4 \%$. This set of parameters leads to $\mathrm{r}^{*}=0 \%$. Notice that this combination is virtually the same as for the

[^124]discrete time model. ${ }^{208}$ However, in the present section, we formally closed the model by adding the productivity element.

Thus, zero natural rate of interest is achieved if $\rho=-\theta \mathrm{g}$. If the population growth is zero and the condition for the convergence of lifetime utility is satisfied, $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$ (see Appendix 7, condition A7_14), zero interest rate requires that the technological progress (and the growth rate of income) is negative. In other words, even if people prefer the given satisfaction to be achieved as soon as possible ( $\rho>0$ ), the natural rate of interest might fall to zero provided that they expect to become poorer over time due to the exogenous fall in their income endowment. Owing to this anticipated decline in income, wants of higher intensity are expected to remain unsatisfied in the future. As a result, present goods might lose their superiority over future goods, and the real natural rate of interest could be easily depressed to zero as people are rushing to move their present income to the less abundant future via huge saving. ${ }^{209}$

It can be seen again that the time shape of income is crucial for the eventual level of the natural rate of interest. Even this extended general equilibrium model gives the result repeated many times before. At the same time, an incredibly deep insight in the works of early economists must be praised. Böhm-Bawerkian three causes of interest are still present in the modern economic growth models, although under the disguise of different names. Fisher's primacy of the flow of income is present here as well. It is absolutely fascinating that their intuition arrived at similar results as modern sophisticated models, whose solution requires many complicated steps and the employment of dynamic optimization techniques.

It should be stressed that the RCK model is also consistent with positive rate of interest and zero subjective discount rate. ${ }^{210}$ Consider again equation (48) and the condition for the convergence of lifetime utility A7_14. For the ZPG and $\rho=0$, the following condition must be satisfied: $\theta>1$ and $\mathrm{g}>0$. As was discussed in the discrete-time version, a positive growth in income endowment is required along with a relatively low elasticity of substitution. People will try to move their higher future income to the present via reduced saving if they prefer a smooth path of consumption $(\theta>1)$. Thus, the interest rate may increase to the positive region even if the subjective discount rate is zero. A relative abundance of goods in the future resulting from positive technological progress gives the present goods premium over future goods even though there is no underestimation of future wants. ${ }^{211}$ As a result, if the second Böhm-Bawerkian cause for interest is not present $(\rho=0)$, the first cause must be effective for a positive rate of interest to emerge (together with positive marginal productivity of capital).
As we can see again, the PTPT is a special case of a more general theory. The natural rate of interest at the steady state would be solely determined by the time preference (in sense two) if the income per person was stable ( $\mathrm{g}=0 \%$ ). This situation closely resembles the Misesian ERE (evenly rotating economy). However, if income varies over time, the subjective discount of future utilities is not the only determinant of the natural rate of interest.

Let us now analyse in more detail the behaviour of the natural rate of interest in an economy with non-constant income. Hayek (1941) in one of his most difficult work envisioned an idea

[^125]of dynamic equilibrium - equilibrium for a growing economy. In this particular respect, modern growth models are much closer to the Hayekian vision compared with the Misesian theory. Furthermore, the idea of dynamic equilibrium seems to be much closer to the real world economies since in normal times, they are growing. Thus, the assumption of the growing income endowment is of particular importance in the theory of interest. The PTPT authors neglected this very important aspect, which resulted in the fact that their approach seems to be rather incomplete.
The behaviour of the natural rate of interest in an economy with non-constant income is best understood when the economy is not at its steady state (or the balanced growth path - BGP). Yet, we will also consider an economy that is initially at the BGP, but then it is suddenly hit by a time-preference or a technological shock. We will discuss again the relative importance of time preference and productivity in determining the natural rate of interest. Before using the RCK model, however, we will introduce a very insightful approach developed by Hayek (1941).

Hayek (1941) spent many pages to investigate the relative importance of time preference and productivity in an economy that is accumulating capital. He used an ingenious extension of the Fisher model (see Figure No. 51). As we can see, his model contains indifference curves and investment opportunity lines. However, Hayek tried to make the Fisher model more dynamic, so he added a $45^{\circ}$ line representing the same amount of consumption in both periods. The axes represent any two consecutive periods $(t$ and $t+1) .{ }^{212}$ Furthermore, the curves are more consistent with a net concept rather than a gross concept. Thus, for example, a movement along the opportunity line portrays a net increase in future return.


Figure No. 51 Hayek's representation of the process of the accumulation of capital (1941:222).

[^126]Hayek assumed that for shorter periods, the investment opportunity line is less curved than the indifference curve. The reason lies in the fact that investment made within a relatively short period of time cannot much affect the schedule of the marginal productivity of capital as the investment is only a negligible part of the entire capital stock. On the other hand, the given amount of saving in the short period represents a relatively significant part of income in that period so "the sacrifice of successive parts of the income of this interval of time in the interests of the future will meet with a rapidly increasing resistance"(1941:233). ${ }^{213}$
Point $P$ represents an invariant flow of income earned only due to permanent resources (i.e. land and labour) without any use of capital. At this start, capital as a factor of production is very productive, so the investment opportunity curve has a large slope. In the first period, a sacrifice of present consumption may generate a very high increase in future output. For a reasonable rate of time preference (in sense two), the economy moves to point A. Due to a relatively low curvature of the investment opportunity curve, the interest rate is determined by the productivity of capital rather than by the time preference (in sense one). Time preference (in the sense one, i.e. MRS-1) will only adjust to the given rate of return.
To determine the position of the economy in the next period, let us shift the system one period forward (a movement from point A to point B). Note that as future becomes present, consumption is higher compared with the situation in which no permanent resources were used in the creation of capital (B versus P). Hayek assumed that the marginal productivity of capital gradually falls, so the next period investment curve is not as favourable as the previous one. As we can see, the economy finds the new equilibrium at point C , closer to the $45^{\circ}$ line.

This process might be repeated in further rounds. Along the path to the steady state equilibrium, the interest rate is determined by the productivity of capital. According to Hayek (1941:233), time preference will only affect the speed at which the capital will be accumulated. However, as the marginal product of capital gradually falls, the next period increments in output are still lower and lower. Finally, the process stops at point E. At this point, there is no net accumulation of capital, and consumption is stable over time. Furthermore, at this point and only at this point of long-run equilibrium, the natural rate of interest is determined by the time preference (the slope of the indifference curve at the $45^{\circ}$ line). It is also obvious that for more patient people, point E is posited further from the origin, and the process of capital accumulation will continue for a longer period of time.
As we can see, Hayek introduced a novel theory. Over the process of the accumulation of capital, the natural rate of interest depends on the productivity of capital. ${ }^{214}$ However, at the eventual steady state, it is determined solely by the time preference. What is even more interesting, the modern RCK model gives analogous predictions. If we look at the convergence of the economy in this model, the story seems to be very similar.

Figure No. 52 portrays a convergence process in the RCK model. For simplicity, we assume zero technological progress ( $\mathrm{g}=0 \%$ ). The economy starts with capital stock $\mathrm{k}(0)$ and consumption $c(0)$. It could have moved here after a sudden and unexpected decrease in the subjective discount rate. ${ }^{215}$ The shape of the saddle path along which the economy moves to the steady state and the speed of convergence depend on $\theta$ and $\rho$ - the time preference and the elasticity of substitution parameters, where the first parameter affects the slope of the indifference curve at the $45^{\circ}$ line and the second parameter the curvature of the indifference

[^127]curve at any point (apart from the $45^{\circ}$ line). Similar predictions were made by Hayek. In the end, the economy should reach the steady state level of the natural rate of interest $r^{*}=\rho$. Hence, in the steady state the natural rate of interest depends solely on the time preference (in sense two). However, along the convergence path, the interest rate behaves according to the diminishing marginal product of capital (Figure No. 52, panel b). As we can see, this behaviour is also in line with Hayek's model. As regards the evolution of consumption, it gradually increases as the economy accumulates more capital. In the eventual steady state, the accumulation stops, and consumption is stable. It is quite fascinating that there is virtually no difference in predictions of the intuitive approach of Hayek and this modern growth model. ${ }^{216}$


Figure No. 52 Convergence of the natural rate of interest in the RCK model after a decrease in $\rho$.

From equation (48), it is perfectly clear that the rate of technological progress positively affects the steady state level of the natural rate of interest. Let us now consider a sudden increase in the growth rate of technological progress from $\mathrm{g}_{1}$ to $\mathrm{g}_{2}$. It is obvious that the steady state level of the natural rate of interest rises. However, what is the evolution of this variable in the transition process? Figure No. 53 shows that the natural rate of interest will gradually increase to the new steady state level. As can be seen in panel (b), in the transition process, the demand for capital grows faster than the supply of capital. The key reason is that

[^128]the marginal product of capital grows immediately at the faster rate of $g_{2}$, whereas the supply of capital is driven up by a higher growth rate in income which only gradually rises to the new steady state (BGP) level of $g_{2}$. In other words, the growth rate in income (and saving) and the resulting growth rate in the supply of capital only gradually increase from $\mathrm{g}_{1}$ to $\mathrm{g}_{2} .{ }^{217}$


Figure No. 53 Convergence of the natural rate of interest in the RCK model after a sudden increase in the growth rate of technological progress.

Furthermore, it should be stressed that every intersection of the demand and supply in Figure No. 53 represents the equilibrium natural rate of interest $\mathrm{r}_{\mathrm{E}, \mathrm{t}}$ in that given period. The steady state level $\mathrm{r}^{*}$ is only one special equilibrium, which is characterised by the fact that it is invariant over time. One may say that $\mathrm{r}_{\mathrm{E}, \mathrm{t}}$ is a static equilibrium in the given period, whereas $\mathrm{r}^{*}$ is the very long-run dynamic equilibrium. However, any artificial attempt to narrow the difference between $\mathrm{r}_{\mathrm{E}, \mathrm{t}}$ and $\mathrm{r}^{*}$ at any moment in the transition period should result only in the disparity between demand and supply of capital and consequently in the misallocation of resources. More on this will be said in Chapter 4, which deals with business cycle considerations in a growing economy.
At the new steady state level, the natural rate of interest is higher $\mathrm{r}_{2}{ }^{*}=\rho+\theta \mathrm{g}_{2}$. Faster technological progress affects the new level of the natural rate of interest via two channels. The first one is the higher (growth rate of) productivity of capital. This can be identified with the third Böhm-Bawerkian cause for interest. However, the more fundamental is the channel of the first cause of interest. Higher growth rate of income-endowment $\left(\mathrm{g}_{2}\right)$ gives present goods higher premium over future goods owing to the abundance of future goods in the more remote periods. If the rate of technological progress suddenly rises, the determining factor in the transition process is the marginal product of capital. In the eventual steady state level, it is the time preference (in sense one, i.e. the joint influence of the first and the second cause of interest) that is of key importance. Nonetheless, as was remarked by Brown (1913), the increasing income-endowment associated with the first reason for interest is caused by the

[^129]growing productivity. Hence, it is the cooperation of the time preference and productivity that ultimately determines the natural rate of interest.

In other words, without the increasing productivity, the first cause will not be effective at the steady state of $\mathrm{r}^{*}$. Yet, it is quite problematic to attach this phenomenon to the third cause of interest as it mainly reflects higher (but diminishing) (marginal) productivity of longer methods of production. Thus, we may say that the role of productivity at the eventual steady state is performed through the first cause of interest, i.e. through a permanently increasing income endowment.
The final question, studied even by Fisher (1930) in great detail, is the evolution of the natural rate of interest when the increase in technologies takes the form of a one-time shock. In other words, instead of the growth rate, we will assume a sudden one-time increase in the level of technologies. ${ }^{218}$ The prediction of the RCK model is given in Figure No. 54. A higher level of technologies will immediately increase the marginal product of capital and consequently the natural rate of interest. However, the accumulation of capital resulting from higher income and saving should gradually decrease the natural rate of interest to the initial level. In the new steady state, the natural interest is again determined only by the subjective discount rate (i.e. time preference in sense two), not by the marginal productivity of capital as in the transition period. ${ }^{219}$


Figure No. 54 Evolution of the natural rate of interest in the RCK model after a sudden increase in the level of technologies A.

[^130]As was indicated in section 2.4, Fisher (1930) suggested that higher average income might be associated with lower impatience. He speculated that better technologies should eventually depress the natural rate of interest below its initial level due to improved living standards. This possibility is depicted by a dashed line in Figure No. 54. Higher income eventually decreases the subjective discount rate, which is the ultimate determinant of the natural rate of interest at the steady state. ${ }^{220}$ As a result, the dynamics of the natural rate of interest after a sudden increase in the level of technologies might be very complicated.
In the previous section, we noticed that the PTPT authors (Mises, Rothbard) deny that the productivity shock should have any influence on the rate of interest. However, in the present section, we demonstrated that the answer depends on the nature and permanence of the shock. If the productivity shock is a discrete positive jump, the natural rate of interest suddenly increases. It then gradually falls back to the level dictated by the time preference $r^{*}=\rho$. This decline, however, is most probably too slow compared with the beliefs of the Austrian authors. Thus, a sudden increase in the level of technologies (e.g. a new invention) keeps the real natural rate of interest higher for a considerable period of time.
Furthermore, if the shock to the technological progress takes the form of an increase in the growth rate $g$, then the impact on the natural rate of interest is permanent. This conclusion is at variance with the Mises and Rothbard theory. The reason lies in the fact that the Austrian authors neglected the influence of a growing income endowment on the rate of interest. In the case of higher $g$, the income endowment should grow at a higher rate, thus the real rate of interest must be affected permanently.
Even more complicated behaviour of the interest rate is obtained if we allow for a stochastic element in the model. Consider an economy without permanent technological progress (i.e. $\mathrm{g}=0 \%$ ) and with stationary population ( $\mathrm{n}=0 \%$ ). Suppose that the level of technologies follows a simple $\operatorname{AR}(1)$ or $\operatorname{AR}(2)$ process. The resulting behaviour not only of the natural rate of interest but also of other most important variables is presented in Appendix 7, section H. However, the most interesting conclusion is that the natural interest is strongly affected by the shocks to productivity.
In the previous sections, we stressed the fact that the analysis of interest must distinguish between the nominal approach and the real approach. The nominal approach is focused on the value difference between output and the expended inputs, whereas the centre of the real approach grounds in the exchange ratio between present goods and future goods. In the continuous infinite horizon model presented here, we have analysed the real natural rate of interest. However, the discussion of a simple model of the nominal interest rate from section 3.1 can be easily extended to this more complicated model.

The behaviour of the nominal rate of interest critically depends not only on the real interest rate but also on the evolution of prices. Prices are in turn affected by the development of output and the money supply. Here, we assume again constant money supply and the velocity of circulation. ${ }^{221}$ Thus, the behaviour of the price level will depend only on the growth rate of output. The exact size of the nominal interest rate can be found as the sum of the real interest rate and the inflation rate. Figures No. 21_A7 and 22_A7 in Appendix 7 demonstrate that when the economy is growing at a positive rate, the nominal rate of interest is lower than the real rate.

[^131]Furthermore, in the stochastic model, the nominal rate of interest seem to be much more volatile compared with the real interest rate as it depends not only on the real rate but also on the growth rate of the economy (Figure No. 31_A7).

Figures No. 8_A7, 9_A7 (and 21_A7, 22_A7), and Figures No. 16_A7 and 17_A7 in Appendix 7 show the evolution of the nominal interest rate and the real interest rate after the shock to the level (and the growth rate) of technology and the subjective discount rate. We have already seen that the steady state value of the real rate of interest might be zero (or negative) even in the RCK model. However, can this conclusion be applied also to the nominal rate of interest? In other words, can the value of output be permanently lower than the value of the expended inputs - a state that is absolutely unthinkable not only for the PTPT authors but also (to a lower extent) for modern mainstream economists due to the belief in the zero lower bound on nominal interest?

We will see that for constant money, the answer is definitely negative. In Appendix 7, we demonstrate that the Ramsey economy cannot be dynamically inefficient. At the steady state, the growth rate of output can never exceed the real rate of interest. The technical reason lies in the fact that the lifetime utility must not diverge, i.e. $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$ by assumption. The steady state level of the real interest rate is $r^{*}=\rho+\theta g$, which is definitely higher than the BGP growth rate of output $\mathrm{n}+\mathrm{g}$. The proof is very simple: If $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$, then $\rho+\theta \mathrm{g}>$ $\mathrm{n}+\mathrm{g}$. Thus, $\mathrm{r}^{*}>\mathrm{n}+\mathrm{g}$.
As a result, in the RCK model with the constant money supply, the nominal rate of interest is always positive because the real rate of interest is higher than the rate of price deflation $(\mathrm{n}+\mathrm{g}) .{ }^{222}$ There is always a positive difference between the value of output and the value of expended inputs. The zero bound can never be hit. This holds not only on the BGP but also on the entire saddle path. There might be an exception for the moment of a shock that suddenly and unexpectedly increases the growth rate of the economy. However, such a shock could not have been taken into account beforehand owing to its unpredictable nature, so the nominal rate of interest is most probably unaffected at that particular moment.

If we assume constant money and velocity, the nominal interest rate at the steady state in the RCK model can be determined as follows:

$$
\begin{equation*}
i^{*}=r^{*}+\pi \tag{49}
\end{equation*}
$$

Since $\mathrm{r}^{*}=\rho+\theta \mathrm{g}$ and $\pi=-[\mathrm{dY}(\mathrm{t}) / \mathrm{dt}] / \mathrm{Y}(\mathrm{t})=-(\mathrm{n}+\mathrm{g})$, equation (49) gives us:

$$
\begin{equation*}
i^{*}=\rho+\theta \cdot \mathrm{g}-\mathrm{n}-\mathrm{g} \tag{50}
\end{equation*}
$$

(50) may be written as:

$$
\begin{equation*}
i^{*}=\rho-n-(1-\theta) g>0 \tag{51}
\end{equation*}
$$

Thus, the nominal interest rate at steady state of the Ramsey model is positive even for constant money supply as long as the condition for the convergence of life-time utility holds. Surprisingly, it is even numerically equal to the specific combination of parameters required.

[^132]Furthermore, constant money, dynamically efficient economy and positive nominal interest rate are closely interconnected also in the RCK model. It is quite interesting that such a result is obtained again. Dynamic efficiency is in the first place a state that guarantees that consumption of one generation cannot be expanded without a sacrifice of consumption of some other generation. However, it also leads to a positive nominal interest and a positive difference between the value of output and the value of the expended inputs. This property is quite unexpected and novel. To the knowledge of the present author, it has been never mentioned by the PTPT authors or in the Austrian literature in general.
Moreover, equation (51) allows us to discuss whether changes in the growth rate of technologies will permanently affect the nominal interest rate, i.e. the value difference between output and the expended inputs. As can be seen, the answer critically depends on the value of the elasticity of substitution ( $1 / \theta$ ). For constant money and velocity, an increase in $g$ may raise the nominal interest rate if this elasticity is rather low (high $\theta$ ). On the other hand, a relatively low preference for consumption smoothing results in a decrease in the steady state nominal interest rate after the rise in $g$ (see Figure No. 22_A7 in Appendix 7). And finally, for the logarithmic utility function, the steady state nominal interest rate is not affected by the change in the rate of technological progress. Even though the transition period with a lower level of nominal interest seems to be significant, the Austrian pure time preference theory could be valid for this specific case if it is defined as a theory of the value difference between output and expended inputs (i.e. in terms of the nominal interest rate) and not as a theory of the intertemporal exchange ratio between present goods and future goods (i.e. in terms of the real interest rate). As has been seen in previous sections, only the logarithmic utility function seems to be favourable to the Austrian PTPT. The dynamic general equilibrium model from the present section confirms this conclusion. ${ }^{223}$
Nevertheless, all these properties critically depend on the assumption of the convergence of the lifetime utility: $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$. At first glance, such a condition seems to be rather technical without any economic background. However, let us now present the economic foundation of this assumption. With zero population growth and stationary technology (or with changing technology and unitary elasticity of substitution, i.e. $1 / \theta=1$ ), the RCK model requires a positive subjective discount rate. This means that people must discount future utilities. Such a requirement is perfectly consistent with the Misesian maxim that people prefer the given satisfaction to be delivered as soon as possible.
If the size of the representative dynasty grows over time ( $n>0$ ), the discount of future utilities must exceed the growth rate of the expansion of the family. However, even when the subjective discount rate is rather low (even zero), the important properties mentioned above might be obtained if the technological progress is fast enough and people have "drastically diminishing marginal utility", i.e. $\theta>1$. In other words, the non-divergence of the utility function, the dynamically efficient character of the model, and the resulting positivity of the nominal interest are guaranteed if the elasticity of substitution between present and future consumption is not very high. The economic explanation of that fact is as follows: Positive technological progress leads to an increasing time shape of the income stream. For a relatively high $\theta$, people prefer a smooth consumption path. Thus, higher income in the future compared with the present leads to the reduction in the saving rate (see the expression of the saving rate in Appendix 7, equation A7_79D). ${ }^{224}$ As a result, the real rate of interest must grow. It will grow high enough to guarantee dynamic efficiency in the first place and a positive nominal rate of interest in the second place.

[^133]On the other hand, a very high elasticity of substitution (i.e. low $\theta$ ) is not consistent with this model, unless the subjective discount rate is high enough ( $\rho \gg 0$ ) or the technological decay is fast enough ( $\mathrm{g} \ll 0$ ). ${ }^{225}$ Thus, we can see that the analysis of the RCK model closely resembles the discussion of the discrete-time model with constant MPK in section 3.1.3. Under normal conditions with positive technological progress, a positive nominal rate of interest (for constant MV) requires either a sufficient discount of future utilities or a low elasticity of substitution.

Furthermore, in the foregoing paragraphs we have shown that the real rate of interest in the RCK model might be zero on the balanced growth path even for a positive time preference (in sense two) provided that $\rho=-\theta \mathrm{g}$. Thus, to achieve zero real interest, the technological progress must be negative at the rate of $g=-\rho / \theta$. Nevertheless, even for constant money, the nominal rate of interest may be positive because the gradual fall in aggregate output will result in the positive inflation rate that will drive up the nominal rate of interest above zero. A graphical representation of this process is given in Appendix 7, Section G, Figure No. 28_A7.

At the end of this paper, we will present key characteristics that in normal times most probably prevail in the economy. First, the Misesian statement that people a priori prefer the given satisfaction to be achieved as soon as possible is represented in standard models by a positive subjective discount rate $\rho$. Its numerical value reflects the intensity of this preference. Second, the marginal productivity of capital must be decreasing at least from some point on the production function. All profit maximizing (non-monopolistic) firms must operate beyond this point (Strigl 2000).

If there was no technological progress, the economy would stabilize in the state in which the real rate of interest would be positive and equal to $\rho$. If money was constant, then prices would be stable, and the nominal rate of interest would coincide with the real rate of interest. Thus, the nominal interest would be definitely positive as well as the value difference between output and the expended inputs. In such a state, the Austrian analysis would be almost indistinguishable from the usual neoclassical theory since the Misesian idea of the ERE closely resembles the notion of stationary equilibrium.
However, a picture that is closer to the real world is more consistent with positive technological progress. In such a case, the economy should sooner or later find its balanced growth path. At this state, the real rate of interest depends not only on the subjective discount rate but also on the growth rate of income and the intertemporal elasticity of substitution in consumption. However, the real interest is definitely positive - "present goods are valued more than future goods of the same kind and number". If money is constant, prices must be naturally falling. Nevertheless, the nominal rate of interest along with the value difference between output and inputs are positive because the economy is dynamically efficient (real interest rate is higher than the growth rate in GDP). Yet its numerical value should be lower than that of the real rate due to the secular deflation of the price level. Even though it may be unthinkable for the majority of economists that prices should be falling in the period of economic growth, Chapter 4 shows that this might be a normal state of a prosperous economy in a dynamic general equilibrium.

[^134]
## 6. CONCLUSIONS

In this chapter, we tried to demonstrate that the pure time preference theory contains a fundamental inconsistency. Even though it might give sufficient reasons for the statement that individuals always prefer the given want to be gratified as soon as possible, this fact alone does not give present goods any superiority over future goods. In other words, it might be true that there is always an a priori positive difference in "the value assigned to want-satisfaction in the immediate future and the value assigned to want-satisfaction in remote periods of the future" (Mises 1996:526) in the minds of value-giving human beings. Yet, this implies neither that the originary interest is necessarily positive, nor that " $i l] t$ manifests itself in the market economy in the discount of future goods as against present goods" (ibid). Since goods may satisfy different wants in various periods and since it is goods, not human wants, that are being exchanged in the intertemporal markets, originary interest can be zero or even negative as there might exist no discount of future goods as against present goods.
A simple graphical and mathematical apparatus developed in this paper not only presented the Böhm-Bawerkian theory in a traditional neoclassical language, but it also demonstrated that the pure time preference theory is valid only under very specific conditions. If the inconsistency indicated in the pure time preference theory is truly present, the productivity element is not only important for the explanation of interest, but it can also be the only determinant of the rate of interest. This chapter tried to show that it is the objective exchange ratio between present goods and future goods, which we associated with the real interest rate, that is the fundamental and genuine centre in the theory of interest. This magnitude might be completely independent of the size of time preference of acting people.

Even though it is the subjective valuations of acting man that give value to present goods and future goods and that give the relative value to present goods as against future goods, the time preference is neither necessary nor sufficient for such valuations. If more is preferred to less and if the marginal utility is diminishing, the objective element of productivity might be the sole determinant of not only the size of interest but also of the emergence of the interest phenomenon as such.
It seems that the defence of the pure time preference theory that arose in the literature put too much emphasis on nominal variables. In other words, the pure time preference theorist considered mainly the interest on money, or the value difference between output and expended inputs. We tried to show that such an approach is not accurate as the nominal variables are always derived from real variables. As a result, any sound general theory must always put aside the veil of nominal variables and focus on the explanation of real phenomena.
The interest on money so heavily stressed by some Austrians is then derived from real phenomena after the introduction of money into the given model. We demonstrated that in the two-period model with constant productivity of capital and constant money supply, a positive value difference between output and expended inputs might emerge with no reference to the size of the time preference provided that the economy is dynamically efficient. This condition is guaranteed if the discount of future utilities is large enough compared with the constant marginal productivity of capital and/or if the elasticity of substitution is sufficiently low.

Hence, one cannot escape the conclusion that the pure time preference theory is only a special case of a more general neoclassical theory that incorporates not only the inherent tendency of people to gratify wants as soon as possible but also the flow of their income over time that might be critically dependent on the objective element of productivity.
As we have seen, the conclusions about insufficiency of the pure time preference theory were valid also in the continuous time and in the infinite horizon model. As a result, the critique
seems to be rather general, even though the models here presented are still very simple as regards their structure and the number of variables they took into account.

This chapter may also serve as a first step to integrate the Austrian theory in a wider neoclassical framework. It must be stressed that it was the neoclassical theory itself that developed from the Austrian origins. The analysis showed that both schools of economic thought can be reconciled at least in the theory of interest if it is honestly admitted that the Austrian theory is only a special case of a more general framework. However, the ideas presented here provided a natural basis for more complex research that may include not only a productivity element but also a time horizon that is extended to any (even infinite) number of periods.

## Appendix 1 Condition for Convex Indifference Curves

In section 2.3 in the main text we operate with convex, downward sloping indifference curves. In this appendix, we will derive a condition that is required for such a specific shape. Consider a lifetime utility function of a representative agent who lives for two periods.
$U=U\left(C_{0}, C_{1}\right)$
This utility function has usual properties; it is increasing in both $\mathrm{C}_{0}$ and $\mathrm{C}_{1}$, but marginal utility of $\mathrm{C}_{0}$ and $\mathrm{C}_{1}$ is diminishing:

$$
\begin{align*}
& \frac{\partial U\left(C_{0}, C_{1}\right)}{\partial C_{0}} \equiv U_{0}>0  \tag{A1_2a}\\
& \frac{\partial^{2} U(\cdot)}{\partial C_{0}^{2}} \equiv U_{00}<0  \tag{A1_2b}\\
& \frac{\partial U(\cdot)}{\partial C_{1}} \equiv U_{1}>0  \tag{A1_2c}\\
& \frac{\partial^{2} U(\cdot)}{\partial C_{1}^{2}} \equiv U_{11}<0 \tag{A1_2d}
\end{align*}
$$

Any indifference curve represents combinations of present and future consumption giving the same level of utility: ${ }^{226}$

$$
\begin{equation*}
U\left(C_{0}, C_{1}\right)=\bar{U} \tag{A1_3}
\end{equation*}
$$

The slope of the indifference curve may be represented by $\mathrm{dC}_{1} / \mathrm{dC}_{0}$ keeping U constant. Since $U_{1}$ is positive by assumption (i.e. non-zero), we may apply the implicit function theorem to express the magnitude of $\mathrm{dC}_{1} / \mathrm{dC}_{0}$ at any given point on the indifference curve (i.e. the slope of the intertemporal indifference curve):

$$
\begin{equation*}
\frac{d C_{1}}{d C_{0}}=-\frac{U_{0}}{U_{1}}<0 \tag{A1_4}
\end{equation*}
$$

Since $U_{0}$ and $U_{1}$ are positive, the indifference curves are downward sloping. Let us define the marginal rate of substitution of future consumption for present consumption, MRS $\mathrm{Cl}_{\mathrm{C}, \mathrm{C} 0}$ :

$$
\begin{equation*}
M R S_{C 1, C 0} \equiv \frac{U_{0}}{U_{1}}=-\left.\frac{d C_{1}}{d C_{0}}\right|_{U_{\_} \text {constant }} \tag{A1_5}
\end{equation*}
$$

Obviously, the MRS can be considered as a ratio of marginal utilities. We require that the MRS declines along the indifference curve in the southeastern direction, i.e. the MRS falls with greater present consumption and lower future consumption:

$$
\begin{equation*}
\frac{d M R S}{d C_{0}}<0 \tag{A1_6}
\end{equation*}
$$

[^135]$\frac{d^{2} C_{1}}{d C_{0}^{2}}>0$
Condition (A1_7) implies that the indifference curve is strictly convex, and the utility function $\mathrm{U}\left(\mathrm{C}_{0}, \mathrm{C}_{1}\right)$ is therefore strictly quasi-concave. Condition (A1_7) is satisfied for: ${ }^{227}$
$\frac{d^{2} C_{1}}{d C_{0}^{2}}=-\frac{d\left(\frac{U_{0}\left(C_{0}, C_{1}\left(C_{0}\right)\right)}{U_{1}\left(C_{0}, C_{1}\left(C_{0}\right)\right)}\right)}{d C_{0}}=-\frac{\left(U_{00}+U_{10} \frac{d C_{1}}{d C_{0}}\right) \cdot U_{1}-U_{0}\left(U_{01}+U_{11} \frac{d C_{1}}{d C_{0}}\right)}{U_{1}^{2}}$
Using (A1_4):
$\frac{d^{2} C_{1}}{d C_{0}^{2}}=-\frac{\left[U_{00}+U_{10}\left(-\frac{U_{0}}{U_{1}}\right)\right] \cdot U_{1}-U_{0}\left[U_{01}+U_{11}\left(-\frac{U_{0}}{U_{1}}\right)\right]}{U_{1}^{2}}$

Since we assume that $U(\cdot)$ is twice continuously differentiable, we may apply Young's theorem and write $\mathrm{U}_{10}=\mathrm{U}_{01}$. Hence, ( $\mathrm{A} 1 \_8 b$ ) gives us (after multiplying both numerator and denominator by $\mathrm{U}_{1}$ ):
$\frac{d^{2} C_{1}}{d C_{0}^{2}}=-\frac{U_{00} U_{1}^{2}-U_{01} U_{0} U_{1}-U_{01} U_{0} U_{1}+U_{11} U_{0}^{2}}{U_{1}^{3}}$
Due to the condition (A1_7), (A1_8c) implies:
$\frac{U_{00} U_{1}^{2}-2 U_{01} U_{0} U_{1}+U_{11} U_{0}^{2}}{U_{1}^{3}}<0$
Since the denominator in (A1_9) is always positive, (A1_10) must apply:
$U_{00} U_{1}^{2}+U_{11} U_{0}^{2}<2 U_{01} U_{0} U_{1}$
(A1_10) states that if (A1_2a) - (A1_2d) hold, $\mathrm{U}_{01}=\mathrm{U}_{10}$ must not be too negative. In economic terms, the marginal utility of present consumption good may not (too much) decrease with more units of future consumption goods and vice versa. However, it is quite difficult to imagine such a situation in which the schedule of the marginal utility of consumption of present consumption goods would be negatively related to the amount of future consumption goods.

Yet, this theoretical situation is plotted in Figure No. 1_A1. In this figure, greater amount of future consumption $\left(\mathrm{C}_{1}{ }^{\mathrm{Z}}>\mathrm{C}_{1}{ }^{\mathrm{Y}}>\mathrm{C}_{1}{ }^{\mathrm{X}}\right)$ ) reduces marginal utility from present consumption and vice versa $\left(\mathrm{C}_{0}{ }^{\mathrm{Z}}>\mathrm{C}_{0}{ }^{\mathrm{Y}}>\mathrm{C}_{0} \mathrm{X}\right)$. Here, we encounter an interesting phenomenon. Indifference curves are decreasing (see Figure No. 2_A1) since marginal utility schedules of $\mathrm{C}_{0}$ and $\mathrm{C}_{1}$ are both diminishing (see Figure No. $1 \_$A1). However, they are not

[^136]convex to the origin due to the violation of condition (A1_10). In such a case, the utility function is not quasi-concave, it is quasi-convex. As a result, if we added the intertemporal budget constraint (IBC), the resulting tangent point would not represent an intertemporal optimum of the consumer (see Figure No. 2_A1). As can be seen, attainable point D would give much higher utility than point $B$ even though the tangent rule applies to point $B$ and not to point D .


Figure No. 1_A1, The response of marginal utility of $\mathrm{C}_{\mathrm{i}}$ to the change in $\mathrm{C}_{\mathrm{j}}$..


Figure No. 2_A1 ... that would not generate convex indifference curves

Appendix 2 Intertemporal elasticity of substitution in consumption of the CRRA utility function

Generally, the elasticity of substitution has the following formula:
$\sigma_{Y, X}=\frac{\% \Delta \frac{Y}{X}}{\% \Delta M R S_{Y, X}}=\frac{\frac{d \frac{Y}{X}}{\frac{Y}{X}}}{\frac{d M R S_{Y, X}}{M R S_{Y, X}}}=\frac{d \frac{Y}{X}}{d M R S_{Y, X}} \frac{M R S_{Y, X}}{\frac{Y}{X}}$
Regarding the analysis of the intertemporal choice, the intertemporal elasticity of substitution in consumption might be expressed as:

$$
\begin{equation*}
\sigma_{C_{-} t+1, C_{-} t}=\frac{\% \Delta \frac{C_{t+1}}{C_{t}}}{\% \Delta M R S_{C_{-} t+1, C_{-} t}}=\frac{\frac{d \frac{C_{t+1}}{C_{t}}}{\frac{C_{t+1}}{C_{t}}}}{\frac{d M R S}{M R S}}=\frac{d \frac{C_{t+1}}{C_{t}}}{d M R S} \frac{M R S}{\frac{C_{t+1}}{C_{t}}} \tag{A2_2}
\end{equation*}
$$

The MRS for the CRRA (or CIES as Barro called it) is (see equations 10-12 in the main text):
$M R S=\left(\frac{C_{t+1}}{C_{t}}\right)^{\theta}(1+\rho)$
Thus, the ratio of consumption levels in (A2_3) might be expressed as:

$$
\begin{equation*}
\frac{C_{t+1}}{C_{t}}=\left(\frac{M R S}{1+\rho}\right)^{1 / \theta} \tag{A2_4}
\end{equation*}
$$

By applying (A2_2), we get:
$\sigma=\frac{d \frac{C_{t+1}}{d M R S}}{\frac{M R S}{C_{t+1}}}=\left[\frac{1}{\theta}\left(\frac{M R S}{1+\rho}\right)^{(1 / \theta)-1} \frac{1}{1+\rho}\right] \frac{M R S}{\frac{C_{t+1}}{C_{t}}}=\frac{1}{\theta} \frac{\left(\frac{M R S}{1+\rho}\right)^{(1 / \theta)}}{M R S} \frac{1+\rho}{1+\rho} \frac{M R S}{\frac{C_{t+1}}{C_{t}}}=\frac{1}{\theta}$

As can be seen, the elasticity of substitution is constant $1 / \theta$. For the case of perfect substitutes, $\theta \rightarrow 0$, the elasticity is infinite. For perfect complements, $\theta \rightarrow \infty$, the elasticity is zero. Furthermore, unitary elasticity is reached for the logarithmic utility function $(\theta \rightarrow 1)$.

More technical representation of $\sigma$ can be expressed as follows: It measures the percentage change in the ratio between consumption at time $t+1$ and at time $t$ (say future and present consumption) due to a one percent change in the MRS. Thus, $\theta$ measures the inverse of this. It represents the constant percentage change in the MRS resulting from a one percent change in the ratio between consumption at time $t+1$ and at time $t$. Hence, the name constant intertemporal elasticity of substitution (CIES) is appropriate for the utility function (10) in the main text.

Furthermore, $\theta$ might be geometrically represented as follows: By what percent the slope of the indifference curve changes, if the slope of the ray coming from the origin changes by one percent. Figure No. 1_A2 illustrates this idea for two magnitudes of $\theta$. In both pictures, the percentage change in the ratio between future and present consumption is the same. However, for the case of very low $\theta$, the MRS is almost unaffected, so the intertemporal substitution is very large (panel a), whereas for high $\theta$ the MRS dramatically drops (panel b). ${ }^{228}$


Figure No. 1_A2, Geometric representation of the intertemporal substitution in consumption; the impact of $\theta$.

Appendix 3 Proof that for $\theta=1$ the utility function is logarithmic
It can be easily shown that if $\theta=1$, the instantaneous utility function (10) from the main text is logarithmic. First, plug unitary $\theta$ to (10):

$$
\begin{equation*}
\lim _{\theta \rightarrow 1} \frac{C^{1-\theta}}{1-\theta}=\frac{1}{0} \tag{A3_1}
\end{equation*}
$$

Expression (A3_1) makes no sense in the theory of consumer, hence (9) must be modified to a more tractable form:

$$
\begin{equation*}
u(C)=\frac{C^{1-\theta}}{1-\theta} \rightarrow v(C)=\frac{C^{1-\theta}}{1-\theta}-\frac{1}{1-\theta} \tag{A3_2}
\end{equation*}
$$

[^137]$v(C)$ is a simple monotonic transformation of $u(C)$ that represents the same preferences as $u(C)$. It can be easily shown that the MRS is identical for both forms. ${ }^{229}$ Thus, the behavior of the consumer is unaffected by this transformation.

By inserting $\theta=1$ into (A3_2), we get:

$$
\begin{equation*}
\lim _{\theta \rightarrow 1} v(C)=\lim _{\theta \rightarrow 1} \frac{C^{1-\theta}-1}{1-\theta}=\frac{0}{0} \tag{A3_3}
\end{equation*}
$$

We may apply the L'Hospital rule:

$$
\begin{equation*}
\lim _{\theta \rightarrow 1} v(C)=\lim _{\theta \rightarrow 1} \frac{C^{1-\theta}-1}{1-\theta}=\lim _{\theta \rightarrow 1} \frac{\frac{d\left(C^{1-\theta}-1\right)}{d \theta}}{\frac{d(1-\theta)}{d \theta}}=\lim _{\theta \rightarrow 1} \frac{(-1) C^{1-\theta} \ln C}{-1}=\ln C \tag{A3_4}
\end{equation*}
$$

As a result, preferences in (A3_1) can be also represented by a logarithmic utility function.

## Appendix 4

In this Appendix, we will solve the optimization problem of a representative shipwrecked sailor from section 3, who lives for two periods. His budget constraint in the first period is represented by equation (A4_1):
$A=C_{0}+S_{0}$
This equation states that the initial stock of hard-tacks A can be used either for present consumption $\mathrm{C}_{0}$ or some part can be also saved. Saved hard-tacks might be lent to somebody else for the interest rate r . The budget constraint in the next period, which assumes no debts or assets at the end of his life, is as follows:
$C_{1}=S_{0}(1+r)$
Thus, the only source for consumption in the next period is the amount of saving from the previous period increased by the accrued interest (if it exists). Inserting the budget constraint of the present period (A4_1) to the budget constraint of the future period (A4_2) will give us his intertemporal budget constraint:

$$
\begin{equation*}
C_{1}=\left(A-C_{0}\right)(1+r) \tag{A4_3}
\end{equation*}
$$

Rearranging the terms gives us the key idea that the flow of consumption in the present value may not exceed the initial endowment of hard-tacks:
$C_{0}+\frac{C_{1}}{1+r}=A$

[^138]His objective is to find the optimum path of consumption so as to maximize his lifetime utility function (A4_5):
$U=u\left(C_{0}\right)+\frac{u\left(C_{1}\right)}{1+\rho}$
subject to his intertemporal budget constraint (A4_4). Let us set up a simple Lagrange function:

$$
\begin{equation*}
L=u\left(C_{0}\right)+\frac{u\left(C_{1}\right)}{1+\rho}+\lambda\left(A-C_{0}-\frac{C_{1}}{1+r}\right) \tag{A4_6}
\end{equation*}
$$

The first order conditions for optimum consumption are: ${ }^{230}$
FOC:

$$
\begin{align*}
& \frac{\partial L}{\partial C_{0}}=u^{\prime}\left(C_{0}\right)-\lambda=0  \tag{A4_7}\\
& \frac{\partial L}{\partial C_{1}}=\frac{u^{\prime}\left(C_{1}\right)}{1+\rho}-\lambda \frac{1}{1+r}=0  \tag{A4_8}\\
& \frac{\partial L}{\partial \lambda}=A-C_{0}-\frac{C_{1}}{1+r}=0 \tag{A4_9}
\end{align*}
$$

From (A4_7) and (A4_8), it is perfectly clear that the optimum is where the marginal rate of substitution in consumption MRS is equal to the interest factor ( $1+\mathrm{r}$ ), or that the marginal rate of time preference $\varepsilon=$ MRS -1 is equal to the interest rate:

$$
\begin{align*}
& \frac{u^{\prime}\left(C_{1}\right)}{1+\rho}=u^{\prime}\left(C_{0}\right) \frac{1}{1+r}  \tag{A4_10}\\
& \frac{u^{\prime}\left(C_{0}\right)}{\frac{u^{\prime}\left(C_{1}\right)}{1+\rho}} \equiv M R S=1+r  \tag{A4_11}\\
& \frac{u^{\prime}\left(C_{0}\right)}{\frac{u^{\prime}\left(C_{1}\right)}{1+\rho}}-1 \equiv M R S-1 \equiv \varepsilon=r \tag{A4_12}
\end{align*}
$$

From (A4_11) the Euler equation can be easily derived:

[^139]\[

$$
\begin{equation*}
\frac{u^{\prime}\left(C_{0}\right)}{u^{\prime}\left(C_{1}\right)}=\frac{1+r}{1+\rho} \tag{A4_13}
\end{equation*}
$$

\]

For the CRRA, (A4_11) turns out to be:

$$
\begin{equation*}
M R S \equiv \frac{\left(\frac{1}{C_{0}}\right)^{\theta}}{\frac{\left(\frac{1}{C_{1}}\right)^{\theta}}{(1+\rho)}}=1+r \tag{A4_14}
\end{equation*}
$$

$\left(\frac{C_{1}}{C_{0}}\right)^{\theta}(1+\rho)=1+r$

And the corresponding Euler equation is:

$$
\begin{equation*}
\frac{C_{1}}{C_{0}}=\left(\frac{1+r}{1+\rho}\right)^{1 / \theta} \tag{A4_16}
\end{equation*}
$$

## Appendix 5



Figure No. 1_A5, Nominal rate of interest for various $\rho$ and $\theta$.
Note: $r=$ MPK $=10 \%$ by assumption, money supply and velocity are constant.


Figure No. 2_A5, Nominal rate of interest and the growth rate in output for $\rho=0 \%$ Note: $r=$ MPK $=10 \%$ by assumption, money supply and velocity are constant.


Figure No. 3_A5, Nominal rate of interest and the growth rate in output for $\rho=5 \%$ Note: $r=$ MPK $=10 \%$ by assumption, money supply and velocity are constant.


Figure No. 4_A5, Nominal rate of interest and the growth rate in output for $\rho=10 \%$ Note: $r=$ MPK $=10 \%$ by assumption, money supply and velocity are constant.


Figure No. 5_A5, Nominal rate of interest and the growth rate in output for $\rho=15 \%$ Note: $r=$ MPK $=10 \%$ by assumption, money supply and velocity are constant.


Figure No. 6_A5, Subjective discount rate required for positive nominal rate of interest Note: Money supply and velocity are constant by assumption.

## Appendix 2 B - Natural Rate of Interest in the Economy with the Given Flow of Income

Consider a representative consumer A maximizing his life-time utility in a simple two-period model (Equation 1). For simplicity, assume that $\theta=1$, hence the utility function is logarithmic. Equation (2) represents his intertemporal budget constraint. $\mathrm{Y}_{0}$ and $\mathrm{Y}_{1}$ stand for his (labour) income in the present and in the future.

$$
\begin{align*}
& U^{A}=\ln C_{0}^{A}+\frac{1}{1+\rho_{A}} \ln C_{1}^{A}  \tag{1}\\
& C_{0}^{A}+\frac{1}{1+r} C_{1}^{A}=Y_{0}^{A}+\frac{1}{1+r} Y_{1}^{A} \tag{2}
\end{align*}
$$

Set up a simple Lagrangian function and solve for the first order conditions (FOC).

$$
\begin{equation*}
L=\ln C_{0}^{A}+\frac{1}{1+\rho_{A}} \ln C_{1}^{A}+\lambda\left(Y_{0}^{A}+\frac{1}{1+r} Y_{1}^{A}-C_{0}^{A}-\frac{1}{1+r} C_{1}^{A}\right) \tag{3}
\end{equation*}
$$

FOC:

$$
\begin{align*}
& \frac{\partial L}{\partial C_{0}^{A}}=\frac{1}{C_{0}^{A}}-\lambda=0  \tag{4}\\
& \frac{\partial L}{\partial C_{1}^{A}}=\frac{1}{1+\rho_{A}} \frac{1}{C_{1}^{A}}-\lambda \frac{1}{1+r}=0 \tag{5}
\end{align*}
$$

$$
\begin{equation*}
\frac{C_{1}^{A}}{C_{0}^{A}}=\frac{1+r}{1+\rho_{A}} \tag{6}
\end{equation*}
$$

Equation (6) represents the Euler equation for this problem. It describes the optimal allocation of consumption over time. Substituting (6) into the IBC (eq. 2) and after simple manipulations, we get the optimum consumption in the present and in the future (eq. 7 and 8):

$$
\begin{align*}
& C_{0}^{A *}=\frac{(1+r) Y_{0}^{A}+Y_{1}^{A}}{1+r} \frac{1+\rho_{A}}{2+\rho_{A}}  \tag{7}\\
& C_{1}^{A *}=\frac{(1+r) Y_{0}^{A}+Y_{1}^{A}}{2+\rho_{A}} \tag{8}
\end{align*}
$$

Suppose for simplicity that there are only two individuals in the economy (or two groups of representative individuals). They differ in their income streams and their subjective discount rates. We could of course extend the analysis by including $n$ individuals. However, this will only complicate things without giving more insight that might be obtained even with a simple example with two individuals. Thus, the optimum of individual B is described by similar equations as in (7) and (8).
Equations (9) and (10) characterize resource constraints in the economy in the present and in the future. Equation (9) basically states that the aggregate consumption at time 0 may not exceed the aggregate income at time 0 . An alternative interpretation is that saving/borrowing of A must be equal to borrowing/saving of B. In other words, in the endowment economy without investment opportunities, aggregate saving must be equal to zero. Equation (10) is a corresponding aggregate constraint in the future. Both constraints might be easily constructed for $n$ individuals, yet we will adhere to a simple 2-person model.

$$
\begin{align*}
& C_{0}^{A} *+C_{0}^{B *}=Y_{0}^{A}+Y_{0}^{B}  \tag{9}\\
& C_{1}^{A} *+C_{1}^{B *}=Y_{1}^{A}+Y_{1}^{B} \tag{10}
\end{align*}
$$

Our system consists of 5 unknowns ( $\mathrm{C}_{0}{ }^{\mathrm{A}^{*}}, \mathrm{C}_{1} \mathrm{~A}^{\mathrm{A}^{*}}, \mathrm{C}_{0}{ }^{\mathrm{B}^{*}}, \mathrm{C}_{1}{ }^{\mathrm{B}^{*}}, \mathrm{r}$ ) and 6 equations ( 7 and 8 both for A and B , and 9 and 10). Thus, one equation is not independent. Let us use (10) and substitute optimum consumption levels from equation (8) for both individuals. This yields:

$$
\begin{gather*}
\frac{Y_{0}^{A}+r Y_{0}^{A}+Y_{1}^{A}}{2+\rho_{A}}+\frac{Y_{0}^{B}+r Y_{0}^{B}+Y_{1}^{B}}{2+\rho_{B}}=Y_{1}^{A}+Y_{1}^{B} \\
\left(Y_{0}^{A}+r Y_{0}^{A}+Y_{1}^{A}\right)\left(2+\rho_{B}\right)+\left(Y_{0}^{B}+r Y_{0}^{B}+Y_{1}^{B}\right)\left(2+\rho_{A}\right)=\left(Y_{1}^{A}+Y_{1}^{B}\right)\left(2+\rho_{A}\right)\left(2+\rho_{B}\right) \\
r\left[Y_{0}^{A}\left(2+\rho_{B}\right)+Y_{0}^{B}\left(2+\rho_{A}\right)\right]=\left(Y_{1}^{A}+Y_{1}^{B}\right)\left(2+\rho_{A}\right)\left(2+\rho_{B}\right)-\left(Y_{0}^{A}+Y_{1}^{A}\right)\left(2+\rho_{B}\right)-\left(Y_{0}^{B}+Y_{1}^{B}\right)\left(2+\rho_{A}\right) \tag{13}
\end{gather*}
$$

$$
\begin{equation*}
r=\frac{\left(Y_{1}^{A}+Y_{1}^{B}\right)\left(2+\rho_{A}\right)\left(2+\rho_{B}\right)-Y_{1}^{A}\left(2+\rho_{B}\right)-Y_{1}^{B}\left(2+\rho_{A}\right)-Y_{0}^{A}\left(2+\rho_{B}\right)-Y_{0}^{B}\left(2+\rho_{A}\right)}{Y_{0}^{A}\left(2+\rho_{B}\right)+Y_{0}^{B}\left(2+\rho_{A}\right)} \tag{14}
\end{equation*}
$$

$r=\frac{\left(Y_{1}^{A}+Y_{1}^{B}\right)\left(2+\rho_{A}\right)\left(2+\rho_{B}\right)-Y_{1}^{A}\left(2+\rho_{B}\right)-Y_{1}^{B}\left(2+\rho_{A}\right)}{Y_{0}^{A}\left(2+\rho_{B}\right)+Y_{0}^{B}\left(2+\rho_{A}\right)}-1$

$$
\begin{equation*}
r=\frac{Y_{1}^{A}\left(2+2 \rho_{A}+\rho_{B}+\rho_{A} \rho_{B}\right)+Y_{1}^{B}\left(2+\rho_{A}+2 \rho_{B}+\rho_{A} \rho_{B}\right)}{Y_{0}^{A}\left(2+\rho_{B}\right)+Y_{0}^{B}\left(2+\rho_{A}\right)}-1 \tag{16}
\end{equation*}
$$

As can be seen, equilibrium real interest rate $r$ rises with higher future income $\left(\mathrm{Y}_{1}{ }^{\mathrm{A}}\right.$ or $\left.\mathrm{Y}_{1}{ }^{\mathrm{B}}\right)$, lower present income ( $\mathrm{Y}_{0}{ }^{A}$ or $\mathrm{Y}_{0}{ }^{\mathrm{B}}$ ) and higher subjective discount rates ( $\rho_{A}$ or $\rho_{B}$ ). If we substitute r into (7) and compare $\mathrm{C}_{0} *$ with $\mathrm{Y}_{0}$, we can decide whether the given individual is a lender or a borrower.

Furthermore, natural real interest rate may fall below zero if the future income of individuals is relatively low compared with the present income. Hence, $r<0$ if:

$$
\begin{equation*}
Y_{1}^{A}\left(2+2 \rho_{A}+\rho_{B}+\rho_{A} \rho_{B}\right)+Y_{1}^{B}\left(2+\rho_{A}+2 \rho_{B}+\rho_{A} \rho_{B}\right)<Y_{0}^{A}\left(2+\rho_{B}\right)+Y_{0}^{B}\left(2+\rho_{A}\right) \tag{17}
\end{equation*}
$$

A) For a constant flow of income, the same income for all individuals $\left(Y_{0}{ }^{A}=Y_{1}{ }^{A}=Y_{0}{ }^{B}=Y_{1}{ }^{B}\right.$ $=\mathrm{Y}$ ), and for the identical subjective discount rate ( $\rho_{\mathrm{A}}=\rho_{\mathrm{B}}=\rho$ ), (16) will turn into:

$$
\begin{align*}
& r=\frac{Y\left(2+2 \rho+\rho+\rho^{2}\right)+Y\left(2+2 \rho+\rho+\rho^{2}\right)}{Y(2+\rho)+Y(2+\rho)}-1  \tag{18}\\
& r=\frac{2\left(2+3 \rho+\rho^{2}\right)}{2(2+\rho)}-1  \tag{19}\\
& r=\frac{2+3 \rho+\rho^{2}-2-\rho}{2+\rho}  \tag{20}\\
& r=\frac{\rho(2+\rho)}{2+\rho}  \tag{21}\\
& r=\rho \tag{22}
\end{align*}
$$

Thus, for constant and identical income for all individuals and identical subjective discount rate (i.e. for a homogenous agent model in stationary conditions) the equilibrium real rate of interest is solely determined by the rate of time preference (in sense two) and it cannot fall below zero, unless $\rho$ is negative. The level of income plays no role if $\rho$ itself is taken as an exogenous constant that does not depend on the size of Y.

Figure No. 1_A2 demonstrates how the new equilibrium $r$ is established after an increase in the subjective discount rate. Higher $\rho$ makes the indifference curve steeper at the $45^{\circ}$ line (panel a). The new optimum lies to the right of the original one. This, however, creates an excess of demand for present goods over their supply ( $\mathrm{C}_{0}{ }^{*}-\mathrm{Y}$ ). Thus, the interest rate must go up to equalize the demand and supply of present goods. The new equilibrium is depicted in panel (b). The new budget line is steeper, reflecting higher interest rate.


Figure No. 1_A2 Increase in the subjective discount rate will lead to a higher interest rate.
B) If the subjective discount rate is the same for all individuals ( $\rho_{A}=\rho_{B}=\rho$ ), if all have a constant flow of income but of different size (i.e. $\mathrm{Y}_{0}{ }^{\mathrm{A}}=\mathrm{Y}_{1}{ }^{\mathrm{A}}=\mathrm{Y}^{\mathrm{A}}$ and $\mathrm{Y}_{0}{ }^{\mathrm{B}}=\mathrm{Y}_{1}{ }^{\mathrm{B}}=\mathrm{Y}^{\mathrm{B}}$ ), (16) might be written as:

$$
\begin{align*}
& r=\frac{Y^{A}\left(2+2 \rho+\rho+\rho^{2}\right)+Y^{B}\left(2+2 \rho+\rho+\rho^{2}\right)}{Y^{A}(2+\rho)+Y^{B}(2+\rho)}-1  \tag{23}\\
& r=\frac{\left(Y^{A}+Y^{B}\right)\left(2+3 \rho+\rho^{2}\right)-\left(Y^{A}+Y^{B}\right)(2+\rho)}{\left(Y^{A}+Y^{B}\right)(2+\rho)}  \tag{24}\\
& r=\frac{\rho(2+\rho)}{(2+\rho)}  \tag{25}\\
& r=\rho \tag{26}
\end{align*}
$$

As we can see, even if people have different size of income, its constancy over time leads to the fact that the equilibrium rate of interest depends only on the subjective discount rate.
C) If the subjective discount rate is the same for all individuals $\left(\rho_{A}=\rho_{B}=\rho\right)$, but their flows of income differ being of any shape, (16) is modified to:

$$
\begin{align*}
& r=\frac{Y_{1}^{A}\left(2+2 \rho+\rho+\rho^{2}\right)+Y_{1}^{B}\left(2+2 \rho+\rho+\rho^{2}\right)}{Y_{0}^{A}(2+\rho)+Y_{0}^{B}(2+\rho)}-1  \tag{27}\\
& r=\frac{\left(Y_{1}^{A}+Y_{1}^{B}\right)(2+\rho)(1+\rho)}{\left(Y_{0}^{A}+Y_{0}^{B}\right)(2+\rho)}-1  \tag{28}\\
& r=\frac{\left(Y_{1}^{A}+Y_{1}^{B}\right)(1+\rho)}{\left(Y_{0}^{A}+Y_{0}^{B}\right)}-1 \tag{29}
\end{align*}
$$

By substituting $r$ to (7) we can determine whether the particular individual is a debtor or a creditor. Debtors (e.g. individuals A) are characterized by the condition that the growth rate of their income stream $\left(\mathrm{Y}_{1}{ }^{\mathrm{A}} / \mathrm{Y}_{0}{ }^{\mathrm{A}}-1\right)$ is higher than the growth rate of the income stream of creditors $\left(Y_{1}{ }^{\mathrm{B}} / \mathrm{Y}_{0}{ }^{\mathrm{B}}-1\right)$.
Because $\left(\mathrm{Y}_{0}{ }^{\mathrm{A}}+\mathrm{Y}_{0}{ }^{\mathrm{B}}\right)$ and $\left(\mathrm{Y}_{1}{ }^{\mathrm{A}}+\mathrm{Y}_{1}{ }^{\mathrm{B}}\right)$ are equal to the aggregate income in the economy in the given period, i.e. $\mathrm{Y}_{0}$ and $\mathrm{Y}_{1}$ respectively, (29) might be written as:

$$
\begin{equation*}
r=\frac{Y_{1}(1+\rho)}{\mathrm{Y}_{0}}-1 \tag{30}
\end{equation*}
$$

As in (16), the equilibrium real rate of interest is positively related to future income and the subjective discount rate and negatively related to present income. Furthermore, negative real rate of interest is possible (see 30 and 31) if the ratio of present income to future income is greater than $(1+\rho)$ or alternatively, if the ratio of future income to present income is lower than the subjective discount factor $\beta \equiv 1 /(1+\rho)$. In other words, future income must be sufficiently low compared with the present income to achieve a premium of future goods over present goods.

$$
\begin{equation*}
\frac{Y_{0}}{\mathrm{Y}_{1}}>(1+\rho) \tag{31}
\end{equation*}
$$

D) If people differ in their subjective discount rates, but have the same and constant flow of income, (16) might be written as:

$$
\begin{align*}
& r=\frac{Y\left(2+2 \rho_{A}+\rho_{B}+\rho_{A} \rho_{B}\right)+Y\left(2+\rho_{A}+2 \rho_{B}+\rho_{A} \rho_{B}\right)}{Y\left(2+\rho_{B}\right)+Y\left(2+\rho_{A}\right)}-1  \tag{32}\\
& r=\frac{4+3 \rho_{A}+3 \rho_{B}+2 \rho_{A} \rho_{B}}{4+\rho_{A}+\rho_{B}}-1  \tag{33}\\
& r=\frac{2 \rho_{A}+2 \rho_{B}+2 \rho_{A} \rho_{B}}{4+\rho_{A}+\rho_{B}} \tag{34}
\end{align*}
$$

As in A), the equilibrium natural real rate of interest does not depend on income, if it is constant and the same for all individuals. Only the subjective discount rates matter. They raise
the equilibrium rate of interest, which cannot fall below zero, unless they become negative too.
E) Consider heterogeneous agents with different subjective discount rates and different incomes that is, however, constant over time. Hence $\mathrm{Y}_{0}{ }^{\mathrm{A}}=\mathrm{Y}_{1}{ }^{\mathrm{A}}=\mathrm{Y}^{\mathrm{A}}$ and $\mathrm{Y}_{0}{ }^{\mathrm{B}}=\mathrm{Y}_{1}{ }^{\mathrm{B}}=\mathrm{Y}^{\mathrm{B}}$. (16) will take the form:
$r=\frac{Y^{A}\left(2+2 \rho_{A}+\rho_{B}+\rho_{A} \rho_{B}\right)+Y^{B}\left(2+\rho_{A}+2 \rho_{B}+\rho_{A} \rho_{B}\right)}{Y^{A}\left(2+\rho_{B}\right)+Y^{B}\left(2+\rho_{A}\right)}-1$
$r=\frac{Y^{A}\left(2+2 \rho_{A}+\rho_{B}+\rho_{A} \rho_{B}\right)+Y^{B}\left(2+\rho_{A}+2 \rho_{B}+\rho_{A} \rho_{B}\right)-Y^{A}\left(2+\rho_{B}\right)-Y^{B}\left(2+\rho_{A}\right)}{Y^{A}\left(2+\rho_{B}\right)+Y^{B}\left(2+\rho_{A}\right)}$
$r=\frac{2 \rho_{A} Y^{A}+2 \rho_{B} Y^{B}+\rho_{A} \rho_{B} Y^{A}+\rho_{A} \rho_{B} Y^{B}}{Y^{A}\left(2+\rho_{B}\right)+Y^{B}\left(2+\rho_{A}\right)}$
$r=\frac{\rho_{A} Y^{A}\left(2+\rho_{B}\right)+\rho_{B} Y^{B}\left(2+\rho_{A}\right)}{Y^{A}\left(2+\rho_{B}\right)+Y^{B}\left(2+\rho_{A}\right)}$

Again, the equilibrium real rate of interest cannot fall below zero if the subjective discount rates are positive. Its value is between $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$. Moreover, the higher the income of individual A compared with B , the closer is the real rate of interest to the subjective discount rate of individual A . Thus, in this case the size of the constant flow of income might affect the equilibrium real rate of interest, whose limits are, however, determined by particular subjective discount rates. Constant income flows of different size therefore give different weights to the particular discount rate in determination of the equilibrium real rate of interest.

## Simulations

1) 

| Growth rate of income A | Growth rate of income B | $\rho_{A}$ | $\rho_{B}$ | r |
| :--- | :--- | :--- | :--- | :--- |
| $-2 \%$ | $2 \%$ | $5 \%$ | $6 \%$ | $5,5 \%$ |



Figure No. 1_A2B Natural rate of interest within the interval ( $\rho_{\mathrm{A}}, \rho_{\mathrm{B}}$ ).

Initial income is the same for both individuals. Income stream of A is decreasing, of B it is increasing.
$r$ is between $\rho_{A}$ and $\rho_{B}$.
Consumption flow of $A$ is increasing $\left(r>\rho_{A}\right)$, of $B$ it is decreasing $\left(r<\rho_{B}\right)$ Individual $A$ is a lender, B is a borrower. ${ }^{231}$
This situation corresponds to Figure No. 42 in the main text.
2)

| Growth rate of income A | Growth rate of income B | $\rho_{\text {A }}$ | $\rho_{B}$ | $\mathbf{r}$ |
| :--- | :--- | :--- | :--- | :--- |
| $-\mathbf{4 \%}$ | $2 \%$ | $5 \%$ | $\mathbf{6 \%}$ | $\mathbf{4 , 4 5 \%}$ |

[^140]

Figure No. 2_A2B Natural rate of interest below the interval ( $\rho_{A}, \rho_{B}$ ).

Initial income is the same for both individuals. Income stream of A is decreasing more than income of $B$ is increasing.
$r$ is below $\rho_{A}$ and $\rho_{B}$.
Consumption flow of $A$ is decreasing ( $r<\rho_{A}$ ), of $B$ it is decreasing ( $r<\rho_{B}$ ) too. Individual $A$ is a lender, B is a borrower.
3)

| Growth rate of income A | Growth rate of income B | $\rho_{\mathrm{A}}$ | $\rho_{\mathrm{B}}$ | $\mathbf{r}$ |
| :--- | :--- | :--- | :--- | :--- |
| $-\mathbf{2 \%}$ | $\mathbf{4 \%}$ | $\mathbf{5 \%}$ | $\mathbf{6 \%}$ | $\mathbf{6 , 5 6 \%}$ |



Figure No. 3_A2B Natural rate of interest above the interval ( $\rho_{\mathrm{A}}, \rho_{\mathrm{B}}$ ).

Initial income is the same for both individuals. Income stream of A is decreasing less than income of $B$ is increasing.
$r$ is above $\rho_{A}$ and $\rho_{B}$.
Consumption flow of $A$ is increasing ( $r>\rho_{A}$ ), of $B$ it is increasing ( $r>\rho_{B}$ ) too. Individual $A$ is a lender, B is a borrower.
4)

| Growth rate of income A | Growth rate of income B | $\rho_{\text {A }}$ | $\rho_{\text {B }}$ | r |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{2 \%}$ | $-2 \%$ | $5 \%$ | $\mathbf{6 \%}$ | $\mathbf{5 , 5 \%}$ |



Figure No. 4_A2B Uniform evolution of income and consumption for both individuals

Initial income is the same for both individuals. Income stream of A is increasing, of B is decreasing.
$r$ is between $\rho_{A}$ and $\rho_{B}$.
Consumption flow of $A$ is increasing $\left(r>\rho_{A}\right)$, of $B$ it is decreasing ( $r<\rho_{B}$ ). Individual $A$ is a borrower, B is a lender.
5)

| Growth rate of income A | Growth rate of income B | $\rho_{\mathrm{A}}$ | $\rho_{\mathrm{B}}$ | $\mathbf{r}$ |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{0 \%}$ | $\mathbf{0 \%}$ | $\mathbf{5 \%}$ | $\mathbf{6 \%}$ | $\mathbf{5 , 5 \%}$ |



Figure No. 5_A2B Opposite evolution of consumption of each individual for identical and constant income streams

Initial income is the same for both individuals. Income streams of A and B are constant. $r$ is between $\rho_{A}$ and $\rho_{B}$.
Consumption flow of $A$ is increasing ( $r>\rho_{A}$ ), of $B$ it is decreasing ( $r<\rho_{B}$ ). Individual $A$ is a lender, B is a borrower.
This situation corresponds to Figure No. 41 in the main text.
6)

| Growth rate of income A | Growth rate of income B | $\rho_{\mathrm{A}}$ | $\rho_{B}$ | $\mathbf{r}$ |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{2 \%}$ | $\mathbf{2 \%}$ | $\mathbf{5 \%}$ | $\mathbf{6 \%}$ | $\mathbf{7 , 6 \%}$ |



Figure No. 6_A2B Uniformly increasing income streams for both individuals.

Initial income is the same for both individuals. Income streams of A and B are increasing at the same rate.
$r$ is higher than $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$.
Consumption flow of $A$ is increasing ( $r>\rho_{A}$ ), of $B$ it is increasing ( $r>\rho_{B}$ ) too. Individual $A$ is a lender, B is a borrower.
7)

| Growth rate of income A | Growth rate of income B | $\rho_{\mathrm{A}}$ | $\rho_{\mathrm{B}}$ | $\mathbf{r}$ |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{1 0 \%}$ | $\mathbf{1 0 \%}$ | $\mathbf{5 \%}$ | $\mathbf{6 \%}$ | $\mathbf{1 6 \%}$ |



Figure No. 7_A2B Uniformly and sharply increasing income streams for both individuals.

Initial income is the same for both individuals. Income streams of A and B are increasing at the same and a very high rate.
$r$ is higher than $\rho_{A}$ and $\rho_{B}$.
Consumption flow of $A$ is increasing ( $r>\rho_{A}$ ), of $B$ it is increasing ( $r>\rho_{B}$ ) too. Individual $A$ is a lender, B is a borrower, but both positions are very close to zero.
8)

| Growth rate of income A | Growth rate of income B | $\rho_{\mathrm{A}}$ | $\rho_{\mathrm{B}}$ | r |
| :--- | :--- | :--- | :--- | :--- |
| $-\mathbf{- 5 , 2 1 \%}$ | $-5,21 \%$ | $5 \%$ | $\mathbf{6 \%}$ | $0 \%$ |



Figure No. 8_A2B Uniformly and sharply decreasing income streams for both individuals leading to zero natural rate of interest.

Initial income is the same for both individuals. Income streams of A and B are decreasing at the same rate. This rate is chosen intentionally to reach:
$r=0 \%$
Consumption flow of A is decreasing $\left(\mathrm{r}<\rho_{\mathrm{A}}\right)$, of B it is decreasing $\left(\mathrm{r}<\rho_{\mathrm{B}}\right)$ too. Individual A is a lender, B is a borrower. Thus, there exists an intertemporal market, an exchange of present goods for future goods. A similar situation (even though for negative interest) is depicted in Figure No. 40 in the main text.

## Appendix 3 B - Time Preference and the Intertemporal Substitution of Labour

A) In this appendix, we will add the assumption that people enjoy also their leisure time, not only consumption. Furthermore, we will explicitly assume that the only source of income is their labour income. We will develop a similar two-period model as was presented in the main
text. In the first version of this model, the phenomenon of the intertemporal substitution of labour will generate some kind of the PPF curve displayed in the main text. However, as we will see, even this curve will depend on utility (or rather disutility), not technical productivity. We will also see that the subjective discount rate will affect not only the shape of the (intertemporal consumption) indifference curves, but also the position of the endowment point(s). Thus, all important outcomes in this sub-model of the theory of interest will depend solely on subjective phenomena.
Consider a representative consumer maximizing his life-time utility in a simple two-period model (Equation 1). For simplicity, assume that $\theta=1$, hence the utility function is logarithmic. His utility depends on consumption C and leisure time H in both periods. Future utilities are discounted by subjective discount rate $\rho$. The relative weight of consumption and leisure in the utility function is represented by parameter $b$. This parameter might also play a role that distinguishes discounting of utility from consumption and from leisure. Alternatively, both terms might be discounted by a different subjective discount rate (i.e. $\rho_{\mathrm{C}}$ and $\rho_{\mathrm{H}}$ ). ${ }^{232}$

Equation (2) represents his intertemporal budget constraint. $\mathrm{W}_{0}$ and $\mathrm{W}_{1}$ stand for real wage earned exogenously in period one and two, respectively. In the second version of this model (B), we will relax the assumption of constant real wage and we explicitly add a production function in both periods that exhibits diminishing marginal product of labour. Alternatively, $\mathrm{W}_{0}$ and $\mathrm{W}_{1}$ might be understood as parameters in linear production function $\mathrm{Y}_{\mathrm{t}}=\mathrm{A}_{\mathrm{t}} \mathrm{L}_{\mathrm{t}}$, i.e. $\mathrm{W}_{0}$ $=\mathrm{dY} / \mathrm{dL}_{0}=\mathrm{A}_{0}$ and $\mathrm{W}_{1}=\mathrm{dY} \mathrm{Y}_{1} / \mathrm{dL}_{1}=\mathrm{A}_{1}$. Furthermore, labour might be used only in short production processes, i.e. it may be used only in the creation of the given period output (in earning the given period income).

$$
\begin{align*}
& U=\ln C_{0}+b \ln H_{0}+\frac{1}{1+\rho} \ln C_{1}+\frac{b}{1+\rho} \ln H_{1}  \tag{1}\\
& C_{0}+\frac{1}{1+r} C_{1}=\mathrm{W}_{0} \mathrm{~L}_{0}+\frac{1}{1+r} \mathrm{~W}_{1} \mathrm{~L}_{1} \tag{2}
\end{align*}
$$

The time constraint in both periods is given by:

$$
\begin{align*}
& \mathrm{L}_{0}+\mathrm{H}_{0}=1  \tag{3}\\
& \mathrm{~L}_{1}+\mathrm{H}_{1}=1 \tag{4}
\end{align*}
$$

We normalized the time endowment to 1 . Thus, time spent by working ( L ) and relaxing ( H ) gives 1 altogether. Substituting (3) and (4) into (1), the lifetime utility function might be written as:

$$
\begin{equation*}
U=\ln C_{0}+b \ln \left(1-L_{0}\right)+\frac{1}{1+\rho} \ln C_{1}+\frac{b}{1+\rho} \ln \left(1-L_{1}\right) \tag{5}
\end{equation*}
$$

Set up a simple Lagrangian function and solve for the first order conditions (FOC).

[^141]\[

$$
\begin{equation*}
\mathrm{L}=\ln C_{0}+b \ln \left(1-L_{0}\right)+\frac{1}{1+\rho} \ln C_{1}+\frac{b}{1+\rho} \ln \left(1-L_{1}\right)+\lambda\left(\mathrm{W}_{0} \mathrm{~L}_{0}+\frac{1}{1+r} \mathrm{~W}_{1} \mathrm{~L}_{1}-C_{0}-\frac{1}{1+r} C_{1}\right) \tag{6}
\end{equation*}
$$

\]

FOCs for consumption are:

$$
\begin{align*}
& \frac{\partial L}{\partial C_{0}}=\frac{1}{C_{0}}-\lambda=0  \tag{7}\\
& \frac{\partial L}{\partial C_{1}}=\frac{1}{1+\rho} \frac{1}{C_{1}}-\lambda \frac{1}{1+r}=0 \tag{8}
\end{align*}
$$

(7) and (8) imply:

$$
\begin{equation*}
\frac{C_{1}}{C_{0}}=\frac{1+r}{1+\rho} \tag{9}
\end{equation*}
$$

Equation (9) represents the Euler (consumption) equation for this problem.
FOCs for labour are given by:

$$
\begin{align*}
& \frac{\partial L}{\partial L_{0}}=\frac{-b}{1-L_{0}}+\lambda W_{0}=0  \tag{10}\\
& \frac{\partial L}{\partial L_{1}}=\frac{1}{1+\rho} \frac{-b}{1-L_{1}}+\lambda \frac{W_{1}}{1+r}=0 \tag{11}
\end{align*}
$$

(10) and (11) imply:

$$
\begin{equation*}
\frac{1-L_{1}}{1-L_{0}}=\frac{W_{0}}{W_{1}} \frac{1+r}{1+\rho} \tag{12}
\end{equation*}
$$

Equation (12) represents the Euler (employment) equation for this problem. It describes the optimal allocation of leisure (labour) over time. This system has five unknowns ( $\mathrm{C}_{0}, \mathrm{C}_{1}, \mathrm{~L}_{0}$, $\mathrm{L}_{1}, \lambda$ ) in five equations ( $7,8,10,11,2$ ). Leisure time can be then easily determined from time constraints (3) and (4).
(7) and (10) imply:

$$
\begin{equation*}
\frac{1}{C_{0}}=\frac{b}{1-L_{0}} \frac{1}{W_{0}} \tag{13}
\end{equation*}
$$

This problem might be more easily solved for the leisure time. Hence (13) becomes:

$$
\begin{equation*}
\frac{1}{C_{0}}=\frac{b}{H_{0}} \frac{1}{W_{0}} \tag{14}
\end{equation*}
$$

Similar manipulations can be done with (8) and (11), which yields:

$$
\begin{equation*}
\frac{1}{C_{1}}=\frac{b}{H_{1}} \frac{1}{W_{1}} \tag{15}
\end{equation*}
$$

Substituting (14) and (15) into (2) and using time constraints (3) and (4), equation (2) becomes:

$$
\begin{equation*}
\frac{W_{0}}{b} H_{0}+\frac{W_{1}}{b(1+r)} H_{1}=\mathrm{W}_{0}\left(1-\mathrm{H}_{0}\right)+\frac{1}{1+r} \mathrm{~W}_{1}\left(1-\mathrm{H}_{1}\right) \tag{16}
\end{equation*}
$$

Using (12), equation (16) takes the form:

$$
\begin{equation*}
\frac{W_{0}}{b} H_{0}+\frac{W_{0}}{b(1+\rho)} H_{0}=\mathrm{W}_{0}-\mathrm{W}_{0} \mathrm{H}_{0}+\frac{1}{1+r} \mathrm{~W}_{1}-\frac{\mathrm{W}_{0}}{1+\rho} \mathrm{H}_{0} \tag{17}
\end{equation*}
$$

A simple (but time-consuming) rearrangement of terms above gives us:

$$
\begin{align*}
& H_{0}\left[\frac{W_{0}}{b}+\mathrm{W}_{0}+\frac{W_{0}}{b(1+\rho)}+\frac{\mathrm{W}_{0}}{1+\rho}\right]=\mathrm{W}_{0}+\frac{1}{1+r} \mathrm{~W}_{1}  \tag{18}\\
& H_{0}^{*}=\frac{1+\frac{1}{1+r} \frac{\mathrm{~W}_{1}}{\mathrm{~W}_{0}}}{1+\frac{1}{b}+\frac{1}{1+\rho}+\frac{1}{b(1+\rho)}} \tag{19}
\end{align*}
$$

Optimum $\mathrm{H}_{1}$ is given by (19) and (12):

$$
\begin{align*}
& H_{1}=\frac{\mathrm{W}_{0}}{\mathrm{~W}_{1}} \frac{1+r}{1+\rho} \times \frac{1+\frac{1}{1+r} \frac{\mathrm{~W}_{1}}{\mathrm{~W}_{0}}}{1+\frac{1}{b}+\frac{1}{1+\rho}+\frac{1}{b(1+\rho)}}  \tag{20}\\
& H_{1}^{*}=\frac{1+(1+r) \frac{\mathrm{W}_{0}}{\mathrm{~W}_{1}}}{(2+\rho)+\frac{1+\rho}{b}+\frac{1}{b}} \tag{21}
\end{align*}
$$

Thus, leisure time (labour) in the present increases (decreases) and leisure time (labour) in the future decreases (increases) if the interest rate falls, the relative intertemporal wage $\mathrm{W}_{1} / \mathrm{W}_{0}$ rises, or if the subjective discount rate grows. As a result, for the given interest rate, higher impatience ( $\rho$ ) moves the endowment point closer to the vertical axis in the $\mathrm{C}_{1}-\mathrm{C}_{0}$ space and further from the horizontal axis, because present labour supply (and therefore labour income) falls and future rises. This represents another channel that might raise the interest rate in
equilibrium after the increase in $\rho$, because the time shape of the income stream will become even more increasing.

If we realise that $\mathrm{L}^{*}=\left(1-\mathrm{H}^{*}\right)$ in every period, the total labour income in the given period is $\mathrm{Y}_{\mathrm{t}}=\mathrm{W}_{\mathrm{t}} \mathrm{L}_{\mathrm{t}}{ }^{*}$. The impact of various rates of interest (for given $\rho$ ) on the endowment point is portrayed in the graph below.

Response of endowment on the interest rate


Figure No. 0_A3. An impact of a decrease in the interest rate on consumption and on income endowment, if the intertemporal substitution of labour is effective.

As can be seen, for a constant wage over time the income stream is smoothed, if the interest rate is equal to the subjective discount rate. This picture closely resembles a usual PPF curve presented in the main text. Lower interest rate moves the endowment closer to the vertical axis. However, there is no element of productivity in our analysis. This PPF depends only on the utility of leisure time. As a result, the equilibrium real rate of interest will also depend only on subjective phenomena.
A decrease in the interest rate and the resulting change in the budget line of an individual are presented in Figure No. 1_A3 below. As can be seen, lower interest rate moves the income endowment point closer to the vertical axis from $\mathrm{A}^{1}$ to $\mathrm{A}^{2}$. At the same time, the budget line becomes flatter. In standard analysis, the budget line rotates around the endowment point A. Here, however, the pivot point itself is being moved. ${ }^{233}$
It is obvious that an increase in the interest rate decreases the growth rate in income over time, because it is more profitable to work in the present and relax in the future. Thus, at the individual level we found an inverse relationship between the interest rate and the shape of the income stream. The phenomenon of the intertemporal substitution of labour introduces a new channel that partly offsets the impact of the income stream on the interest rate presented in

[^142]our previous analysis. To find the ultimate impact on the interest rate, we have to analyse the optimum consumption stream in this model. ${ }^{234}$
By substituting Euler consumption equation (9) and equations (14) and (15) into the intertemporal budget constraint (2), we get:
\[

$$
\begin{align*}
& C_{0}+\frac{1}{1+\rho} C_{0}=\mathrm{W}_{0}\left(1-\frac{b C_{0}}{\mathrm{~W}_{0}}\right)+\frac{\mathrm{W}_{1}}{1+r}\left(1-\frac{\mathrm{b}}{\mathrm{~W}_{1}} \frac{1+\mathrm{r}}{1+\rho} C_{0}\right)  \tag{22}\\
& \frac{(1+\rho) C_{0}+C_{0}}{1+\rho}=\mathrm{W}_{0}-b C_{0}+\frac{\mathrm{W}_{1}}{1+r}-\frac{\mathrm{b}}{1+\rho} C_{0}  \tag{23}\\
& \frac{(1+\rho) C_{0}+C_{0}+b(1+\rho) C_{0}+b C_{0}}{1+\rho}=\mathrm{W}_{0}+\frac{\mathrm{W}_{1}}{1+r}  \tag{24}\\
& C_{0}(2+\rho+2 b+b \rho)=\frac{\mathrm{W}_{0}(1+r)+\mathrm{W}_{1}}{(1+r)}(1+\rho)  \tag{25}\\
& C_{0}(2+\rho)(1+b)=\frac{\mathrm{W}_{0}(1+r)+\mathrm{W}_{1}}{(1+r)}(1+\rho)  \tag{26}\\
& C_{0}(2+\rho)(1+b)=\frac{\mathrm{W}_{0}(1+r)+\mathrm{W}_{1}}{(1+r)}(1+\rho)  \tag{27}\\
& C_{0}^{*}=\frac{\mathrm{W}_{0}(1+r)+\mathrm{W}_{1}}{(1+r)} \frac{(1+\rho)}{(2+\rho)(1+b)} \tag{28}
\end{align*}
$$
\]

If $b=0$, this expression perfectly coincides with equation (7) in Appendix 2. Optimum future consumption is derived, if we substitute (28) into (9):

$$
\begin{equation*}
C_{1}^{*}=\frac{\mathrm{W}_{0}(1+r)+\mathrm{W}_{1}}{(2+\rho)(1+b)} \tag{29}
\end{equation*}
$$

[^143]As is perfectly clear from (28) and (29) parameter $b$ (preference for leisure time) decreases consumption in both periods. Furthermore, by comparing $\mathrm{C}_{0}{ }^{*}$ and $\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}$, we can determine whether the given individual is a lender or a borrower for the given $r$. Figure No. 1_A3 below shows a consumer, whose subjective discount rate is higher than the real interest rate. With regard to the stream of wages, it is either increasing ( $\mathrm{W}_{1}>\mathrm{W}_{0}$ ) or constant. As a result, according to (12) his endowment point is above the $45^{\circ}$ line because he works relatively more in the future. Thus, he also earns relatively more in the future $\left(\mathrm{Y}_{1}=\mathrm{W}_{1} \mathrm{~L}_{1}{ }^{*}>\mathrm{Y}_{0}=\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}\right)$. At the same time, according to (9) his optimal consumption stream is decreasing $\left(\mathrm{C}_{1}{ }^{*}<\mathrm{C}_{0}{ }^{*}\right)$. This particular consumer is a borrower because $\mathrm{C}_{0}{ }^{*}>\mathrm{Y}_{0}=\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}$.

Now, consider a reduction in the interest rate ( $\mathrm{r}_{2}<\mathrm{r}_{1}$ ). As can be seen in Figure No. 1_A3, it will raise present consumption from $\mathrm{C}_{0}{ }^{1 *}$ to $\mathrm{C}_{0}{ }^{2 *}$ and reduce present labour supply (and increase present leisure), which will consequently decrease present labour income from $\mathrm{Y}_{0}{ }^{1}$ to $\mathrm{Y}_{0}{ }^{2}$. Decline in the interest rate is beneficial for a debtor, as the new optimum is posited at a higher indifference curve. ${ }^{235}$ Moreover, a reduction in the interest rate decreases the amount of saving to a greater extent when the intertemporal substitution of labour exists compared with its absence. The reason lies in a decline in present income endowment, which drives up the difference between present income $\left(\mathrm{Y}_{0}\right)$ and present optimal consumption $\left(\mathrm{C}_{0}{ }^{*}\right)$.


Figure No. 1_A3. An impact of a decrease in the interest rate on consumption and on income endowment when the intertemporal substitution of labour is effective.

Thus, the saving curve is more elastic when the intertemporal substitution in labour (ISL) is included in the model. The reason lies in the fact that the reduction in the interest rate

[^144]decreases present labour supply and hence the present labour income and raises future labour supply and future labour income. Both changes in income shift the traditional saving curve to the left. As a result, the saving curve that includes both the intertemporal substitution in consumption (ISC) and in labour might be constructed as follows: A drop in the interest rate moves the optimum saving along the traditional saving curve, which neglects ISL, from point $\mathrm{E}^{1}$ to point B . The second round effect on the income stream shifts the entire traditional saving curve to the left. The new point of optimum can be found at point $\mathrm{E}^{2}$. Connecting points $\mathrm{E}^{1}$ and $\mathrm{E}^{2}$, the more general saving curve can be found $\left(\mathrm{S}^{\text {ISL }}\right)$. This curve reflects both the ISC and the ISL. As can be seen, our representative consumer makes negative saving, since present consumption exceeds present income. ${ }^{236}$


Figure No. 1B_A3. Construction of the saving curve, which includes both the intertemporal substitution in consumption and in labour.

Furthermore, if we relax the assumption that the subjective discount rate and the stream of wages are the same for all individuals, we can derive the equilibrium real interest rate for a general intertemporal model. The aggregate constraint for such an economy would be analogous to (9) and (10) from Appendix 2B:

$$
\begin{equation*}
C_{0}^{A^{*}}+C_{0}^{B^{*}}=\mathrm{W}_{0}^{\mathrm{A}} \mathrm{~L}_{0}^{\mathrm{A}^{*}}+\mathrm{W}_{0}^{\mathrm{B}} \mathrm{~L}_{0}^{\mathrm{B}^{*}} \tag{30}
\end{equation*}
$$

[^145]\[

$$
\begin{equation*}
C_{1}^{A^{*}}+C_{1}^{B^{*}}=\mathrm{W}_{1}^{\mathrm{A}} \mathrm{~L}_{1}^{\mathrm{A}^{*}}+\mathrm{W}_{1}^{\mathrm{B}} \mathrm{~L}_{1}^{\mathrm{B}^{*}} \tag{31}
\end{equation*}
$$

\]

However, such an analysis would be too complicated compared with the results acquired, since these would not surpass those already discussed in Appendix 2. Thus, let us assume that all individuals are identical with regard to their subjective discount rate and their exogenous stream of wages. Such homogeneity implies that individual saving is zero on the part of each individual.

As a result, neither the interest rate $r_{1}$ nor $r_{2}$ in Figure No. 1_A3 is a good candidate for an equilibrium rate of interest. Both are too low since they result in the excess of demand for present goods over their available supply $\left(\mathrm{C}_{0} *>\mathrm{Y}_{0}\right)$. The condition for an equilibrium rate of interest is thus given by:

$$
\begin{equation*}
C_{0}^{*}=\mathrm{W}_{0} \mathrm{~L}_{0}^{*}=Y_{0} \tag{32}
\end{equation*}
$$

From (28), (19) and (3) we get:

$$
\begin{equation*}
\frac{\mathrm{W}_{0}(1+r)+\mathrm{W}_{1}}{(1+r)} \frac{(1+\rho)}{(2+\rho)(1+b)}=\mathrm{W}_{0}\left(1-\frac{1+\frac{1}{1+r} \frac{\mathrm{~W}_{1}}{\mathrm{~W}_{0}}}{1+\frac{1}{b}+\frac{1}{1+\rho}+\frac{1}{b(1+\rho)}}\right) \tag{33}
\end{equation*}
$$

The only unknown is the real interest rate $r$. However, instead of solving (33) we can directly substitute (32) into (13) and a similar constraint $C_{1}^{*}=\mathrm{W}_{1} \mathrm{~L}_{1}^{*}$ into (15), which yields:

$$
\begin{align*}
& L_{0}=\frac{1}{1+b}  \tag{34}\\
& L_{1}=\frac{1}{1+b} \tag{35}
\end{align*}
$$

Thus, the labour supply will be the same in both periods. As a result, the equilibrium interest rate will depend only on the flow of wages and the subjective discount rate. Substitute (34) and (35) into (12):

$$
\begin{equation*}
(1+r)=\frac{W_{1}}{\mathrm{~W}_{0}}(1+\rho) \tag{36}
\end{equation*}
$$

(36) closely resembles equation (30) from Appendix 2. There is therefore no need to repeat the analysis again. If the flow of wages is constant, the equilibrium real interest rate will be equal to the subjective discount rate. If the stream of wages is increasing, the interest rate will be greater than the subjective discount rate.
However, in this particular case the equilibrium income endowment will not be affected by the intertemporal substitution of labour. The reason is as follows: An increase in the average intertemporal wage ( $\mathrm{W}_{1} / \mathrm{W}_{0}$ ) will benefit present leisure time. However, higher $\mathrm{W}_{1} / \mathrm{W}_{0}$ will accordingly increase the real interest rate, which perfectly offsets the original tendency. Hence, the equilibrium of a representative individual might be represented by Figures 38 or 39 in the main text. The endowment point and the resulting equilibrium interest rate will depend only on the time shape of wages, not on the allocation of labour over time, since it is constant. Moreover, parameter $b$ (the relative importance of leisure in the utility function) does not affect the equilibrium interest rate either.


Figure No. 2_A3. The impact of an increase in the subjective discount rate on consumption, income endowment, and eventually on the interest rate

In other words, in this homogenous-agent model the intertemporal allocation of labour will not be affected by the time shape of the stream of wages, because any shape will accordingly modify the equilibrium interest rate, which will eventually leave the optimal intertemporal allocation of labour at the previous level that is characterised by $\mathrm{L}=1 /(1+\mathrm{b})$ in every period.

A similar analysis can be done for the subjective discount rate. Its rise will increase the real interest rate by the same amount keeping the equilibrium intertemporal allocation of labour unaffected. The only outcome will be a steeper indifference curve and a steeper budget line. There will be no impact on the representative endowment point.
This analysis is presented in Figure No. 2_A3. We assume a constant stream of wages ( $\mathrm{W}_{1}=\mathrm{W}_{0}=\mathrm{W}$ ). The labour supply in both periods is the same $\mathrm{L}^{*}=1 /(1+\mathrm{b})$. As a result, the time shape of the income stream is constant ( $\mathrm{Y}_{0}=\mathrm{Y}_{1}=\mathrm{Y}=\mathrm{WL}$ *). According to equation (36), the interest rate must be equal to the subjective discount rate. Thus, consumption is also smoothed over time (see equation 9). Now, consider an increase in the subjective discount rate. This will benefit present leisure time at the expense of future leisure time, so present labour falls and future labour increases. As a result, present income decreases from Y to $\mathrm{Y}_{0}=\mathrm{WL}_{0}{ }^{*}$ and consequently the income endowment point moves from $\mathrm{A}^{1}$ to $\mathrm{A}^{2}$ (see panel a). ${ }^{237}$ At the same time, the indifference curve will become steeper (a similar analysis was made in Appendix 2). The resulting excess of demand for present goods over their supply $\left(\mathrm{C}_{0}{ }^{2 *}>\mathrm{Y}_{0}=\mathrm{WL}_{0}{ }^{*}\right)$ is greater compared with the situation if leisure is not included in the utility function (compare the size of borrowing represented by the red solid line and the dashed line). The reason is that the intertemporal substitution of labour moves the income endowment point to the top left. Yet, to equilibrate the demand for present goods ( $\mathrm{C}_{0}{ }^{*}$ ) with their available supply ( $\mathrm{Y}_{0}=\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}$ ) the interest rate must go up. In the end, the interest rate is equal to the new subjective discount rate $\left(r_{2}=\rho_{2}\right)$. Furthermore, labour supply is the same in both periods. The same holds for income and consumption. Thus, the endowment point is eventually in the same position as it was in the beginning (see panel b).
However, the individual intertemporal substitution of labour might play an important role in equilibrium if there is heterogeneity across individuals. Equation (33) for heterogeneous agents would be modified to (see equation 30):

$$
\begin{align*}
& \frac{\mathrm{W}_{0}^{A}(1+r)+\mathrm{W}_{1}^{\mathrm{A}}}{(1+r)} \frac{\left(1+\rho_{A}\right)}{\left(2+\rho_{A}\right)(1+b)}+\frac{\mathrm{W}_{0}^{B}(1+r)+\mathrm{W}_{1}^{\mathrm{B}}}{(1+r)} \frac{\left(1+\rho_{B}\right)}{\left(2+\rho_{B}\right)(1+b)}= \\
& =\mathrm{W}_{0}^{\mathrm{A}}\left(1-\frac{1+\frac{1}{1+r} \frac{\mathrm{~W}_{1}^{\mathrm{A}}}{\mathrm{~W}_{0}^{A}}}{1+\frac{1}{b}+\frac{1}{1+\rho_{A}}+\frac{1}{b\left(1+\rho_{A}\right)}}\right)+\mathrm{W}_{0}^{\mathrm{B}}\left(1-\frac{1+\frac{1}{1+r} \frac{\mathrm{~W}_{1}^{\mathrm{B}}}{\mathrm{~W}_{0}^{B}}}{1+\frac{1}{b}+\frac{1}{1+\rho_{B}}+\frac{1}{b\left(1+\rho_{B}\right)}}\right) \tag{37}
\end{align*}
$$

We do not have the ambition to solve this complicated equation for the equilibrium interest rate $r$. Yet, it is obvious that it will depend negatively on present wages $W_{0 i}$ and positively on future wages $\mathrm{W}_{1 \mathrm{i}}$ and subjective discount rates $\rho_{\mathrm{i}}$. Furthermore, more patient people (low $\rho_{\mathrm{i}}$ )

[^146]will consume less and work more in the present. As a result, their net lending position will be positive. It will be more positive than if the intertemporal substitution of labour does not exist. The opposite result would hold for less patient individuals (high $\rho_{\mathrm{i}}$ ). Next, an increasing stream of wages will lead to a lower present labour supply and therefore even to a lower present income. Thus, this channel will further raise borrowing of people with an increasing time shape of wages.
As can be seen, the inclusion of leisure into the utility function and the resulting intertemporal substitution of labour reinforce the results obtained in Appendix 2B. The reason lies in the fact that, in the first place, the subjective discount rate influences the position of the individual endowment point (provided that $r$ does not move one-for-one with $\rho_{\mathrm{i}}$ ). In the second place, the shape of the exogenous stream of wages affects the position of the endowment point ( $\mathrm{Y}_{0 \mathrm{i}}=$ $\left.\mathrm{W}_{0 \mathrm{i}} \times \mathrm{L}_{0 \mathrm{i}}{ }^{*} ; \mathrm{Y}_{1 \mathrm{i}}=\mathrm{W}_{1 \mathrm{i}} \times \mathrm{L}_{1 \mathrm{i}}{ }^{*}\right)$ not only directly due to the magnitude of $\mathrm{W}_{1 \mathrm{i}} / \mathrm{W}_{0 \mathrm{i}}$, but also due the impact on the optimum allocation of labour ( $\mathrm{L}_{1 \mathrm{i}}{ }^{*} ; \mathrm{L}_{0 \mathrm{i}}{ }^{*}$ ). Thus, both the subjective discount rate and the exogenous flow of wages will in turn affect the individual net borrowing/lending position and maybe the resulting equilibrium interest rate. ${ }^{238}$ In other words, each individual exogenous parameter might have a stronger impact on the equilibrium interest rate, if the intertemporal substitution of labour exists.
Pure time preference theorists have never discussed the possibility of the intertemporal substitution of labour. Yet, this channel might amplify the link between the time preference (in sense two) and the natural rate of interest. The reason is that time preference favours not only present satisfaction from consumption goods, but it also favours present leisure. As a result, relatively greater leisure time in the present (and lower in the future) reduces the provision of present goods and improves their future provision. This phenomenon therefore supports the first Böhm-Bawerkian ground for interest. It can be said that owing to the preference for present leisure time (and the resulting intertemporal substitution of labour) the second cause for interest reinforces the first cause for interest due to the impact on the relative provision of goods over time.
B) In Part A, we assumed a constant wage in each period that is not affected by changes in the labour supply. This assumption is rather strong especially in the general equilibrium model, however, it helped us to focus on specific aspects in the theory of interest. In the present section, we will relax this assumption as we introduce production function that exhibits diminishing marginal product of labour. Output (and income) will depend on the amount of labour expended in the given period, and the real wage will be equal to the marginal product of labour. In this section, labour can be used only in short processes, so present labour might not be used in a longer process that will mature next period. This extension will be postponed to section C.

The structure of this model is the same as in section A with only one exception. Output in the present period and in the future period respectively depends on the amount of labour expended in the given period and the level of technologies $\mathrm{A}_{\mathrm{t}}$ :

$$
\begin{align*}
& Y_{0}=A_{0} L_{0}^{\alpha}  \tag{38}\\
& Y_{1}=A_{1} L_{1}^{\alpha} \tag{39}
\end{align*}
$$

The marginal product of labour is decreasing because $0<\alpha<1$. The intertemporal budget constraint (2) is then modified to:

[^147]\[

$$
\begin{equation*}
C_{0}+\frac{1}{1+r} C_{1}=A_{0} L_{0}^{\alpha}+\frac{1}{1+r} A_{1} L_{1}^{\alpha} \tag{40}
\end{equation*}
$$

\]

Furthermore, the time endowment will be generalized to $\mathrm{T}=\mathrm{L}+\mathrm{H}$. The Euler (employment) equation (12) will result in:

$$
\begin{equation*}
\frac{T-L_{1}}{T-L_{0}}=\frac{A_{1} L_{1}^{1-\alpha}}{A_{0} L_{0}^{1-\alpha}} \frac{1+r}{1+\rho} \tag{41}
\end{equation*}
$$

The solution of this system will not be presented here, as it seems to be too complicated compared with the results obtained. However, the main conclusions from the previous sections are preserved. Present labour supply increases with a lower subjective discount rate and a higher interest rate. Thus, the PPF curve will be generated again, even though it will depend also on the decreasing marginal productivity of labour (not capital), and its shape will be most probably concave due to this property. As a result, the endowment points will then critically depend on the subjective discount rate (and the level of technologies). Thus, the interest rate in this economy will depend mainly on subjective psychological elements, although the marginal productivity of labour (not capital!) will also affect its size.
C) In the last section, we will only briefly outline a model, in which present labour might be used not only in the production of the given period output, but in which the present labour can be employed also in a longer (and more productive) process that will, however, provide output in the next period. Furthermore, the longer process will also require application of labour in the next period to be fully completed.
The time constraint in each period is given by the following equations:

$$
\begin{align*}
& L_{0}^{S}+L_{0}^{L}+H_{0}=T  \tag{42}\\
& L_{1}^{S}+L_{1}^{L}+H_{1}=T \tag{43}
\end{align*}
$$

We assume that the time endowment in each period is the same $\mathrm{T} . \mathrm{L}_{0}{ }^{\mathrm{S}}$ stands for the amount of labour applied in the present in the short production process. $\mathrm{L}_{0}{ }^{\mathrm{L}}$ represents the amount of present labour that is applied in the long process that will mature in the future. $\mathrm{L}_{1}{ }^{s}$ is applied in the short process in the future, whereas $L_{1}{ }^{L}$ is the amount of future labour that is used to finish the output, whose production started in the present.
We will assume that longer processes are more productive. However, some amount of future labour must be employed to finish the longer process. We also allow for a change in the level of technologies over time, hence $A_{0}$ need not equal $A_{1}$. Furthermore, we assume that technology (knowledge) is non-rival and it might be used in full in both production methods. As a result, the output of consumption goods in each period is given by the following production functions:

$$
\begin{align*}
& Y_{0}=A_{0}\left(L_{0}^{S}\right)^{\alpha}  \tag{44}\\
& Y_{1}^{S}=A_{1}\left(L_{1}^{S}\right)^{\alpha}  \tag{45}\\
& Y_{1}^{L}=A_{1}\left(L_{1}^{L}\right)^{\beta}\left(L_{0}^{L}\right)^{\gamma} \tag{46}
\end{align*}
$$

The total output of consumable goods in the future is:

$$
\begin{equation*}
Y_{1}=Y_{1}^{S}+Y_{1}^{L} \tag{47}
\end{equation*}
$$

We assume that the marginal product of labour is decreasing, thus $\alpha, \beta, \gamma$ are all between 0 and 1. However, it is assumed that longer process is more productive than the shorter process, therefore $\alpha<\beta$ and $\alpha<\gamma$. Furthermore, labour applied in the present in the longer process is more remunerative than labour applied in the future in the same process, hence $\beta<\gamma$. Finally, we assume that the returns to scale in the longer process are not increasing, which means that $\beta+\gamma \leq 1$.
The lifetime utility function (1) is preserved and the intertemporal budget constraint is given by:

$$
\begin{equation*}
C_{0}+\frac{1}{1+r} C_{1}=\mathrm{Y}_{0}+\frac{1}{1+r} \mathrm{Y}_{1} \tag{48}
\end{equation*}
$$

We could set up a similar Lagrangian function as in section A. Such a system has 13 unknowns: $\mathrm{C}_{0}, \mathrm{C}_{1}, \mathrm{~L}_{0}{ }^{\mathrm{S}}, \mathrm{L}_{0}{ }^{\mathrm{L}}, \mathrm{H}_{0}, \mathrm{~L}_{1}{ }^{\mathrm{S}}, \mathrm{L}_{1}{ }^{\mathrm{L}}, \mathrm{H}_{1}, \lambda, \mathrm{Y}_{0}, \mathrm{Y}_{1}{ }^{\mathrm{S}}, \mathrm{Y}_{1}{ }^{\mathrm{L}}, \mathrm{Y}_{1}$, in 13 equations:
a) FOC for $\mathrm{C}_{0}$
b) FOC for $\mathrm{C}_{1}$
c) FOC for $\mathrm{L}_{0}{ }^{\mathrm{S}}$
d) FOC for $L_{0}{ }^{L}$
e) FOC for $L_{1}{ }^{\mathrm{S}}$
f) FOC for $L_{1}{ }^{L}$
g) intertemporal budget constraint (48)
h) production function (44)
i) production function (45)
j) production function (46)
k) total future output (47)

1) time constraint (42)
m) time constraint (43)

The fundamental goal of this analysis would be to find the determinants of the equilibrium rate of interest $r$. We can again consider a homogeneous or heterogeneous-agent model. For a homogenous model, only one more equation is required. Namely, that the individual saving is zero, i.e. $\mathrm{C}_{0}=\mathrm{Y}_{0}$. As can be seen, present goods cannot be saved, but present labour can in the form of a longer production process. Although we will not solve this problem, several observations will surely emerge.

First, the equilibrium rate of interest will be determined by various parameters of the model. It will depend not only on the time preference parameter $\rho$, but also on the productivity parameters $\alpha, \beta, \gamma$. A diminishing marginal product of capital (i.e. of longer methods) might appear here, if $\beta+\gamma<1$. Thus, the entire picture of this economy might be represented by convex (consumption) indifference curves and a concave investment opportunity line, whose shape depends not only on the productivity of longer methods (46), but also on the diminishing marginal productivity of labour. Furthermore, its shape will be surely affected by the subjective discount rate. As a result, the natural rate of interest in this more comprehensive model will depend on the time preference and productivity phenomena.

To conclude this appendix, the inclusion of the intertemporal substitution of labour might open new fields in the analysis of the natural rate of interest. In section A, the natural rate of interest was determined by purely subjective phenomena, even though we generated a typical PPF curve, whose nature was, however, also purely subjective.
In section B, we allowed for a decreasing marginal productivity of labour. We suggested that this phenomenon must modify our analysis from the previous section. In final section C, the idea of higher productivity of roundabout methods was introduced. It is highly probable, that the natural rate of interest in such a model must be co-determined by the time preference and diminishing marginal productivity (of longer methods).

## Supplement 1 to section A. Endowment point after a change in the subjective discount rate

In this supplement, we will prove that after the change in the subjective discount rate, the endowment point must move along the original budget line.
The optimum amount of present leisure time (equation 19) might be expressed as:

$$
\begin{align*}
& H_{0}^{*}=\frac{\frac{(1+r) W_{0}+W_{1}}{(1+r) W_{0}}}{\frac{b(1+\rho)+1+\rho+b+1}{b(1+\rho)}}  \tag{49}\\
& H_{0}^{*}=\frac{(1+r) W_{0}+W_{1}}{(1+r) W_{0}} \times \frac{b(1+\rho)}{(1+\rho)(b+1)+b+1}  \tag{50}\\
& H_{0}^{*}=\frac{(1+r) W_{0}+W_{1}}{(1+r) W_{0}} \times \frac{b(1+\rho)}{(1+b)(2+\rho)} \tag{51}
\end{align*}
$$

Similarly for the optimal future leisure time, equation (21) might be written as:

$$
\begin{align*}
& H_{1}^{*}=\frac{W_{0}}{W_{1}} \frac{(1+r)}{(1+\rho)} \frac{(1+r) W_{0}+W_{1}}{(1+r) W_{0}} \times \frac{b(1+\rho)}{(1+b)(2+\rho)}  \tag{52}\\
& H_{1}^{*}=\frac{(1+r) W_{0}+W_{1}}{W_{1}} \times \frac{b}{(1+b)(2+\rho)} \tag{53}
\end{align*}
$$

Now, let us define the present value of the income stream. From the intertemporal budget constraint (Equation 2) and time constraints (3) and (4), it is clear that:

$$
\begin{equation*}
P V=W_{0}\left(1-H_{0}^{*}\right)+\frac{W_{1}\left(1-H_{1}^{*}\right)}{(1+r)} \tag{54}
\end{equation*}
$$

As can be seen, we consider only optimal levels of leisure in both periods. Using (51) and (53):

$$
\begin{align*}
& P V=W_{0}-\frac{(1+r) W_{0}+W_{1}}{(1+r)} \times \frac{b(1+\rho)}{(1+b)(2+\rho)}+\frac{W_{1}}{(1+r)}-\frac{(1+r) W_{0}+W_{1}}{(1+r)} \times \frac{b}{(1+b)(2+\rho)}  \tag{55}\\
& P V=W_{0}+\frac{W_{1}}{(1+r)}-\frac{(1+r) W_{0}+W_{1}}{(1+r)} \times \frac{b}{(1+b)(2+\rho)}(2+\rho)  \tag{56}\\
& P V=W_{0}+\frac{W_{1}}{(1+r)}-\frac{(1+r) W_{0}+W_{1}}{(1+r)} \times \frac{b}{(1+b)} \tag{57}
\end{align*}
$$

The present value expression from Appendix 2 is here adjusted for the last term. However, if we exclude leisure time from the utility function (i.e. $\mathrm{b}=0$ ), it will be perfectly the same as in Appendix 2. Furthermore, PV of the (optimum) income stream does not depend on the subjective discount rate. This is the crucial result of the foregoing analysis:

$$
\begin{equation*}
\frac{\partial P V}{\partial \rho}=0 \tag{58}
\end{equation*}
$$

In the first round, a change in the subjective discount rate does not alter the real interest rate. If both the PV and the interest are constant, then the new endowment point must lie on the original budget line, because neither the slope of the budget line ( $r$ is constant), nor the position of the budget line ( PV is constant) changed. In other words, the new budget line must coincide with the initial one, even though the endowment point is at a different position.
Alternatively, we can demonstrate that the endowment point $\left(\mathrm{Y}_{1}=\mathrm{W}_{1} \mathrm{~L}_{1}{ }^{*} ; \mathrm{Y}_{0}=\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}\right)$ is moved in the same direction as is the slope of the budget constraint. This slope $\left(\mathrm{dC}_{1} / \mathrm{dC}_{0}\right)$ is specifically given by $-(1+r)$ as can be seen from the explicit form of the budget constraint:

$$
\begin{equation*}
C_{1}=(1+r) \mathrm{W}_{0} \mathrm{~L}_{0}+\mathrm{W}_{1} \mathrm{~L}_{1}-(1+r) C_{0} \tag{59}
\end{equation*}
$$

We will solve this problem with the help of the optimum leisure time rather than labour supply. It will be also useful to define the following relationship:

$$
\begin{equation*}
W_{0} H_{0}^{*}=W_{0}-W_{0} L_{0}^{*} \tag{60}
\end{equation*}
$$

The last term in (60) is the optimum present income. Furthermore, the response of present income to a change in the subjective discount rate is, using (60), given by:

$$
\begin{equation*}
\frac{\partial\left(W_{0} H_{0}^{*}\right)}{\partial \rho}=-\frac{\partial\left(W_{0} L_{0}^{*}\right)}{\partial \rho} \tag{61}
\end{equation*}
$$

Similar relationship holds for the future period:

$$
\begin{equation*}
\frac{\partial\left(W_{1} H_{1}^{*}\right)}{\partial \rho}=-\frac{\partial\left(W_{1} L_{1}^{*}\right)}{\partial \rho} \tag{62}
\end{equation*}
$$

Using (50), we can find the optimal response of the present leisure time to a change in the time preference (in sense two):

$$
\begin{align*}
& \frac{\partial\left(W_{0} H_{0}^{*}\right)}{\partial \rho}=\frac{(1+r) W_{0}+W_{1}}{(1+r)} \times \frac{b}{(1+b)} \times \frac{(2+\rho)-(1+\rho)}{(2+\rho)^{2}}  \tag{63}\\
& \frac{\partial\left(W_{0} H_{0}^{*}\right)}{\partial \rho}=\frac{(1+r) W_{0}+W_{1}}{(1+r)} \times \frac{b}{(1+b)} \times \frac{1}{(2+\rho)^{2}} \tag{64}
\end{align*}
$$

Using (53), we can find the optimal response of future leisure time:

$$
\begin{equation*}
\frac{\partial\left(W_{1} H_{1}^{*}\right)}{\partial \rho}=-\left[(1+r) W_{0}+W_{1}\right] \times \frac{b}{(1+b)} \times \frac{1}{(2+\rho)^{2}} \tag{65}
\end{equation*}
$$

Applying (61) and (62) and dividing (65) by (64), the optimal relative change in future income $\left(\mathrm{Y}_{1}=\mathrm{W}_{1} \mathrm{~L}_{1}{ }^{*}\right)$ to present income $\left(\mathrm{Y}_{0}=\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}\right)$, i.e. the movement of the endowment point A , in response to a change in the subjective discount rate is given by:

$$
\begin{equation*}
\frac{\frac{\partial\left(W_{1} L_{1}^{*}\right)}{\partial \rho}}{\frac{\partial\left(W_{0} L_{0}^{*}\right)}{\partial \rho}}=-(1+r) \tag{66}
\end{equation*}
$$

Thus, a change in the subjective discount rate will move the endowment point along the original budget line. Q.E.D.

## Supplement 2 to section A. Endowment point after a change in the real interest rate

In this second supplement, we will demonstrate that the response of the endowment point to a change in the real interest rate is more complicated than the response to a change in $\rho$.
Using (51), a response of present leisure time to a change in the interest rate is given by:

$$
\begin{align*}
& \frac{\partial\left(W_{0} H_{0}^{*}\right)}{\partial r}=\frac{b(1+\rho)}{(1+b)(2+\rho)} \times \frac{W_{0}(1+r)-\left[(1+r) W_{0}+W_{1}\right]}{(1+r)^{2}}  \tag{67}\\
& \frac{\partial\left(W_{0} H_{0}^{*}\right)}{\partial r}=\frac{b(1+\rho)}{(1+b)(2+\rho)} \times \frac{-W_{1}}{(1+r)^{2}} \tag{68}
\end{align*}
$$

Using (53), a response of future leisure time might be expressed as:

$$
\begin{equation*}
\frac{\partial\left(W_{1} H_{1}^{*}\right)}{\partial r}=\frac{b}{(1+b)(2+\rho)} \times W_{0} \tag{69}
\end{equation*}
$$

Applying (61) and (62) and dividing (69) by (68), we can easily derive the relative movement of the endowment point after a change in the real interest rate:

$$
\begin{equation*}
\frac{\frac{\partial\left(W_{1} L_{1}^{*}\right)}{\partial r}}{\frac{\partial\left(W_{0} L_{0}^{*}\right)}{\partial r}}=-\frac{W_{0}}{W_{1}} \frac{(1+r)^{2}}{(1+\rho)} \tag{70}
\end{equation*}
$$

If $\mathrm{W}_{0}=\mathrm{W}_{1}$ and $\mathrm{r}=\rho$, the derivative above is equal to $(1+\mathrm{r})$ in absolute value. Hence, after the decrease (or increase) in the interest rate, the new endowment point would lie on the original budget line. If $r<\rho$, the derivative is lower than $(1+r)$. In Figure No. 1_A3, we assumed that $r$ $<\rho$ and $\mathrm{W}_{0}<\mathrm{W}_{1}$ (or $\mathrm{W}_{0}=\mathrm{W}_{1}$ ). Thus, the new endowment point must be below the original budget line because the directional shift of the endowment point has a lower slope than the budget line. If $r>\rho$, the derivative would be higher than $(1+r)$, and the new endowment point would lie above the initial budget line. ${ }^{239}$

## Appendix 4 B Optimization Problem in the Finite and Infinite Horizon

In this Appendix, we will solve the optimization problem of a representative consumer from section 5 . As usual, his objective is to find the optimum path of consumption so as to maximize his lifetime utility (A4_1), subject to his lifetime intertemporal budget constraint (A4_2). ${ }^{240}$

$$
\begin{align*}
& U=\sum_{t=0}^{T} \frac{u\left(C_{t}\right)}{(1+\rho)^{t}}=u\left(C_{0}\right)+\frac{u\left(C_{1}\right)}{1+\rho}+\frac{u\left(C_{2}\right)}{(1+\rho)^{2}}+\ldots+\frac{u\left(C_{T}\right)}{(1+\rho)^{T}}  \tag{A4_1}\\
& \mathrm{C}_{0}+\frac{\mathrm{C}_{1}}{(1+\mathrm{r})}+\frac{\mathrm{C}_{2}}{(1+\mathrm{r})^{2}}+\ldots+\frac{\mathrm{C}_{T}}{(1+\mathrm{r})^{\mathrm{T}}}=\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{(1+\mathrm{r})}+\frac{\mathrm{Y}_{2}}{(1+\mathrm{r})^{2}}+\ldots+\frac{\mathrm{Y}_{T}}{(1+\mathrm{r})^{\mathrm{T}}} \tag{A4_2}
\end{align*}
$$

Set up a simple Lagrangian function for this problem:

$$
\begin{align*}
& L=u\left(C_{0}\right)+\frac{u\left(C_{1}\right)}{1+\rho}+\frac{u\left(C_{2}\right)}{(1+\rho)^{2}}+\ldots+\frac{u\left(C_{T}\right)}{(1+\rho)^{T}}+\lambda\left(Y_{0}+\frac{Y_{1}}{1+r}+\frac{Y_{2}}{(1+r)^{2}}+\ldots+\frac{1}{(1+r)^{T}} Y_{T}-\right. \\
& \left.C_{0}-\frac{1}{1+r} C_{1}-\frac{1}{(1+r)^{2}} C_{2}-\ldots-\frac{1}{(1+r)^{T}} C_{T}\right) \tag{A4_3}
\end{align*}
$$

Let us find the first order conditions for optimum consumption at any time $t$ and $t+1$ :

[^148]FOC:

$$
\begin{align*}
& \frac{\partial L}{\partial C_{t}}=\frac{u^{\prime}\left(C_{t}\right)}{(1+\rho)^{t}}-\lambda \frac{1}{(1+r)^{t}}=0  \tag{A4_4}\\
& \frac{\partial L}{\partial C_{t+1}}=\frac{u^{\prime}\left(C_{t+1}\right)}{(1+\rho)^{t+1}}-\lambda \frac{1}{(1+r)^{t+1}}=0 \tag{A4_5}
\end{align*}
$$

Expressing $\lambda$ in both periods and dividing (A4_5) by (A4_4) we get:

$$
\begin{equation*}
\frac{u^{\prime}\left(C_{t+1}\right)}{u^{\prime}\left(C_{t}\right)}=\frac{1+\rho}{1+r} \tag{A4_6}
\end{equation*}
$$

(A4_6) is the Euler equation for this problem. Alternatively, FOC for $\mathrm{C}_{0}$ is just $\lambda=\mathrm{u}^{\prime}\left(\mathrm{C}_{0}\right)$. Thus, using (A4_4) the Euler equation might be written as:

$$
\begin{equation*}
\frac{u^{\prime}\left(C_{t}\right)}{u^{\prime}\left(C_{0}\right)}=\left(\frac{1+\rho}{1+r}\right)^{t} \tag{A4_7}
\end{equation*}
$$

Applying a specific CRRA utility function, for which the marginal utility of consumption is $\mathrm{C}^{-\theta}$, (A4_6) yields (see equation 25 in the main text):

$$
\begin{equation*}
\frac{C_{t}}{C_{t+1}}=\left(\frac{1+\rho}{1+r}\right)^{1 / \theta} \tag{A4_8}
\end{equation*}
$$

Similarly, (A4_7) is modified to:

$$
\begin{equation*}
\frac{C_{0}}{C_{t}}=\left(\frac{1+\rho}{1+r}\right)^{t / \theta} \tag{A4_9}
\end{equation*}
$$

(A4_9) might be used to solve the initial optimal level of consumption and then consumption at any time. Assuming $\theta=1$ (i.e. logarithmic utility function), (A4_9) for time 0 and 1 is:

$$
\begin{equation*}
C_{1}=\frac{1+r}{1+\rho} C_{0} \tag{A4_10}
\end{equation*}
$$

(A4_8) and (A4_9) for time 1 and 2 are:

$$
\begin{equation*}
C_{2}=\frac{1+r}{1+\rho} C_{1}=\left(\frac{1+r}{1+\rho}\right)^{2} C_{0} \tag{A4_11}
\end{equation*}
$$

Substituting $\mathrm{C}_{1}, \mathrm{C}_{2}$ etc. into the intertemporal budget constraint (A4_2) and for infinite T, we may write:

$$
\begin{align*}
& \mathrm{C}_{0}+\frac{\frac{1+r}{1+\rho} C_{0}}{(1+\mathrm{r})}+\frac{\left(\frac{1+r}{1+\rho}\right)^{2} C_{0}}{(1+\mathrm{r})^{2}}+\ldots=\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{(1+\mathrm{r})}+\frac{\mathrm{Y}_{2}}{(1+\mathrm{r})^{2}}+\ldots  \tag{A4_12}\\
& \mathrm{C}_{0}+\frac{\mathrm{C}_{0}}{1+\rho}+\frac{\mathrm{C}_{0}}{(1+\rho)^{2}}+\ldots=\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{(1+\mathrm{r})}+\frac{\mathrm{Y}_{2}}{(1+\mathrm{r})^{2}}+\ldots \tag{A4_13}
\end{align*}
$$

The right-hand side of (A4_13) represents the present value of the flow of income ( $\mathrm{PV}_{\text {income }}$ ). Using the formula for the sum of the infinite geometric series, (A4_13) becomes:

$$
\begin{align*}
& \frac{\mathrm{C}_{0}}{1-\frac{1}{1+\rho}}=P V_{\text {income }}  \tag{A4_14}\\
& \mathrm{C}_{0}=\frac{\rho}{1+\rho} P V_{\text {income }} \tag{A4_15}
\end{align*}
$$

(A4_15) and (A4_9) might be used to derive consumption in any period. For time 1,2 and t (see A4_10 and A4_11), we get:

$$
\begin{align*}
& C_{1}=\frac{\rho(1+r)}{(1+\rho)^{2}} P V_{\text {income }}  \tag{A4_16}\\
& C_{2}=\frac{\rho(1+r)^{2}}{(1+\rho)^{3}} P V_{\text {income }}  \tag{A4_17}\\
& C_{t}=\frac{\rho(1+r)^{t}}{(1+\rho)^{t+1}} P V_{\text {income }} \tag{A4_18}
\end{align*}
$$

For a zero time preference in sense two $(\rho=0)$, i.e. if people do not prefer the given want to be gratified as soon as possible, a man will not consume in the present. As can be seen in (A4_15), $\mathrm{C}_{0}=0$. But he will not consume in the next period either. Moreover, he will not consume in any future period, expect for infinity. Consumption will be postponed forever. Thus, it seems that Mises was right that with zero time preference and positive interest rate, the act of consumption will never occur.
Yet, if we look at (A4_15) again, $\mathrm{C}_{0}$ might be non-zero (or more precisely it may take on any value), if PV of income is infinite. This might occur either if the real interest rate declines to zero and the flow of income is sufficiently non-decreasing over time, or if the growth rate in income exceeds (or is equal to) the real interest rate. However, the later statement would indicate dynamic inefficiency. Thus, we will exclude the possibility of an infinite present value of the income stream.

Olson and Bailey (1981) suggested that zero time preference is consistent with positive real interest rate and positive present consumption if the marginal utility of consumption "drastically" decreases with higher consumption and the labour income endowment grows
over time. (A4_15) and (A4_18) are derived for $\theta=1$, i.e. for a logarithmic utility function. As we will see below, $\theta=1$ might be too low to generate such a property.

As a result, let us try to recalculate the key equations for any value of $\theta$. However, it might be more instructive to start with a finite horizon to grasp the key idea. Thus, consider the budget constraint (A4_2). If the income process obeys the following process $\mathrm{Y}_{\mathrm{t}}=(1+\mathrm{g}) \mathrm{Y}_{\mathrm{t}-1}$, the present value of the income stream is given by (see A5_24 in Appendix 5 B):

$$
\begin{equation*}
P V_{\text {income }}=\frac{(1+r)^{T+1}-(1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_{0} \tag{A4_19}
\end{equation*}
$$

Using the equation for the optimal flow of consumption (A4_9), the left hand side of the intertemporal budget constraint (A4_2) might be written as:

$$
\begin{align*}
& P V_{\text {consunption }}=C_{0}+\frac{C_{0}\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}}{1+r}+\frac{C_{0}\left(\frac{1+r}{1+\rho}\right)^{2 / \theta}}{(1+r)^{2}}+\ldots+\frac{C_{0}\left(\frac{1+r}{1+\rho}\right)^{T / \theta}}{(1+r)^{T}}  \tag{A4_20}\\
& P V_{\text {consumption }}=C_{0}+C_{0} \frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}+C_{0}\left[\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}\right]^{2}+\ldots+C_{0}\left[\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}\right]^{T} \tag{A4_21}
\end{align*}
$$

According to the formula of the sum of the finite geometric sequence, we may write:

$$
\begin{align*}
& P V_{\text {consumption }}=C_{0} \frac{1-\left[\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}\right]^{T+1}}{1-\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}}  \tag{A4_22}\\
& P V_{\text {consumption }}=\frac{(1+\rho)^{(T+1) / \theta}-(1+r)^{(T+1)(1-\theta) / \theta}}{(1+\rho)^{(T+1) / \theta}} \frac{(1+\rho)^{1 / \theta}}{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}} C_{0} \tag{A4_23}
\end{align*}
$$

From (A4_19), (A4_23) and (A4_2) we can easily determine the optimal level of present consumption $\mathrm{C}_{0}$ :

$$
\begin{gather*}
P V_{\text {consumption }}=P V_{\text {income }} \\
\frac{(1+\rho)^{(T+1) / \theta}-(1+r)^{(T+1)(1-\theta) / \theta}}{(1+\rho)^{(T+1) / \theta}} \frac{(1+\rho)^{1 / \theta}}{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}} C_{0}=\frac{(1+r)^{T+1}-(1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_{0} \\
C_{0}=\frac{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}} \frac{(1+\rho)^{(T+1) / \theta}}{(1+\rho)^{(T+1) / \theta}-(1+r)^{(T+1)(1-\theta) / \theta}} \frac{(1+r)^{T+1}-(1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_{0}
\end{gather*}
$$

$$
\begin{equation*}
C_{0}=P V_{\text {income }} \frac{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}} \frac{(1+\rho)^{(T+1) / \theta}}{(1+\rho)^{(T+1) / \theta}-(1+r)^{(T+1)(1-\theta) / \theta}} \tag{A4_27}
\end{equation*}
$$

Using (A4_9), the value of consumption in any period might be then determined. As can be seen, the present consumption depends on the interest rate, time preference in sense two (i.e. the subjective discount rate), the elasticity of substitution $1 / \theta$, the time horizon T and the present value of the income stream, which in turn depends on the initial income, interest rate, and the growth rate in income $g$.
According to Olson and Bailey, the present consumption in the infinite horizon model must be depressed to zero if the time preference is zero, unless income endowment grows over time and people have a drastically diminishing marginal utility. Before analyzing the infinite horizon model, let us use our finite horizon model to explore their argument.

We introduce a set of various simulations. In the first section, we assume a positive growth in the income endowment at the rate of $\mathrm{g}=1 \%$. This rate was assumed in Olson and Bailey (1981).


Figure No. 1_A4
Figure No. 1_A4 is designed for real interest rate $\mathrm{r}=5 \%$, initial labour income $\mathrm{Y}_{0}=100$, its growth rate of $1 \%$, zero time preference $(\rho=0)$ and relatively high $\theta=5$, which was assumed by Olson and Bailey (1981). Thus, for such a high $\theta$, the marginal utility is surely drastically diminishing. We start with a relatively short time horizon, only 10 years.

Our model predicts that the optimal present consumption is slightly above the initial income level. By no means, it is depressed to a zero level. It grows at the optimal rate of $1 \%$. Because consumption exceeds income in the initial periods, saving is negative. ${ }^{241}$ Hence, debt is issued $\left(\mathrm{B}_{0}<0\right)$ and it is gradually accumulated. However, due to the No-Ponzi-Game condition imposed on our model (see Appendix 5 for a thorough discussion), its terminal value must be zero, i.e. $\mathrm{B}_{\mathrm{T}=10}=0$ (see the final point of the yellow curve). As can be seen, in period 6 savings are positive and debt might be reduced, as it reaches its maximum in period 5 .
Figure No. 2_A4 extends the time horizon to 50 years. Nevertheless, similar conclusions might be said here as in Figure No.1_A4. The same holds for time horizons 100 years and 500 years. In neither case is the present consumption depressed to zero, even though people do not prefer the given satisfaction to be delivered as soon as possible (i.e. they do not discount future utilities, $\rho=0 \%$ ). The reason for a non-zero present consumption might be an increasing profile of their income stream and a dramatically decreasing MU, so the analysis of Olson and Bailey seems to be accurate. Future is better provided for and people prefer a relatively smoothed profile of consumption, which results in a positive premium on present goods, whose consumption thus cannot fall to zero. Future higher income is therefore moved closer to the present and saving is negative in the first part of the relevant time horizon.


Figure No. 2_A4

[^149]

Figure No. 3_A4


Figure No. 4_A4

In the next part, let us reduce parameter $\theta$ and thus increase the elasticity of substitution. Keeping all the other parameters at the same level, optimum consumption growth is then not in accordance with the exogenous income growth; it is higher. Figure No. 5_A4 represents an individual with $\theta=2$ and 100-year planning horizon.


Figure No. 5_A4
As can be seen in Figure No. 5_A4, higher elasticity of substitution (i.e. less dramatically decreasing MU) reduces present consumption to $68 \%$ of present income. As a result, saving is positive for a considerable part of the planning horizon and assets are accumulated to relatively high levels, which enables the future consumption to exceed future labour income by a large amount. This point is of crucial importance and it will be stressed in further sections again. Long planning horizons lead to the fact that compounded interest allows for very high future consumption levels quite independent of the future levels of the labour income. Thus, it seems that an increasing profile of the labour income stream might not be the crucial reason for non-zero present consumption if future consumption is to reach very high levels.

Figure No. 6_A4 displays simulation of logarithmic utility function (i.e. $\theta \rightarrow 1$ ) and a planning horizon of 50 years. Longer horizons would give us an analogous picture, but of an inferior clarity. As can be seen, similar conclusions might be derived as in the previous case. Yet, present consumption is depressed to zero even more for the benefit of future consumption. Thus, we may conclude that the higher the elasticity of substitution (i.e. the lower the parameter $\theta$ ), the lower the optimum present consumption, the higher the optimal growth rate in consumption and the higher the optimum future consumption, whose astronomical future values are quite independent of future-period labour income.


Figure No. 6_A4

Nonetheless, exactly zero present consumption and postponement of all consumption to the future, as was predicted by Mises for zero time preference, might be achieved only for the case of perfect substitutes (i.e. $\theta \rightarrow 0$ ). This can be seen in Figure No. 7_A4, which is designed for only a 20 -year horizon. The analysis also suggests that longer time horizons might also considerably depress present consumption provided that $\theta$ is very low (but not necessarily zero). ${ }^{242}$ Thus, an infinite horizon model might give the Misesian analysis some credit.


Figure No. 7_A4

[^150]In this section, we relax the assumption of an increasing income, thus we set $\mathrm{g}=0 \%$. Olson and Bailey (1981) suggested that if time preference is zero, constant income stream must lead to zero present consumption when the planning horizon is infinite. Before we test their prediction, let us examine a finite horizon model for the same set of assumptions, i.e. zero income growth and zero time preference.
Figure No. 8_A4 is designed for real interest rate $\mathrm{r}=5 \%$, constant labour income stream of $\mathrm{Y}_{0}=100$, zero time preference ( $\rho=0 \%$ ) and $\theta=5$, which was assumed by Olson and Bailey (1981).We start again with a relatively short time horizon of 10 years. As can be seen in this figure, the optimum growth rate of consumption is $1 \%$ as in Figure No. $1 \_$A4 since it depends on the real interest rate, subjective discount rate and the elasticity of substitution, not on the specific shape of the income stream. However, the constancy of income requires that present consumption must be below the present labour income to generate sufficient growth in consumption because the level of future consumption must exceed the level of future income, if the entire lifetime income is to be completely exhausted.
As a result, saving is positive in the first part of the planning horizon and assets are being accumulated. The accrued interest in a 10 -year horizon is quite modest, so the future consumption is quite close to future labour income. The stock of assets (i.e. $\mathrm{B}_{\mathrm{t}}$ ) reaches its maximum in period 5 since from period 6 saving is negative, and assets gradually decline to a zero level. As is stressed in Appendix 5, for a monotonically increasing utility function it cannot be optimal to hold any positive assets at the end of the planning horizon.


Figure No. 8_A4

A similar picture can be deduced from longer time horizons (see Figures No. 9_A4, 10_A4, $\left.11 \_A 4,12 \_A 4\right) .{ }^{243}$ The only difference is a relatively larger reduction in the present consumption compensated by a higher level of future consumption. However, even a very long time horizon of 500 years will not depress the present consumption to zero, although more assets must be accumulated in the first part of the planning horizon. The reason is a relatively low elasticity of substitution that results in a very modest optimum growth rate in consumption that is below the real interest rate. The initial accumulation of assets then provides enough capital income to finance very high levels of future consumption that will considerably exceed future-period labour income. Thus, it seems that an increasing profile in labour income is not necessary as was predicted by Olson and Bailey (1981). A positive difference between the real interest rate and the growth rate in consumption, and sufficient saving at the beginning of the planning horizon suffice to generate very high levels of future consumption even without the reduction of present consumption to zero. ${ }^{244}$
Another interesting observation, discussed in Appendix 5 B in more detail, is that saving might be positive even if the given period consumption $\left(\mathrm{C}_{\mathrm{t}}\right)$ is greater than the given period labour income $\left(\mathrm{Y}_{\mathrm{t}}\right)$. The reason lies in the fact that the relevant income source for saving is the disposable income $\left(\mathrm{Y}_{\mathrm{t}}+\mathrm{r}_{1} \mathrm{~B}_{\mathrm{t}-1}\right)$, i.e. the sum of labour income and capital income, not the simple labour income. Until the difference between the disposable income and consumption is positive, the stock of assets might be increasing. ${ }^{245}$


Figure No. 9_A4

[^151]

Figure No. 10_A4


Figure No. 11_A4


Figure No. 12_A4


Figure No. 13_A4


Figure No. 14_A4

Figures No.15_A4 and 16_A4 are designed for a higher elasticity of substitution ( $\theta=2$ and $\theta=1$ ). As can be seen, with lower preference for consumption smoothing, present optimal consumption is lower because optimum consumption growth is higher. But it never declines to zero. Thus, even for the case of constant labour income, which is closest to the idea of an evenly rotating economy envisioned by Mises, zero time preference and positive real interest rate do not lead to a complete reduction of present consumption and postponement of all consumption to the end of the planning horizon. Even though this tendency might be deduced from lower $\theta$ and an extending planning horizon, the Misesian story does not hold apart from a perfect-substitute case, which is characterised by a non-decreasing marginal utility of consumption (see Figure No. 17_A4).


Figure No. 15_A4


Figure No. 16_A4


Figure No. 17_A4

It can be said that Mises underestimated the power of the law of diminishing marginal utility in his intertemporal analysis. This law results in an effort to smooth consumption over time and to provide enough consumption goods in every period. Thus, no period can be oversupplied with consumption goods at the expense of other periods even if the time preference is zero and the real interest rate is positive.

Only very long planning horizons and a moderately diminishing MU (i.e. low $\theta$ ) suggest that present consumption might be depressed to zero. Thus, let us examine an infinite horizon model whether it gives credit to the Misesian reasoning. At the same time, we will explore the key predictions of Olson and Bailey about the necessity of drastically diminishing MU and an increasing profile of the income stream.

## Infinite Horizon

The relevant intertemporal budget constraint for the infinite horizon is (A5_12) or (A5_13) in Appendix 5 B . Let us assume a constant real interest rate over time so as to easily compare our results with those of Olson and Bailey. By substituting the optimal consumption flow represented by (A4_9) into the intertemporal budget constraint (A5_13), we may write:

$$
\begin{equation*}
\mathrm{C}_{0}+\frac{\left(\frac{1+r}{1+\rho}\right)^{1 / \theta} C_{0}}{(1+\mathrm{r})}+\frac{\left(\frac{1+r}{1+\rho}\right)^{2 / \theta} C_{0}}{(1+\mathrm{r})^{2}}+\ldots=\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{(1+\mathrm{r})}+\frac{\mathrm{Y}_{2}}{(1+\mathrm{r})^{2}}+\ldots \tag{A4_28}
\end{equation*}
$$

The left hand side of the intertemporal budget constraint (A4_28) might be written as:

$$
\begin{equation*}
P V_{\text {consumption }}=C_{0}+C_{0} \frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}+C_{0}\left[\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}\right]^{2}+\ldots \tag{A4_29}
\end{equation*}
$$

According to the formula of the sum of the infinite geometric series, (A4_29) gives us:

$$
\begin{align*}
& P V_{\text {consumption }}=\frac{1}{1-\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}} C_{0}  \tag{A4_30}\\
& P V_{\text {consumption }}=\frac{(1+\rho)^{1 / \theta}}{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}} C_{0} \tag{A4_31}
\end{align*}
$$

However, the sum of this infinite series is finite only if:

$$
\begin{align*}
(1+r)^{(1-\theta) / \theta} & <(1+\rho)^{1 / \theta}  \tag{A4_32}\\
\frac{1-\theta}{\theta} \ln (1+r) & <\frac{1}{\theta} \ln (1+\rho) \tag{A4_33}
\end{align*}
$$

If $r$ and $\rho$ are small numbers, (A4_33) yields:

$$
\begin{align*}
& 1-\theta<\frac{\rho}{r}  \tag{A4_34}\\
& \theta>1-\frac{\rho}{r}  \tag{A4_35}\\
& \rho-(1-\theta) r>0 \tag{A4_35b}
\end{align*}
$$

In (A4_10) and the equations that ensued, we set $\theta=1$ and $\rho=0$. We concluded that in such a case the optimal present consumption (and all future consumption levels except for infinity) was zero. Now, we can see that this set of parameters is not consistent even with the convergence of the sum of the flow of optimum consumption in the infinite horizon. ${ }^{246}$ Zero time preference $(\rho=0)$ is consistent only with $\theta>1$, i.e. only with the utility function that exhibits relatively low elasticity of substitution. Alternatively we may say that such a utility function is characterized by a "drastically diminishing marginal utility". Values of $\theta$ lower or equal to one would lead not only to the divergence of the sum of the flow of optimum consumption, but also to negative or zero optimum present consumption, as can be seen in equation (A4_37) below if we substitute $\rho=0, \theta \leq 1$ and any positive real interest rate.
Furthermore, equation (A4_35) also implies that the higher the subjective discount rate (i.e. time preference in sense two), the higher the elasticity of substitution (lower $\theta$ ) is feasible to achieve a convergent sum of the flow of consumption and positive optimum present consumption.

Using the formula for the present value of income and provided that the income process obeys $\mathrm{Y}_{\mathrm{t}}=(1+\mathrm{g}) \mathrm{Y}_{\mathrm{t}-1}$, (A4_28) might be written as (see A5_21 in Appendix 5 B for the PV of income):

$$
\begin{equation*}
C_{0} \frac{(1+\rho)^{1 / \theta}}{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}=Y_{0} \frac{1+r}{r-g} \tag{A4_36}
\end{equation*}
$$

$$
\begin{equation*}
C_{0}=Y_{0} \frac{1+r}{r-g} \frac{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}} \tag{A4_37}
\end{equation*}
$$

Using (A4_37) and (A4_9), the level of optimum consumption in any period might be then determined. Olson and Bailey suggested that if the time preference is zero $(\rho=0)$, the Euler equation (A4_9) in the infinite horizon implies that present consumption is depressed to zero, unless the income endowment grows over time (i.e. $g>0 \%$ ) and the utility function exhibits a dramatically decreasing marginal utility ( $\theta>1$ in our framework). Our analysis confirms their second observation, $\theta>1$ must be positive to obtain positive $\mathrm{C}_{0}$. However, according to (A4_37), positive $g$ might not be required.
Let us first explore the optimal level of present consumption when income grows over time. In the second part, we will keep the income endowment constant. Both Mises (1996) and Olson and Bailey (1981) would predict that present consumption must be necessarily zero and every unit of disposable income must be saved and postponed to indefinite future provided that the time preference is zero. Equation (41) from the main text (here A4_38) seems to be in accordance with this conclusion:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{C_{0}}{C_{T}}=\lim _{T \rightarrow \infty} \frac{1}{(1+r)^{T / \theta}}=0 \tag{A4_38}
\end{equation*}
$$

Olson and Bailey tried to escape from this result by assuming a positive growth in (labour) income endowment, which, according to them, can only guarantee infinite C in infinity and non-zero present consumption.

[^152]Again, let us run several simulations, this time for an infinite horizon. Olson and Bailey suggested that $\mathrm{r}=5 \%, \mathrm{~g}=1 \%, \theta=5$ and $\rho=0$. These parameters lead to the growth rate in consumption of $1 \%$ (see equation 42 in the main text), i.e. to the same growth rate as in the case of income. However, equation (42) is just an approximation. The correct growth rate is $0.98 \%$ for this set of parameters (see 39 in the main text or A4_44 below), which is a little bit less than the growth rate in income. That is the reason why in Figures 1_A4 to 4_A4 we observed that present consumption exceeded present income and that saving was negative in the first part of the planning horizon and positive in the second part. These dynamics then resulted in a typical U -shaped behaviour of total debt $\mathrm{B}_{\mathrm{t}}$. Debt was first accumulated and then it was being paid off.

Nonetheless, the key intention of Olson and Bailey was surely to perfectly equalize the growth rate in income and consumption. Then they could argue that future consumption might be infinite even for a positive present consumption. In the infinite horizon, we must select parameters that will perfectly result in a $1 \%$ growth rate in consumption otherwise the error from the approximation of equation (39) by equation (42) from the main text will expand beyond all limits.
Thus, our analysis of the finite horizon model presented above, which included this error, just demonstrated the optimum behaviour of consumption, saving and debt, if the optimum growth rate of consumption $(0.98 \%)$ is lower than the given growth rate in income ( $1 \%$ ). The subsequent simulations then provided dynamics for the set of parameters resulting in the optimum growth rate in consumption that exceeded the growth rate in income (which was either $1 \%$ or $0 \%$ ). We saw that in such a case, assets were accumulated first and then used to keep the optimum consumption growing independently of the sluggish growth (or stagnation) in the labour income.

However, if $\mathrm{r}=5 \%$ and $\rho=0 \%, \theta$ that is required to generate a $1 \%$ growth rate in consumption is 4.9 (see the procedure below from equation (A4_39) onwards), not 5 as suggested by Olson and Bailey. ${ }^{247}$

$$
\begin{align*}
& \frac{C_{t+1}}{C_{t}}=\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}=(1+g)  \tag{A4_39}\\
& \theta \ln (1+g)=\ln \left(\frac{1+r}{1+\rho}\right)  \tag{A4_40}\\
& \theta=\frac{\ln (1+r)-\ln (1+\rho)}{\ln (1+g)} \tag{A4_41}
\end{align*}
$$

Alternatively, if we want to determine the interest rate that would lead to the growth of consumption $g$ for some given $\theta$, we can modify (A4_39) to:
$\frac{1+r}{1+\rho}=(1+g)^{\theta}$

[^153]$r=(1+g)^{\theta}(1+\rho)-1$

If $r, g$ and $\rho$ are small numbers, (A4_40) might be written as:
$r=\rho+\theta g$

Thus, for $\mathrm{r}=5 \%, \mathrm{~g}=1 \%, \theta=4.9, \rho=0$ (and zero initial assets) the model (both infinite and finite) predicts that optimum consumption perfectly coincides with income in every period. To prove this conclusion, let us notice that equation (A4_8) suggests that the optimum growth rate of consumption $x$ is:
$x=\frac{C_{t+1}}{C_{t}}-1=\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}-1$
(A4_30) then implies that:

$$
\begin{equation*}
P V_{\text {consumption }}=\frac{1}{1-\frac{1+x}{1+r}} C_{0} \tag{A4_45}
\end{equation*}
$$

Because the PV of income must be equal to the PV of consumption, (A4_45) yields:

$$
\begin{equation*}
P V_{\text {income }}=P V_{\text {consumption }}=\frac{1+r}{r-x} C_{0} \tag{A4_46}
\end{equation*}
$$

By substituting the formula for the present value of income (A5_21), (A4_46) might be written as: ${ }^{248}$
$\frac{1+r}{r-x} C_{0}=Y_{0} \frac{1+r}{r-g}$
$C_{0}^{*}=Y_{0} \frac{r-x^{*}}{r-g}$
As can be seen, the optimum present consumption is below present income if the optimum growth rate of consumption is greater than the exogenous growth rate in income ( $\mathrm{x}^{*}>\mathrm{g}$ ). In this case, the consumer is a saver. On the other hand, it is optimal to be a borrower when the opposite assumption holds. The set of parameters leading to $\mathrm{x}^{*}>\mathrm{g}$ or $\mathrm{x}^{*}<\mathrm{g}$ can be found in (A4_71) and (A4_74) respectively, which is, however, derived from (A4_43). To take one example, $\mathrm{x}^{*}>\mathrm{g}$, i.e. the consumer is a lender if the interest rate is large enough compared with the subjective discount rate and the growth rate in labour income. Furthermore, the lower the elasticity of substitution (higher $\theta$ ), the higher interest rate is required to reach the lending position.
(A4_48) clearly demonstrates that the optimum present consumption is equal to the present income $\left(\mathrm{C}_{0}=\mathrm{Y}_{0}=100\right)$, if $\mathrm{x}^{*}=\mathrm{g}$. It then grows at the rate of the labour income ( $\mathrm{g}=1 \%$ ) reaching infinity in the infinite horizon. Saving is zero every period, so is the debt, hence the No-Ponzi-Game condition is satisfied (see A5_15 in Appendix 5 B). Hence, present

[^154]consumption is not depressed to zero in an infinite horizon even in the absence of positive time preference and the presence of positive real interest rate. At the same time, future consumption grows beyond all limits confirming the Euler equation (41) in the main text or (A4_38) in this Appendix. Consumption need not be postponed to indefinite future since the positive growth rate in income will guarantee an infinite future consumption in infinity. Olson and Bailey were perfectly right in this respect, unintentionally refusing the theory of Ludwig von Mises.

The above discussion indicates that the solution of Olson and Bailey leads not only to zero present value of (future) assets (or debts) as time goes to infinity, i.e.:
$\lim _{T \rightarrow \infty} \frac{B_{T}}{(1+\mathrm{r})^{\mathrm{T}}}=0$
but also to zero future value of (future) assets (or debts) as time grows beyond all limits, because no debt is ever issued or no asset is ever accumulated:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} B_{T}=0 \tag{A4_50}
\end{equation*}
$$

However, condition (A4_50) is more restrictive than condition (A4_49). Moreover, the intertemporal budget constraint used here and presented in Olson and Bailey (1981) is consistent with (A4_49). Discussion in Appendix 5 B suggests that the presence of the infinite discounting in (A4_49) is consistent not only with non-zero debt or assets, but even with an infinite debt or assets provided that the numerator grows more slowly than the denominator. Thus, (A4_50) might be non-zero even if (A4_49) is zero. As a result, much wider range of possibilities could be consistent with the Euler equation (A4_38), which requires either zero present consumption or an infinite future consumption, and with the intertemporal budget constraint (A4_2) (or alternatively with the No-Ponzi-Game condition A4_49). However, such a proof demands a very detailed analysis of the behaviour of $B_{t}$ in the infinite horizon.
Before we explore the dynamics of debt (assets) in the infinite horizon, let us present a simple path of consumption that is consistent with non-zero present consumption, infinite future consumption (see A4_38), and with the intertemporal budget constraint (A5_13) (or A4_2 in the infinite horizon). This path also builds on the ideas developed in the finite horizon models presented above. Suppose that a representative agent has a constant flow of income for the entire infinite planning horizon $(\mathrm{g}=0)$ and that the interest rate is positive $(\mathrm{r}>0)$. Suppose further that he reduces his present consumption only by a tiny amount below the level of his present income. Let us assume that in subsequent periods, his consumption will be the same as the given period labour income. Even with no additional restriction on further consumption, the initial saving enables him to reach infinite consumption in the future due to the infinite compounding of interest. A tiny amount of savings in the present results in the infinite value of assets in infinity provided that the interest rate is positive.
Moreover, our agent can consume even more than his given period income $\left(C_{t}>Y_{t}\right)$ provided that the base of his assets is not undermined. The reason lies in the fact that in the infinite horizon, any positive value of assets will grow to infinite value even when part of these assets is consumed in some future periods. Thus, the representative agent can consume even more than is the value of the disposable income in the given period $\left(C_{t}>Y_{t}+r_{t} B_{t-1}\right)$ if the value of his assets is being kept above zero. It should be stressed that such a strategy is required due to condition (A4_49) because assets must grow more slowly than the interest rate to satisfy the intertemporal budget constraint (NPG with equality).

We do not claim that such a strategy will maximize life-time utility (A4_1) in the infinite horizon. However, it clearly demonstrates that there might exist a path of consumption that is consistent with zero time preference, positive real interest rate, positive (i.e. non-zero) present consumption, infinite future consumption, constant flow of income, and also with the intertemporal budget constraint.

We already know that the utility maximizing path of consumption must obey the Euler equation. Equation (A4_44) then determines the optimum growth rate of consumption x. If this growth rate is lower than the interest rate, the present value of consumption in infinity might be zero, even if the size of consumption as such is infinite and equation (A4_38) is thus satisfied.

To find the set of parameters that is consistent with this condition, let us realize that the present value of optimum consumption in infinity is (using A4_9):
$\lim _{T \rightarrow \infty} \frac{C_{T}}{(1+r)^{T}}=\lim _{T \rightarrow \infty} \frac{C_{0}\left(\frac{1+r}{1+\rho}\right)^{T / \theta}}{(1+r)^{T}}$
Zero present value requires:
$\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}<(1+r)$
or
$(1+r)^{(1-\theta) / \theta}<(1+\rho)^{\theta}$
However, (A4_53) leads to condition (A4_35) thoroughly discussed before. Assuming zero time preference ( $\rho=0 \%$ ), the present value of future consumption might be zero (see A4_51) in the infinite horizon even if the future value is infinite (the numerator in A4_51) only for $\theta>1$. Surprisingly, we again arrived at our well known condition of dramatically decreasing marginal utility. Notice that we have not assumed anything about the specific time shape of the labour income.

Thus, let us here use both ideas from previous paragraphs. There might exist a consumption path with non-zero present consumption and infinite future consumption that is both consistent with the intertemporal budget constraint (NPG condition) and with the Euler equation, notwithstanding the time shape of the labour income stream.

Let us now explore the time path of debt (or assets) for the optimum path of consumption in the infinite horizon. We already know the optimum growth rate of consumption $\mathrm{x}^{*}$ (see A4_44) and the optimum level of present consumption $\mathrm{C}_{0}{ }^{*}$ ( $\mathrm{A} 4 \_48$ ). The key question is whether zero time preference requires both an increasing time shape of labour income and dramatically diminishing marginal utility, or if some of these assumptions might be omitted. Our goal is also to determine the resulting development of debt (or assets) in such a case.
The value of optimal assets (or debt) at time $t$ is (see Appendix 5 B for the general law of motion of $B_{t}$ ):

$$
\begin{equation*}
B_{t}^{*}=Y_{t}+B_{t-1}^{*}(1+r)-C_{t}^{*} \tag{A4_54}
\end{equation*}
$$

Alternatively, (A4_54) might be written as:

$$
\begin{equation*}
B_{t}^{*}=Y_{0}(1+g)^{t}+B_{t-1}^{*}(1+r)-C_{0}^{*}\left(1+x^{*}\right)^{t} \tag{A4_55}
\end{equation*}
$$

Let us solve this difference equation iteratively. The optimum size of assets (or debt) in the present is:

$$
\begin{equation*}
B_{0}^{*}=Y_{0}-C_{0}^{*} \tag{A4_56}
\end{equation*}
$$

One period ahead, we may write:

$$
\begin{equation*}
B_{1}^{*}=Y_{1}+B_{0}^{*}(1+r)-C_{1}^{*} \tag{A4_57}
\end{equation*}
$$

This leads to:
$B_{1}^{*}=B_{0}^{*}(1+r)+Y_{0}(1+g)-C_{0}^{*}\left(1+x^{*}\right)$

In period 2, (A4_55) is:
$B_{2}^{*}=Y_{2}+B_{1}^{*}(1+r)-C_{2}^{*}$

By plugging (A4_57):

$$
\begin{equation*}
B_{2}^{*}=(1+r)\left[B_{0}^{*}(1+r)+Y_{0}(1+g)-C_{0}^{*}\left(1+x^{*}\right)\right]+Y_{0}(1+g)^{2}-C_{0}^{*}\left(1+x^{*}\right)^{2} \tag{A4_60}
\end{equation*}
$$

And by inserting (A4_56) and rearranging terms:

$$
\begin{equation*}
B_{2}^{*}=(1+r)^{2} Y_{0}+(1+r)(1+g) Y_{0}+Y_{0}(1+g)^{2}-(1+r)^{2} C_{0}^{*}-(1+r)\left(1+x^{*}\right) C_{0}^{*}-C_{0}^{*}\left(1+x^{*}\right)^{2} \tag{A4_61}
\end{equation*}
$$

At the end of the finite planning horizon T, (A4_61) can be generalized to:

$$
\begin{equation*}
B_{T}^{*}=(1+r)^{T} Y_{0}+(1+g)(1+r)^{T-1} Y_{0}+\ldots+(1+g)^{T} Y_{0}-(1+r)^{T} C_{0}^{*}-\left(1+x^{*}\right)(1+r)^{T-1} C_{0}^{*}-\ldots-\left(1+x^{*}\right)^{T} C_{0}^{*} \tag{A4_62}
\end{equation*}
$$

Adding up all terms with income and consumption separately, (A4_62) might be written as:
$B_{T}^{*}=(1+r)^{T} Y_{0} \frac{1-\left(\frac{1+g}{1+r}\right)^{T+1}}{1-\frac{1+g}{1+r}}-(1+r)^{T} C_{0}^{*} \frac{1-\left(\frac{1+x^{*}}{1+r}\right)^{T+1}}{1-\frac{1+x^{*}}{1+r}}$

$$
\begin{align*}
& B_{T}^{*}=Y_{0} \frac{(1+r)^{T}-\frac{(1+g)^{T+1}}{1+r}}{\frac{1+r-1-g}{1+r}}-C_{0}^{*} \frac{(1+r)^{T}-\frac{\left(1+x^{*}\right)^{T+1}}{1+r}}{\frac{1+r-1-x^{*}}{1+r}}  \tag{A4_64}\\
& B_{T}^{*}=\frac{Y_{0}}{r-g}\left[(1+r)^{T+1}-(1+g)^{T+1}\right]-\frac{C_{0}^{*}}{r-x^{*}}\left[(1+r)^{T+1}-\left(1+x^{*}\right)^{T+1}\right] \tag{A4_65}
\end{align*}
$$

In the infinite time horizon, (A4_65) yields:
$\lim _{T \rightarrow \infty} B_{T}^{*}=\lim _{T \rightarrow \infty}\left\{\frac{Y_{0}}{r-g}\left[(1+r)^{T+1}-(1+g)^{T+1}\right]-\frac{C_{0}^{*}}{r-x^{*}}\left[(1+r)^{T+1}-\left(1+x^{*}\right)^{T+1}\right]\right\}$
Using the expression for the optimum present consumption (A4_48), we may write:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} B_{T}^{*}=\lim _{T \rightarrow \infty}\left\{\frac{Y_{0}}{r-g}\left[(1+r)^{T+1}-(1+g)^{T+1}\right]-\frac{Y_{0} \frac{r-x^{*}}{r-g}}{r-x^{*}}\left[(1+r)^{T+1}-\left(1+x^{*}\right)^{T+1}\right]\right\} \tag{A4_67}
\end{equation*}
$$

$\lim _{T \rightarrow \infty} B_{T}^{*}=\lim _{T \rightarrow \infty}\left\{\frac{Y_{0}}{r-g}\left[\left(1+x^{*}\right)^{T+1}-(1+g)^{T+1}\right]\right\}$

As can be seen, the eventual value of debt (or assets) is zero only if the growth rate of optimum consumption is perfectly equal to the growth rate of labour income ( $\mathrm{x}^{*}=\mathrm{g}$ ). The set of parameters leading to this outcome can be derived from (A4_43):

$$
\begin{equation*}
\ln (1+r)=\theta \ln (1+g)+\ln (1+\rho) \tag{A4_69}
\end{equation*}
$$

If $r, g$, and $\rho$ are small numbers, the resulting interest rate is:

$$
\begin{equation*}
r=\rho+\theta g \tag{A4_69b}
\end{equation*}
$$

However, if these growth rates differ, the total assets grow either to positive infinity or to negative infinity (so to infinite debt). First, if the growth rate of optimum consumption is greater than the growth rate in labour income (i.e. $\mathrm{x}^{*}>\mathrm{g}$ ), assets will be accumulated beyond all limits, even so the NPG condition will be satisfied, as we will show below. We will also see that it is the initial accumulation of assets that will finance relatively greater growth rate of consumption to the infinite future. The formal proof for the statement above is as follows: Provided that $\mathrm{x}^{*}>\mathrm{g}$, (A4_68) might be written as follows:
$\lim _{T \rightarrow \infty} B_{T}^{*}=\frac{Y_{0}}{r-g} \lim _{T \rightarrow \infty}\left(1+x^{*}\right)^{T+1}\left[1-\left(\frac{1+g}{1+x^{*}}\right)^{T+1}\right]=+\infty$

The set of parameters leading to $\mathrm{x}^{*}>\mathrm{g}$ can be derived from (A4_39) and (A4_43):

$$
\begin{equation*}
r>(1+g)^{\theta}(1+\rho)-1 \tag{A4_71}
\end{equation*}
$$

If $r, g$, and $\rho$ are small numbers, (A4_71) yields:

$$
\begin{equation*}
r>\rho+\theta g \tag{A4_72}
\end{equation*}
$$

Thus, the real interest rate must be relatively high to generate an infinite amount of assets in infinity. On the other hand, if the growth rate of optimum consumption is lower than the growth rate of the labour income (i.e. $\mathrm{x}^{*}<\mathrm{g}$ ), an infinite debt will be accumulated even though the rate of the debt accumulation must be lower than the interest rate to satisfy the NPG condition. The proof might use (A4_68) again:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} B_{T}^{*}=\frac{Y_{0}}{r-g} \lim _{T \rightarrow \infty}(1+g)^{T+1}\left[\left(\frac{1+x^{*}}{1+g}\right)^{T+1}-1\right]=-\infty \tag{A4_73}
\end{equation*}
$$

The set of parameters that will lead to such a result is:

$$
\begin{equation*}
r<(1+g)^{\theta}(1+\rho)-1 \tag{A4_74}
\end{equation*}
$$

Or alternatively:

$$
\begin{equation*}
r<\rho+\theta g \tag{A4_75}
\end{equation*}
$$

The real interest rate must be relatively low, people must be impatient enough or the labour income must grow rapidly along with a relatively low elasticity of substitution (low $1 / \theta$ ). Then the debt will increase sky high in the infinite horizon.
Let us now examine the combination of parameters that will satisfy the NPG condition (A4_49). Using (A4_68), the NPG becomes:
$\lim _{T \rightarrow \infty} \frac{B_{T}^{*}}{(1+r)^{T}}=\lim _{T \rightarrow \infty} \frac{\frac{Y_{0}}{r-g}\left[\left(1+x^{*}\right)^{T+1}-(1+g)^{T+1}\right]}{(1+r)^{T}}$

By substituting the optimum growth rate of consumption $x^{*}$ (see A4_44), we can write:
$\lim _{T \rightarrow \infty} \frac{B_{T}^{*}}{(1+r)^{T}}=\frac{Y_{0}}{r-g} \lim _{T \rightarrow \infty} \frac{\left[\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}\right]^{T+1}-(1+g)^{T+1}}{(1+r)^{T}}$
$\lim _{T \rightarrow \infty} \frac{B_{T}^{*}}{(1+r)^{T}}=\frac{Y_{0}}{r-g} \lim _{T \rightarrow \infty} \frac{(1+r)^{-T+(T+1) / \theta}}{(1+\rho)^{(T+1) / \theta}}-\frac{Y_{0}}{r-g} \lim _{T \rightarrow \infty}(1+g)\left(\frac{1+g}{1+r}\right)^{T}$

Because $r$ is greater than $g$ by assumption, the last part in the expression above is zero. (A4_78) might be the then written as:

$$
\begin{align*}
& \lim _{T \rightarrow \infty} \frac{B_{T}^{*}}{(1+r)^{T}}=\frac{Y_{0}}{r-g} \lim _{T \rightarrow \infty} \frac{(1+r)^{(T+1-T \theta) / \theta}}{(1+\rho)^{(T+1) / \theta}}  \tag{A4_79}\\
& \lim _{T \rightarrow \infty} \frac{B_{T}^{*}}{(1+r)^{T}}=\frac{Y_{0}}{r-g}\left(\frac{1+r}{1+\rho}\right)^{1 / \theta} \lim _{T \rightarrow \infty}\left(\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}\right)^{T} \tag{A4_80}
\end{align*}
$$

(A4_80) converges to zero, i.e. the NPG condition is satisfied with equality, if and only if:
$(1+r)^{(1-\theta) / \theta}<(1+\rho)^{1 / \theta}$

But this condition was derived many times before (see A4_53, A4_32 and even A5_29). At this point, we can conclude that zero time preference ( $\rho=0 \%$ ) is consistent with the NPG condition only for a dramatically diminishing marginal utility $(\theta>1)$.
The solution of standard neoclassical growth model implies that on the balanced growth path (at the steady state) the growth rate of the optimum consumption equals the growth rate in labour income. This assumption was also employed by Olson and Bailey. The resulting equilibrium interest rate is then (A4_43). Substituting this equation into (A4_81), we can modify our requirement to:

$$
\begin{equation*}
\left[(1+g)^{\theta}(1+\rho)\right]^{(1-\theta) / \theta}<(1+\rho)^{1 / \theta} \tag{A4_82}
\end{equation*}
$$

$$
\begin{equation*}
(1-\theta) \ln (1+g)+\frac{(1-\theta)}{\theta} \ln (1+\rho)<\frac{1}{\theta} \ln (1+\rho) \tag{A4_83}
\end{equation*}
$$

$$
\begin{equation*}
(1-\theta) \ln (1+g)-\ln (1+\rho)<0 \tag{A4_84}
\end{equation*}
$$

If $g$ and $\rho$ are small numbers, we can write:

$$
\begin{equation*}
\rho-(1-\theta) g>0 \tag{A4_85}
\end{equation*}
$$

On the balanced growth path (where the growth rates of consumption and income are equal), this condition guarantees stability in many parts of our model. If the time preference is zero, income growth must be positive and marginal utility must be dramatically diminishing ( $\theta>1$ ).
However, in our analysis we also explore possibilities of different growth rates of income and optimum consumption. (A4_81) is then the required condition. Before we simulate the behaviour of the key variables, let us examine the optimum rate of debt or asset accumulation in the infinite horizon. In the finite horizon, debt or assets must be zero at the end of the
planning horizon (i.e. $\mathrm{B}_{\mathrm{T}}=0$ ). In the infinite horizon, this is true only if the growth rate of consumption and labour income coincide.

Nevertheless, if these growth rates differ, assets are either infinitely accumulated or debt is infinitely issued. We would like to find the optimum rate of this debt or assets expansion, that is, however, lower than the interest rate, because the NPG condition must be satisfied. Using (A5_16), the optimum growth rate of assets in the infinite horizon is:
$\lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=\lim _{T \rightarrow \infty}\left[\frac{Y_{T}-C_{T}^{*}}{B_{T-1}^{*}}+r\right]$

By inserting (A4_68) for $\mathrm{B}^{*}{ }_{\mathrm{T}-1}$ :
$\lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=\lim _{T \rightarrow \infty}\left[\frac{Y_{0}(1+g)^{T}-C_{0}^{*}\left(1+x^{*}\right)^{T}}{\frac{Y_{0}}{r-g}\left[\left(1+x^{*}\right)^{T}-(1+g)^{T}\right]}+r\right]$
(A4_87)

If we substitute (A4_48) for the optimum present consumption:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=\lim _{T \rightarrow \alpha}\left[\frac{Y_{0}(1+g)^{T}-Y_{0} \frac{r-x^{*}}{r-g}\left(1+x^{*}\right)^{T}}{\frac{Y_{0}}{r-g}\left[\left(1+x^{*}\right)^{T}-(1+g)^{T}\right]}+r\right] \tag{A4_88}
\end{equation*}
$$

A simple rearrangement of terms gives us:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=\lim _{T \rightarrow \infty}\left[\frac{(r-g)(1+g)^{T}-\left(r-x^{*}\right)\left(1+x^{*}\right)^{T}}{\left[\left(1+x^{*}\right)^{T}-(1+g)^{T}\right]}+r\right] \tag{A4_89}
\end{equation*}
$$

If the growth rate of optimum consumption is greater than the growth rate in labour income (i.e. $\left.x^{*}>\mathrm{g}\right),\left(\mathrm{A} 4 \_89\right)$ might be written as:

$$
\begin{align*}
& \lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=\lim _{T \rightarrow \infty}\left[\frac{(r-g)\left(\frac{1+g}{1+x^{*}}\right)^{T}-\left(r-x^{*}\right)}{1-\left(\frac{1+g}{1+x^{*}}\right)^{T}}+r\right]  \tag{A4_90}\\
& \lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=-r+x^{*}+r=x^{*}=\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}-1
\end{align*}
$$

Condition (A4_81) guarantees that $\mathrm{x}^{*}$ is lower than the interest rate r . Thus, if the growth rate of optimum consumption is higher than the growth rate in labour income, the optimum assets grow beyond all limits (see A4_70) at the rate of $\mathrm{x}^{*}<\mathrm{r}$.

On the other hand, if the growth rate of optimum consumption is lower than the growth rate in labour income (i.e. $\mathrm{x}^{*}<\mathrm{g}$ ), (A4_89) can be rearranged as:

$$
\begin{align*}
& \lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=\lim _{T \rightarrow \infty}\left[\frac{(r-g)-\left(r-x^{*}\right)\left(\frac{1+x^{*}}{1+g}\right)^{T}}{\left(\frac{1+x^{*}}{1+g}\right)^{T}-1}+r\right]  \tag{A4_92}\\
& \lim _{T \rightarrow \infty} \frac{\Delta B_{T}^{*}}{B_{T-1}^{*}}=-(r-g)+r=g \tag{A4_93}
\end{align*}
$$

As can be seen, if the growth rate of optimum consumption is lower than the growth rate in labour income, the optimum debt grows beyond all limits (see A4_73) at the rate of $\mathrm{g}<\mathrm{r}$.

Now, we can utilize all the information we have just derived to run simple simulations in an infinite horizon. Our main objective is to test both the Misesian prediction about the reduction of present consumption to zero in the absence of time preference and the Olson and Bailey prediction that the income growth must be positive to obtain non-zero present consumption if the subjective discount rate is zero.
Let us start with the situation when the growth rate in labour income exceeds the optimum growth rate in consumption. This might be achieved, if we use the approximation (42) in the main text instead of the accurate equation (34). Suppose (as Olson and Bailey) that $\mathrm{r}=5 \%, \rho$ $=0 \%, \theta=5$ and $\mathrm{g}=1 \%$. The precise optimum growth rate of consumption is $0.98 \%$. This set of parameters is consistent with condition (A4_32).

Figures No. 18_A4 - 22_A4 show the resulting dynamics of consumption, income, total debt, saving, the growth rate of debt, and the present value of debt. Various time horizons are reported for further clarity. First, the real interest rate ( $\mathrm{r}=5 \%$ ) is greater than the subjective discount rate ( $\rho=0 \%$ ), so the optimum consumption grows over time (see equation A4_8). Second, this optimum growth rate of consumption ( $\mathrm{x}^{*}=0.98 \%$ ) is lower than the growth rate in income ( $\mathrm{g}=1 \%$ ), thus equation (A4_48) implies that the present optimum consumption exceeds present income (see Figure No. 18_A4). As can be seen, present consumption is not depressed to zero, as would be predicted by L. von Mises.


Figure No. 18_A4


Figure No. 19_A4


Figure No. 20_A4


Figure No. 21_A4


Figure No. 22_A4

Third, negative saving in the first part of the planning horizon (the consumer is a borrower) leads to an increasing debt over time. Even though the consumption gradually falls below the given period labour income, saving is still negative due to the repayment of interest from the increasing debt. As a result, debt is not being repaid and it grows ( $B_{t}$ is more and more negative). ${ }^{250}$ In the infinite horizon, the NPG condition, along with the intertemporal budget constraint, only requires that the growth rate in debt is lower than the interest rate. In this particular case, debt reaches an infinite value in infinity (see equation A4_73) growing at the rate of $g$ (see A4_93), even though its present value gradually approaches zero. The optimum growth rate of consumption falls short of the income growth due to the eternal payment of interest. The economic intuition behind this behaviour is as follows: People, in the expectation of higher future income, increase their present consumption above the present period income. This choice is reinforced by relatively high $\theta$, which motivates the individual to smooth consumption over time. High future consumption levels attained due to rapid increase in labour income are moved closer to present owing to negative saving and the accumulation of debt.

Thus, Olson and Bailey's prediction seems to be correct. Zero time preference does not lead to zero present consumption. Its level might even exceed the present labour income, if the preference for consumption smoothing is really high resulting in a low optimum growth rate in consumption.

Figures No. 23_A4 - 25_A4 display optimum paths for $\theta=2$. As we can see, lower preference for consumption smoothing (less dramatically diminishing marginal utility), results in a higher growth in optimum consumption ( $\mathrm{x}^{*}=2.47 \%$ ), which in our case exceeds the growth rate in labour income ( $\mathrm{g}=1 \%$ ). Present consumption is thus depressed below present labour

[^155]income (the consumer is a lender, see A4_48) and assets are gradually accumulated. This asset accumulation is eternal fuelling higher consumption growth. As can be seen, even though the labour income will fall below consumption in the future, the interest income earned from the previous saving allows consumption to grow faster than the labour income. Thus, consumption in very remote future seems to be quite independent of that period labour income. Furthermore, notice that saving need not be especially large in the present and in early periods to generate enough assets. Hence, present consumption is not depressed to zero. Eternal interest earned on assets and the growth of consumption that is lower than the interest rate ( $\mathrm{x}^{*}<\mathrm{r}$ ) result in the fact that assets grow forever (see equation A4_70). The growth rate at which the assets are growing in the infinite horizon is $x^{*}$ (see A4_91).


Figure No. 23_A4


Figure No. 24_A4


Figure No. 25_A4

The previous analysis suggests that the accumulation of interest might suffice to feed high consumption growth regardless of the behaviour of the labour income. Let us suppose that the labour income is constant. This assumption is closest to the Misesian ERE. Other assumptions from the previous sections will be kept. Thus, the real interest rate is positive, $r=5 \%$, and the time preference (in sense two) is zero. Mises would suggest that the present consumption must be depressed to zero and the act of consumption will never occur. Olson and Bailey require a positive income growth for the parameters above. If it is constant, they predict the same outcome as L. von Mises.

However, let us demonstrate that present consumption might be positive even with constant labour income stream. The only requirement is a relatively low elasticity of substitution ( $\theta>1$ ), i.e. what Olson and Bailey called a dramatically diminishing marginal utility. Figures No. 26_A4 - 28_A4 display the development of the key variables for $\theta=5$. As can be seen, optimum consumption growth is positive ( $x^{*}=0.98 \%$ again) because the real interest rate exceeds the subjective discount rate. Present consumption is lower than present income, which leads to positive present saving. The same observation holds for a few early periods. As a result, positive assets are accumulated, which allows consumption to exceed labour income in the future. Compared with the situation of positive income growth ( $\mathrm{g}=1 \%$ ), present consumption cannot be greater than present labour income (see A4_48). Nevertheless, it is not depressed to zero as Mises, and Olson and Bailey would suggest. In the infinite horizon, the initial accumulation of assets generates enough capital income in the future to fuel eternal consumption growth. At the same time, assets are growing at the rate of $x^{*}$, because $x^{*}=$ $0.98 \%$ is greater than $\mathrm{g}=0 \%$ (see the discussion from A4_86 to A4_91), reaching positive infinity in the infinite horizon (see A4_70). Notice that the NPG condition (the intertemporal budget constraint) is satisfied with equality because the present value of assets gradually falls to zero (Figure No. 27_A4).


Figure No. 26_A4


Figure No. 27_A4


Figure No. 28_A4

The economic reason for such behaviour is as follows. If the marginal utility is sufficiently decreasing, the additional units of consumption goods satisfy wants of much lower urgency. Even for zero time preference (in sense two), i.e. if the given want is not preferred to be satisfied as soon as possible, and for the positive interest rate, the representative consumer has a tendency to spread consumption goods evenly over time. Overprovision of goods in the future and starving in the present cannot be optimal, because present wants of very high urgency would stay ungratified, whereas the abundance of future consumption goods will satisfy wants of only very low importance. As a result, the optimum behaviour is not to reduce present consumption to zero, but to some reasonable levels. On the other hand, these levels must be sufficiently low to accumulate enough assets in the early periods. Then, the consumer may enjoy future interest income as a source for eternal consumption growth. Thus, there is no need for the labour income to grow over time to satisfy Olson and Bailey's condition (41) from the main text. The eternal accumulation of interest will guarantee infinite future consumption and the present consumption need not fall to negligible levels.

Our earlier analysis suggested that as $\theta$ approaches 1 , the present consumption is depressed to zero provided that the time preference does not exist $(\rho=0 \%)$. We also demonstrated that if $\rho$ $=0 \%$ the stability of the model requires $\theta>1$. Figures No. 29_A4 - 31_A4 clearly show that if the utility function approaches the logarithmic form $(\theta \rightarrow 1)$, the solution of Mises is much more plausible to emerge. Consumption in the present and in the early periods is negligible for the benefit of the future. Assets are growing at the rate of $x^{*}=4.95 \%$. This figure is very close to the size of the interest rate. As a result, the present value of assets converges to zero at a very slow pace (see Figure No. 30_A4).


Figure No. 29_A4


Figure No. 30_A4


Figure No. 31_A4

We can conclude that the Misesian theory holds only under special circumstances characterised by a relatively high elasticity of intertemporal substitution in consumption $(\theta \leq 1)$, i.e. the utility function must exhibit weakly diminishing marginal utility. In such a case, the absence of time preference leads to zero consumption in all times except for infinity. Thus, Olson and Bailey suggested that the utility function must exhibit "drastically diminishing" marginal utility and income must grow over time. However, we clearly demonstrated that the path of labour income is quite immaterial provided that its growth rate is lower than the interest rate.


Figure No. 32_A4


Figure No. 33_A4


Figure No. 34_A4

Let us even consider a diminishing time shape of the labour income stream. Even in this case, the requirement (A4_53) guarantees an infinite future consumption and positive and relatively
high (depending on $\theta$ ) present consumption. See Figures No. 32_A4 - 34_A4 that report behaviour of the key variables for $\mathrm{r}=5 \%, \theta=5, \rho=0 \%$, and $\mathrm{g}=-1 \%$. As can be seen, they are almost indistinguishable from Figures No. 26_A4 - 28_A4. The only difference is lower present consumption that is required to accumulate enough assets serving as a source for gradually increasing future consumption.
As an extreme case of diminishing labour income endowment, we may consider a Fisherian infinitely lived sailor shipwrecked with a stock of goods that have a constant productive power (e.g. herd of sheep or rice). Suppose that the initial endowment of this good is 100 . It cannot be enlarged, i.e. the future income endowment is $0(\mathrm{~g}=-100 \%)$ unless it is wisely invested. The marginal productivity of this good is constant MPK $=r=5 \%$. Suppose that the subjective discount rate is $0 \%$, and $\theta=5$. As can be seen in Figures No. 35_A4 - 37_A4, the sailor might enjoy an increasing consumption flow ( $x^{*}=0.98 \%$ ) reaching infinity in the infinite horizon even though his labour income is zero over the entire lifetime (formally except for the present). All sources of future consumption consist of capital income that is generated owing to considerable saving in early periods. Although this saving is relatively high, present consumption is not depressed to zero even in this extreme case.


Figure No. 35_A4


Figure No. 36_A4


Figure No. 37_A4

At the end of this Appendix, let us stress that our analysis examined the optimal behaviour of one representative agent. We tried to demonstrate that the theory of L. von Mises (1996) and Olson and Bailey (1981) are inaccurate. However, the solution offered by the latter, i.e. the uniform growth rate of income and consumption, seems to be sensible for the aggregate economy. Our case of constant labour income stream and an increasing path of consumption
led to an eternal accumulation of assets. It is quite difficult to generalize such behaviour to the entire economy. It seems to be quite reasonable to assume that the surplus of saving from the beginning of the planning horizon will depress the interest rate to the level of the subjective discount rate, i.e. to zero. However, this would violate condition (A4_53), and the model would collapse. Thus, zero saving, zero eternal debt (assets), positive interest rate, and zero time preference seem to be a stable situation at the aggregate level only for an increasing income stream and low elasticity of substitution $(\theta>1)$.

## Appendix 5 B Intertemporal Budget Constraint in the Finite and Infinite Horizon

A) This appendix serves as a technical support for Section 5 in the main text and for Appendix 4 B. First, let us derive the intertemporal budget constrain in (A4_2) or (30) step by step. The present labour (i.e. non-capital) income earned in period 0 might be used for present consumption $\mathrm{C}_{0}$ or it might be saved $\left(\mathrm{B}_{0}\right)$. If $\mathrm{B}_{0}$ is positive, the individual is a saver in period 0 . If it is negative, the consumer is a debtor in period 0 .
$Y_{0}=C_{0}+B_{0}$
In the next period, the accumulated saving increased by interest $\mathrm{r}_{1}$ together with the labour income earned that period represent sources for consumption $\mathrm{C}_{1}$ (see A5_2). If sources do not suffice, the individual must issue a debt at the size of $B_{1}<0$. If consumption falls short of the size of sources, he can buy a bond, and $B_{1}$ is positive. It must be stressed that there is no necessary connection between $\mathrm{B}_{0}$ and $\mathrm{B}_{1}$. Thus, the individual might be a creditor in period 0 (i.e. $\mathrm{B}_{0}>0$ ), and he may become a debtor in period 1 (i.e. $\mathrm{B}_{1}<0$ ) if his consumption sufficiently exceeds his sources in that period (i.e. $\left.Y_{1}+B_{0}\left(1+r_{1}\right)<C_{1}\right)$.
$Y_{1}+B_{0}\left(1+r_{1}\right)=C_{1}+B_{1}$
(A5_2) might be rewritten as:
$Y_{1}+r_{1} B_{0}-C_{1}=B_{1}-B_{0}=\Delta B_{1}$

Term ( $\mathrm{Y}_{1}+\mathrm{r}_{1} \mathrm{~B}_{0}$ ) represents his disposable income in period 1. The entire left-hand side stands for his saving as it shows a difference between disposable income and consumption. The right-hand side of the equation indicates a change in his net lending/borrowing position. Thus, saving in the given period has a crucial impact on the change in the lending (or borrowing) position. It should be stressed that we call saving only the difference between disposable income $\left(\mathrm{Y}_{1}+\mathrm{r}_{1} \mathrm{~B}_{0}\right)$ and $\mathrm{C}_{1}$ (i.e. $\left.\mathrm{B}_{1}-\mathrm{B}_{0}=\Delta \mathrm{B}_{1}\right)$ not the difference between his entire wealth $\left(Y_{1}+\left(1+r_{1}\right) B_{0}\right)$ and consumption $C_{1}$ (i.e. this difference would be $\left.B_{1}\right)$. The reason is that $\mathrm{B}_{1}$ we will call a debt if it is negative, or "wealth before income" if it is positive. ${ }^{251}$

As can be seen, debt from the previous period is increased owing to the interest rate that is due in the subsequent period. Suppose that $\mathrm{B}_{0}<0$. This debt will be $\mathrm{B}_{0}\left(1+\mathrm{r}_{1}\right)$ at the beginning

[^156]of the next period. Provided that the entire labour income in period $1\left(\mathrm{Y}_{1}\right)$ is completely consumed (i.e. $\left.\mathrm{Y}_{1}=\mathrm{C}_{1}\right)$, this debt can be easily financed by issuing a new debt $\left(\mathrm{B}_{1}=\left(1+\mathrm{r}_{1}\right) \mathrm{B}_{0}\right)$. As a result, debt will grow at the rate of the respective interest rate $\left(B_{1} / B_{0}-1=r_{1}\right)$ if the labour income in the given period is fully consumed. In such a case, we will say that the debt is being rolled over to the future.
If the individual consumes only his disposable income ( $\mathrm{C}_{1}=\mathrm{Y}_{1}+\mathrm{r}_{1} \mathrm{~B}_{0}$ ), his debt in that period $B_{1}$ will be stabilized at the level of the previous period $B_{0}$, and the change in the borrowing position will be nil ( $\Delta \mathrm{B}_{1}=0$ ). Furthermore, the entire debt burden might be eliminated (i.e. $\left.B_{1}=0\right)$ only if consumption is low enough $\left(C_{1}=Y_{1}+\left(1+r_{1}\right) B_{0}>0\right)$, but still positive.
Thus, a debt is not being rolled over by a simple issuing of a new debt if consumption is lower than the labour income in the given period (i.e. $\mathrm{C}_{1}<\mathrm{Y}_{1}$ ). From (A5_2) it can be seen that in such a case debt in the present value is partly reduced $\left(B_{1} /\left(1+r_{1}\right)>B_{0}\right.$, i.e. $B_{1} /\left(1+r_{1}\right)$ is less negative than $B_{0}$ ) because $B_{1}>B_{0}\left(1+r_{1}\right)$. However, the size of the debt (in the given period value) might increase (i.e. $\mathrm{B}_{1}<\mathrm{B}_{0}$, thus $\Delta \mathrm{B}_{1}<0$ ) if consumption is not lower than the entire disposable income ( $C_{1}>Y_{1}+r_{1} B_{0}$ ). Reduction of debt in the value of the given period therefore requires positive saving in that period $\left(C_{1}<Y_{1}+r_{1} B_{0}\right)$.
If we substitute $B_{0}$ from (A5_1) into (A5_2) and after a simple manipulation, we can write:
\[

$$
\begin{equation*}
Y_{0}+\frac{Y_{1}}{\left(1+r_{1}\right)}=C_{0}+\frac{C_{1}}{\left(1+r_{1}\right)}+\frac{B_{1}}{\left(1+r_{1}\right)} \tag{A5_4}
\end{equation*}
$$

\]

In the two-period model, we implicitly assumed that all debts (in the present value) must be eventually settled, therefore $\mathrm{B}_{1} /\left(1+\mathrm{r}_{1}\right) \geq 0$. Such a condition might be called a No-Ponzi-Game condition for a two-period model. However, this also implies that $\mathrm{B}_{1} \geq 0$. At the same time, for a monotonically increasing utility function, ${ }^{252}$ it would not be optimal to hold positive assets at the end of life (i.e. in period one), since consumption of all assets might increase utility. Thus, in the two-period model the last term in (A5_4) should disappear. As a result, this equation simply states that the flow of consumption in the present value might not exceed (or better, is equal to) the flow of income in the present value.
However, let us now extend the time horizon to T. The budget constraint in period 2 (following the example of A5_2) can be written as:

$$
\begin{equation*}
Y_{2}+B_{1}\left(1+r_{2}\right)=C_{2}+B_{2} \tag{A5_5}
\end{equation*}
$$

Again, a debt in this period might be just rolled over to the future if $\mathrm{Y}_{2}=\mathrm{C}_{2}$. Thus, it will rise by the interest (rate) because $\mathrm{B}_{2}=\left(1+\mathrm{r}_{2}\right) \mathrm{B}_{1}$. Of course, debt can increase even more if that period consumption exceeds that period labour income. In such a case, the growth rate of debt will even exceed the rate that is implied by a simple roll-over strategy ( $\mathrm{B}_{2} / \mathrm{B}_{1}-1>\mathrm{r}_{2}$ ). However, it might be also partly reduced in the present value if $\mathrm{C}_{2}<\mathrm{Y}_{2}$. Furthermore, its size could be stabilized ( $B_{2}=B_{1} ; \Delta B_{2}=0$ ) when consumption is reduced even more to the level of the disposable income ( $\left.C_{2}=Y_{2}+r_{2} B_{1}\right)$. In other words, consumption in that period must be low enough so that the consumer pays off (out of the given period labour income) the interest from the previous debt. And finally, debt could be completely eliminated, if consumption is depressed to such a level that $\mathrm{C}_{2}=\mathrm{Y}_{2}+\left(1+\mathrm{r}_{2}\right) \mathrm{B}_{1}$. However, it might happen that the resulting consumption is negative because ( $1+\mathrm{r}_{2}$ ) $\mathrm{B}_{1}$ is too high (in negative value). In such a case, debt cannot be eliminated in that given period. Its elimination then requires a sufficient reduction

[^157]of consumption also in subsequent periods because consumption cannot fall below zero in any period.

Now, let us substitute (A5_4) into (A5_5) to obtain a similar idea as in (A5_4). A simple rearrangement of terms yields:

$$
\begin{equation*}
Y_{0}+\frac{Y_{1}}{\left(1+r_{1}\right)}+\frac{Y_{2}}{\left(1+r_{1}\right)\left(1+r_{2}\right)}=C_{0}+\frac{C_{1}}{\left(1+r_{1}\right)}+\frac{C_{2}}{\left(1+r_{1}\right)\left(1+r_{2}\right)}+\frac{B_{2}}{\left(1+r_{1}\right)\left(1+r_{2}\right)} \tag{A5_6}
\end{equation*}
$$

If we generalize this procedure to T periods, (A5_6) becomes:

$$
\begin{gather*}
\mathrm{C}_{0}+\frac{\mathrm{C}_{1}}{\left(1+\mathrm{r}_{1}\right)}+\frac{\mathrm{C}_{2}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right)}+\ldots+\frac{\mathrm{C}_{T}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots\left(1+\mathrm{r}_{\mathrm{T}}\right)}+\frac{\mathrm{B}_{T}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots\left(1+\mathrm{r}_{\mathrm{T}}\right)}= \\
=\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{\left(1+\mathrm{r}_{1}\right)}+\frac{\mathrm{Y}_{2}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right)}+\ldots+\frac{\mathrm{Y}_{T}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots\left(1+\mathrm{r}_{\mathrm{T}}\right)} \tag{A5_7}
\end{gather*}
$$

If the interest rate is constant over time, (A5_7) might be written as:

$$
\begin{align*}
& C_{0}+\frac{1}{1+r} C_{1}+\frac{1}{(1+r)^{2}} C_{2}+\frac{1}{(1+r)^{3}} C_{3}+\ldots+\frac{1}{(1+r)^{T}} C_{T}+\frac{1}{(1+r)^{T}} B_{T}= \\
& =Y_{0}+\frac{1}{1+r} Y_{1}+\frac{1}{(1+r)^{2}} Y_{2}+\frac{1}{(1+r)^{3}} Y_{3}+\ldots+\frac{1}{(1+r)^{T}} Y_{T} \tag{A5_8}
\end{align*}
$$

Olson and Bailey (1981:9) assumed the following form of the intertemporal budget constraint:
$\sum_{t=0}^{T} \frac{Y_{0}^{*}-C_{t}}{(1+r)^{t}}=0$

First, they assumed a constant level of income $\mathrm{Y}_{0}{ }^{*}$ and constant interest rate r over time. Second, they implicitly imposed the requirement that all debts must be repaid in the end. Thus, $\mathrm{B}_{\mathrm{T}} /(1+\mathrm{r})^{\mathrm{T}} \geq 0$. However, this also implies that $\mathrm{B}_{\mathrm{T}} \geq 0$, because $(1+\mathrm{r})^{\mathrm{T}}$ is finite and positive. Again, this might be called a No-Ponzi-Game condition in a T-period model. At the same time, it cannot be optimal for an individual to hold any assets at time T. Thus, $\mathrm{B}_{\mathrm{T}}=0$ and this term disappears in (B5_7) or (B5_8). As a result, (B5_7) or (B5_8) might be interpreted as follows: The discounted flow of consumption may not exceed (or better, is equal to) the discounted flow of income. Hence, we arrived at the now familiar form of the intertemporal budget constraint used in the main text ( 34 and 35 ) and in Appendix 4 B (A4_2).
Furthermore, (B5_7) implies that debt must not be rolled-over up to time T. Thus, $\mathrm{C}_{\mathrm{t}}$ must not exceed $Y_{t}$ in too many periods if some debt was issued in the past. But (B5_7) (or B5_8) requires even more. The net borrowing position must gradually decline ( $\Delta \mathrm{B}_{\mathrm{t}}>0$ ). Hence, consumption should be even lower than the disposable income, and saving must be positive $\left(C_{t}<Y_{t}+r_{t} B_{t-1}\right)$ in sufficient number of periods to reach zero debt at time $T$.

One technical note might be mentioned here, as it will be used in an infinite-horizon model. (A5_5) at time T is:

$$
\begin{equation*}
Y_{T}+B_{T-1}\left(1+r_{T}\right)=C_{T}+B_{T} \tag{A5_10}
\end{equation*}
$$

This equation might be rearranged to:

$$
\begin{equation*}
\frac{Y_{T}-C_{T}}{B_{T-1}}+r_{T}=\frac{B_{T}-B_{T-1}}{B_{T-1}} \tag{A5_11}
\end{equation*}
$$

If the debt (or assets) at time $\mathrm{T}-1$ (i.e. $\mathrm{B}_{\mathrm{T}-1}$ ) is big enough, the growth rate in debt (or assets) is almost unaffected by the difference between consumption and labour income and the major determinant is simply the interest rate. This holds for any time $t$ if $\mathrm{B}_{\mathrm{t}-1}$ is large enough compared with $\mathrm{Y}_{\mathrm{t}}$ and $\mathrm{C}_{\mathrm{t}}$.

Olson and Bailey (1981) extended the analysis to an infinite horizon. As $T$ approaches infinity, (A5_7) becomes:

$$
\begin{equation*}
\mathrm{C}_{0}+\frac{\mathrm{C}_{1}}{\left(1+\mathrm{r}_{1}\right)}+\frac{\mathrm{C}_{2}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right)}+\ldots=\mathrm{Y}_{0}+\frac{\mathrm{Y}_{1}}{\left(1+\mathrm{r}_{1}\right)}+\frac{\mathrm{Y}_{2}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right)}+\ldots \tag{A5_12}
\end{equation*}
$$

And if the interest rate is constant, then (see A5_8):

$$
\begin{equation*}
C_{0}+\frac{1}{1+r} C_{1}+\frac{1}{(1+r)^{2}} C_{2}+\frac{1}{(1+r)^{3}} C_{3}+\ldots=Y_{0}+\frac{1}{1+r} Y_{1}+\frac{1}{(1+r)^{2}} Y_{2}+\frac{1}{(1+r)^{3}} Y_{3}+\ldots \tag{A5_13}
\end{equation*}
$$

The No-Ponzi-Game condition imposed on (A5_12) is (following our previous discussion):
$\lim _{T \rightarrow \infty} \frac{B_{T}}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots} \geq 0$

And the No-Ponzi-Game condition imposed on (A5_13) might be written as:
$\lim _{T \rightarrow \infty} \frac{B_{T}}{(1+\mathrm{r})^{\mathrm{T}}} \geq 0$

The NPG in an infinite horizon, as well as in the finite horizon, states that the present value of assets cannot be negative. ${ }^{253}$ However, in the infinite horizon this does not necessarily mean that $\mathrm{B}_{\mathrm{T}}$ is non-negative too. The reason lies in the fact, that the denominator in (A5_14) or (A5_15) is infinite for any positive interest rate. Thus, NPG might be satisfied even for a negative $\mathrm{B}_{\mathrm{T}}$ because discounting from infinity raises the denominator beyond all limits. Moreover, debt can expand beyond all limits too ( $\mathrm{B}_{\mathrm{T}}$ can be negative infinity) if it reaches infinity "later" than the denominator in (A5_14) or (A5_15). Thus, even an infinite debt in infinity is consistent with the intertemporal budget constraints (A5_12) and (A5_13) and the one assumed by Olson and Bailey (1981).

[^158]To determine the growth rate at which the debt might be growing in the infinite horizon, apply (A5_11):

$$
\begin{equation*}
\lim _{T \rightarrow \infty}\left[\frac{Y_{T}-C_{T}}{B_{T-1}}+r_{T}\right]=\lim _{T \rightarrow \infty} \frac{\Delta B_{T}}{B_{T-1}} \tag{A5_16}
\end{equation*}
$$

Thus, if $\mathrm{B}_{\mathrm{T}-1}$ overcomes $\left(\mathrm{Y}_{\mathrm{T}-1}-\mathrm{C}_{\mathrm{T}-1}\right)$ in the infinite horizon, the growth rate of debt might be very close to the interest rate and no saving is needed. Even so, the intertemporal budget constraint will be satisfied.

Let us now discuss the present value of the income flow, i.e. the right hand side of the intertemporal budget constraint (A5_13). We will assume for simplicity that the interest rate is constant over time. Furthermore, it will be required that the present value of income is finite even in the infinite horizon. The reason for this will become obvious below. Next, we assume that the (labour) income process is represented by the following equation:
$Y_{t}=(1+g) Y_{t-1}$
Alternatively, (A5_17) implies:

$$
\begin{equation*}
Y_{t}=Y_{0}(1+g)^{t} \tag{A5_18}
\end{equation*}
$$

Thus, income is growing at some given exogenous and constant rate. When $g$ is zero, the labour income is constant over time. If it is negative, labour income falls over time. Substituting (A5_18) into the right hand side of (A5_13), we get:

$$
\begin{equation*}
P V_{\text {income }}=Y_{0}+\frac{Y_{0}(1+g)}{1+r}+\frac{Y_{0}(1+g)^{2}}{(1+r)^{2}}+\frac{Y_{0}(1+g)^{3}}{(1+r)^{3}}+\ldots \tag{A5_19}
\end{equation*}
$$

A simple formula for the sum of the infinite geometric series gives us:

$$
\begin{align*}
& P V_{\text {income }}=Y_{0} \frac{1}{1-\frac{1+g}{1+r}}  \tag{A5_20}\\
& P V_{\text {income }}=Y_{0} \frac{1+r}{r-g} \tag{A5_21}
\end{align*}
$$

A finite PV in infinite horizon requires that the interest rate is greater than the growth rate in labour income. But this is exactly the condition for a dynamically efficient economy. Thus, $r>g$ seems to be a reasonable assumption.
In Appendix 4 B , a formula for the PV of income in the finite horizon was used. Consider the right-hand side of equation (A5_8) for zero debt at time $T$. Suppose that $\mathrm{Y}_{\mathrm{t}}=(1+\mathrm{g})^{\mathrm{t}} \mathrm{Y}_{0}$. The PV of income is then:

$$
\begin{equation*}
P V_{\text {income }}=Y_{0}+\frac{Y_{0}(1+g)}{1+r}+\frac{Y_{0}(1+g)^{2}}{(1+r)^{2}}+\frac{Y_{0}(1+g)^{3}}{(1+r)^{3}}+\ldots+\frac{Y_{0}(1+g)^{T}}{(1+r)^{T}} \tag{A5_22}
\end{equation*}
$$

According to the formula of the sum of the finite geometric sequence: ${ }^{254}$

$$
\begin{align*}
& P V_{\text {income }}=Y_{0} \frac{1-\left(\frac{1+g}{1+r}\right)^{T+1}}{1-\frac{1+g}{1+r}}  \tag{A5_23}\\
& P V_{\text {income }}=\frac{(1+r)^{T+1}-(1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_{0} \tag{A5_24}
\end{align*}
$$

## B) Present Value of Lifetime Utility

In this part, the lifetime utility function will be studied in more detail. Surprisingly, condition (A4_32) will be derived again. In this case, it will guarantee a convergence of the sum of instantaneous utilities.
(A4_1) for the CRRA might be written as:
$U=\sum_{t=0}^{\infty} \frac{\frac{C_{t}^{1-\theta}}{1-\theta}}{(1+\rho)^{t}}=\frac{C_{0}^{1-\theta}}{1-\theta}+\frac{1}{1+\rho} \frac{C_{1}^{1-\theta}}{1-\theta}+\frac{1}{(1+\rho)^{2}} \frac{C_{2}^{1-\theta}}{1-\theta}+\ldots$

The optimum path of consumption is described by (A4_9). Equation (A5_25) then yields:
$U=\frac{C_{0}^{1-\theta}}{1-\theta}+\frac{1}{1+\rho} \frac{\left[C_{0}\left(\frac{1+r}{1+\rho}\right)^{1 / \theta}\right]^{1-\theta}}{1-\theta}+\frac{1}{(1+\rho)^{2}} \frac{\left[C_{0}\left(\frac{1+r}{1+\rho}\right)^{2 / \theta}\right]^{1-\theta}}{1-\theta}+\ldots$
$U=\frac{C_{0}^{1-\theta}}{1-\theta}+\frac{C_{0}^{1-\theta}}{1-\theta} \frac{1}{1+\rho}\left(\frac{1+r}{1+\rho}\right)^{(1-\theta) / \theta}+\frac{C_{0}^{1-\theta}}{1-\theta} \frac{1}{(1+\rho)^{2}}\left[\left(\frac{1+r}{1+\rho}\right)^{(1-\theta) / \theta}\right]^{2}+\ldots$

The sum of this infinite series is:
$U=\frac{C_{0}^{1-\theta}}{1-\theta} \frac{1}{1-\frac{1}{1+\rho}\left(\frac{1+r}{1+\rho}\right)^{(1-\theta) / \theta}}$
$U=\frac{C_{0}^{1-\theta}}{1-\theta} \frac{1}{1-\frac{(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}}$

[^159]This sum converges if and only if condition (A4_32) holds. Thus, if the time preference is non-existent ( $\rho=0$ ), the marginal utility must be "drastically diminishing" $(\theta>1)$.

Furthermore, by substituting (A4_37) into (A5_29), we get:
$U=\frac{\left[Y_{0} \frac{1+r}{r-g} \frac{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}{(1+\rho)^{1 / \theta}}\right]^{1-\theta}}{1-\theta} \frac{(1+\rho)^{1 / \theta}}{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}$
$U=\frac{\left(Y_{0} \frac{1+r}{r-g}\right)^{1-\theta}}{1-\theta} \frac{\left[(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}\right]^{1-\theta}}{(1+\rho)^{(1-\theta) / \theta}} \frac{(1+\rho)^{1 / \theta}}{(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}}$
$U=\frac{\left(Y_{0} \frac{1+r}{r-g}\right)^{1-\theta}}{1-\theta} \frac{1+\rho}{\left[(1+\rho)^{1 / \theta}-(1+r)^{(1-\theta) / \theta}\right]^{\theta}}$

As can be seen, lifetime utility increases with higher present labour income $\mathrm{Y}_{0}$ and higher growth rate of the labour income $g$. It can be shown that the impact of $r$ on $U$ depends on the net lending/borrowing position of the consumer. If he is a lender (see A4_71), an increase in the interest rate will raise his lifetime utility. On the other hand, the borrowing position (A4_74) will result in the opposite outcome.


Figure No. 1_A5. CRRA utility function, $\rho=r>0$ and $\theta<1$.

Equation (A5_29) (or A4_35) suggests that convergence of lifetime utility is consistent with moderately diminishing marginal utility $(\theta<1)$ only if people discount future utilities $(\rho>0)$. Figure No. 1_A5 outlines this idea. In panel (a), a few discounted instantaneous utility functions are displayed. Supposing that the optimal consumption flow is constant (i.e. $\mathrm{r}=\rho$ ), the resulting discounted instantaneous utility for any time period is plotted in panel (b). As can be seen, the value of the utility decreases with the increasing time horizon approaching zero in infinity. Thus, the lifetime utility represented as an area below this hypothetical curve converges to a finite number.

As has been already said, condition (A4_35) also requires $\theta$ greater than 1 if there is no discount on future utilities $(\rho=0)$. Notice that $\theta>1$ implies that the instantaneous utility function lies below the horizontal axis (and it is more curved than in the previous figure). ${ }^{255}$ Furthermore, a positive interest rate and zero subjective discount rate thoroughly discussed in Appendix 4 B lead to an increasing time shape of the optimum consumption. As time elapses, consumption grows and instantaneous utility gradually approaches zero (see Figure No. 2_A5). Since there is no discounting, the instantaneous utility function in panel (a) is at the same position. However, optimum consumption grows over time (see the horizontal axis in panel (a). In panel (b), the instantaneous utility is plotted for every given time period. As can be seen, the area above this hypothetical curve represents the lifetime utility, and it converges to a finite number.


Figure No. 2_A5. CRRA utility function, $\rho=0, r>0($ e.g. $r=\theta$ g $), \theta>1$

Hence, it can be said that the convergence of the lifetime utility might be induced either by sufficiently high time preference (in sense two), or if the time preference is low (or even zero), by a "drastically diminishing marginal utility" and by an increasing path of

[^160]consumption over time. This increasing stream is then generated by a positive difference between the interest rate and the subjective discount rate.

It should be stressed that from the point of view of our representative consumer, the real interest rate is an exogenous parameter. However, this does not hold for the economy as a whole. As was discussed in Appendix 4 B and as is shown in section 5.1 of the main text, the interest rate $r$ in a general equilibrium dynamic model should gradually reach $\mathrm{r}^{*}=\rho+\theta \mathrm{g}$ (see A4_43). At this specific point, consumption grows at the same rate as the labour income. There is also neither an eternal accumulation of assets nor debt (see A4_68-A4_70). Thus, the positive difference between the real interest rate and the subjective discount rate leading to an increasing consumption over time is eventually caused by an increasing labour income ( $\mathrm{g}>0$ ). In standard growth models, increasing labour income is in turn a consequence of exogenous technological progress. As a result, a positive gap between the real interest rate and the subjective discount rate is caused by advances in technologies. We can also see that if the time preference $(\rho)$ is zero, only positive technological progress can induce positive real interest rate. Furthermore, if the subjective discount rate is zero, condition (A4_85) (or A4_32) is satisfied only for $\theta$ greater than 1 .
We can conclude that the lifetime utility will converge even in the absence of explicit discounting of future utilities $(\rho=0)$ provided that the technological progress is positive ( $\mathrm{g}>$ 0 ) and the utility function exhibits sufficiently diminishing marginal utility $(\theta>1)$.

## Appendix 6 - Continuous Time Model of Fisherian Shipwrecked Sailors

A) In this Appendix, we will solve the problem of a Fisherian (infinitely and finitely lived) shipwrecked sailor who is endowed with a fixed stock of hardtacks $K$.

Let us start with an infinite horizon. The objective of our sailor is to maximize his lifetime utility function expressed in continuous time as: ${ }^{256}$

$$
\begin{equation*}
U=\int_{0}^{\infty} e^{-\rho \mathrm{t}} u(C(t)) d t \tag{A6_1}
\end{equation*}
$$

Subject to his resource (or budget) constraint: ${ }^{257}$

$$
\begin{equation*}
\int_{0}^{\infty} C(t) d t \leq K \tag{A6_2}
\end{equation*}
$$

This condition states that lifetime consumption cannot exceed the initial endowment of hardtacks. Hardtacks have zero productivity, thus the MPK and the interest rate in this economy is zero. Furthermore, assuming the absence of the satiation point, (A6_2) should be satisfied with equality.
We will solve this dynamic problem with the help of calculus of variation. ${ }^{258}$ Set up a Lagrangian function:

[^161]\[

$$
\begin{equation*}
\int_{0}^{\infty} e^{-\rho \mathrm{t}} u(C(t)) d t+\lambda\left[K-\int_{0}^{\infty} C(t) d t\right] \tag{A6_3}
\end{equation*}
$$

\]

The solution of this dynamic problem should obey the Euler equation:

$$
\begin{equation*}
\frac{\partial F(\cdot)}{\partial C(t)}=\frac{d \frac{\partial F(\cdot)}{\partial \dot{C}(t)}}{d t} \tag{A6_4}
\end{equation*}
$$

where $F(\cdot)=e^{-\rho \mathrm{t}} u(C(t))-\lambda C(t) ;$ and $\dot{C}(t)=\frac{d C(t)}{d t}$

Thus, using (A6_4) in solving (A6_3), we get:

$$
\begin{equation*}
e^{-\rho t} u^{\prime}(C(t))-\lambda=0 \tag{A6_5}
\end{equation*}
$$

(A6_5)simply states that in optimum, the discounted marginal utility of consumption in every period must be the same. To find the optimum growth rate in consumption, let us take logarithm of both sides of equation (A6_5):

$$
\begin{equation*}
-\rho \mathbf{t}+\ln u^{\prime}(C(t))=\ln \lambda \tag{A6_6}
\end{equation*}
$$

And differentiate (A6_6) with respect to time:

$$
\begin{equation*}
-\rho+\frac{\frac{d u^{\prime}(C(t))}{d t}}{u^{\prime}(C(t))}=0 \tag{A6_7}
\end{equation*}
$$

Equation (A6_7) reflects the requirement that in optimum, the growth rate in marginal utility of consumption must be equal to the subjective discount rate. To achieve this, consumption must be falling over time.
Solving (A6_7), we get:

$$
\begin{equation*}
\frac{u^{\prime \prime}(C(t)) \dot{C}(t)}{u^{\prime}(C(t))}=\rho \tag{A6_8}
\end{equation*}
$$

(A6_8) might be written as:

$$
\begin{equation*}
-\frac{\dot{C}(t)}{C(t)} \frac{u^{\prime \prime}(C(t)) C(t)}{u^{\prime}(C(t))}=-\rho \tag{A6_9}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\dot{C}(t)}{C(t)}=\frac{-\rho}{-\frac{u^{\prime \prime}(C(t)) C(t)}{u^{\prime}(C(t))}} \tag{A6_10}
\end{equation*}
$$

Equation (A6_10) describes the optimum growth rate in consumption as a function of the subjective discount rate $\rho$ and the Arrow-Pratt measure of the relative risk aversion $\mathrm{v}(\mathrm{C})$, which is represented by the denominator of (A6_10). For the CRRA utility function, v(C) is simply $\theta$ (see section B). As can be seen, the optimum consumption path must be definitely decreasing because the interest rate is lower than the subjective discount rate $(0<\rho)$. The shape of this path then depends on the magnitude of the relative risk aversion (curvature of the utility function).
Furthermore, (A6_7) is a simple differential equation that might be expressed as:

$$
\begin{equation*}
\frac{\mathrm{d} \ln \left[u^{\prime}(C(t))\right]}{d t}=\rho \tag{A6_11}
\end{equation*}
$$

Its solution is:

$$
\begin{equation*}
u^{\prime}(C(t))=A e^{\rho \mathrm{t}} \tag{A6_12}
\end{equation*}
$$

$A$ is an arbitrary constant that must be solved. For the CRRA utility function, (A6_12) might be written as:

$$
\begin{align*}
& C(t)^{-\theta}=A e^{\rho t}  \tag{A6_13}\\
& C(t)=A e^{-\rho t \theta} \tag{A6_14}
\end{align*}
$$

At time 0 , the optimum consumption is:

$$
\begin{equation*}
C(0)=A \tag{A6_15}
\end{equation*}
$$

which must be, however, determined. To do that, let us insert (A6_15) into the resource constraint (A6_2):

$$
\begin{equation*}
\int_{0}^{\infty} C(0) e^{-\rho t \theta} d t=K \tag{A6_16}
\end{equation*}
$$

The solution of (A6_16) is:

$$
\begin{align*}
& {\left[-C(0) \frac{1}{e^{\rho t / \theta}} \frac{\theta}{\rho}\right]_{0}^{\infty}=K}  \tag{A6_17}\\
& C(0)=K \frac{\rho}{\theta} \tag{A6_18}
\end{align*}
$$

Because present consumption cannot exceed the initial endowment, $\theta$ must be greater than $\rho$. Substituting (A6_18) into (A6_14) we obtain the optimal path of consumption in the infinite horizon:

$$
\begin{equation*}
C(t)=K \frac{\rho}{\theta} e^{-\rho t \theta} \tag{A6_19}
\end{equation*}
$$

To obtain the optimum in the finite horizon, (A6_17) might be written as:

$$
\begin{equation*}
\left[-C(0) \frac{1}{e^{\rho \mathrm{t} \theta}} \frac{\theta}{\rho}\right]_{0}^{T}=K \tag{A6_20}
\end{equation*}
$$

Optimal $\mathrm{C}(0)$ is then:

$$
\begin{align*}
& C(0) \frac{\theta}{\rho}\left(1-\frac{1}{e^{\rho \mathrm{T} / \theta}}\right)=K  \tag{A6_21}\\
& C(0)=\frac{\rho}{\theta} \frac{e^{(\rho / \theta) \mathrm{T}}}{e^{(\rho / \theta) \mathrm{T}}}-1 \tag{A6_22}
\end{align*}
$$

Present consumption must not exceed the initial endowment. (A6_22) thus implies:

$$
\begin{align*}
& \frac{e^{(\rho / \theta) \mathrm{T}}}{e^{(\rho / \theta) \mathrm{T}}-1} \leq \frac{\theta}{\rho}  \tag{A6_23}\\
& \frac{\rho}{\theta} \leq \frac{e^{(\rho / \theta) \mathrm{T}}-1}{e^{(\rho / \theta) \mathrm{T}}}  \tag{A6_24}\\
& \frac{\rho}{\theta} \leq 1-\frac{1}{e^{(\rho / \theta) \mathrm{T}}}  \tag{A6_25}\\
& \frac{1}{e^{(\rho / \theta) \mathrm{T}}} \leq 1-\frac{\rho}{\theta}  \tag{A6_26}\\
& -\frac{\rho}{\theta} T \leq \ln \left(1-\frac{\rho}{\theta}\right) \tag{A6_27}
\end{align*}
$$

If $\rho / \theta$ is a small number then:

$$
\begin{equation*}
T \geq 1 \tag{A6_28}
\end{equation*}
$$

If it is not, (A6_27) yields:

$$
\begin{align*}
& T \geq-\frac{\theta}{\rho} \ln \left(1-\frac{\rho}{\theta}\right)  \tag{A6_29}\\
& T \geq \frac{\theta}{\rho} \ln \frac{\theta}{\theta-\rho} \tag{A6_30}
\end{align*}
$$

Again, $\theta$ must be greater than $\rho$. However, there is also a very weak restriction on the length of the planning horizon $T$.

To obtain the optimal path of consumption in the finite horizon $T$, let us substitute (A6_22) into (A6_14):

$$
\begin{align*}
& C(t)=\frac{\rho}{\theta} \frac{e^{(\rho / \theta) \mathrm{T}}}{e^{(\rho / \theta) \mathrm{T}}-1} K e^{-\rho t / \theta}  \tag{A6_31}\\
& C(t)=\frac{\rho}{\theta} \frac{e^{\rho(\mathrm{T}-\mathrm{t}) \theta}}{e^{(\rho / \theta) \mathrm{T}}-1} K \tag{A6_32}
\end{align*}
$$

## B) Proof that $\mathrm{v}(\mathrm{C})$ is equal to $\theta$ for the CRRA:

The Arrow-Pratt measure of the relative risk aversion $\mathrm{v}(\mathrm{C})$ for the CRRA utility function $u(C)=\frac{C^{1-\theta}}{1-\theta}$ is as follows:

$$
\begin{equation*}
v(C)=-\frac{u^{\prime \prime}(C) C}{u^{\prime}(C)}=-\frac{-\theta C^{-\theta-1}}{C^{-\theta}} C=\theta \tag{A6_1B}
\end{equation*}
$$

Equation (A6_10) is then simply:

$$
\begin{equation*}
\frac{\dot{C}(t)}{C(t)}=\frac{-\rho}{\theta} \tag{A6_2B}
\end{equation*}
$$

## Appendix 7 - Natural Rate of Interest in the Ramsey-Cass-Koopmans Model

In this appendix, we will derive a simple textbook Ramsey-Cass-Koopmans model that will complete our comparison with the Misesian theory of interest on one hand and the Hayekian approach on the other.

Aggregate output in a given period depends on capital, labour, and labour-augmenting technological progress in the same period:
$Y(t)=F(K(t), A(t) L(t))$
Capital is essential in production, thus: ${ }^{259}$
$F(0, A(t) L(t))=0$

[^162]Marginal product of capital (MPK) is positive and diminishing for all levels of capital: $F_{K}>0 ; F_{K K}<0$

MPK satisfies usual Inada conditions that guarantee an interior steady state:
$\lim _{K \rightarrow 0} F_{K}=\infty$
$\lim _{K \rightarrow \infty} F_{K}=0$

The labour force grows exogenously at the rate of $n, \mathrm{~L}(0)$ is the initial size of the labour force:
$\frac{\dot{L}(t)}{L(t)}=n ;$ thus $L(t)=L(0) e^{n t}$
Similar idea holds for technological progress growing at rate $g$ :
$\frac{\dot{A}(t)}{A(t)}=g$; hence $A(t)=A(0) e^{g t}$
Capital accumulation is described by a simple neoclassical (and rather "non-Austrian") law of motion of capital:
$\dot{K}(t)=s Y(t)-\delta K(t)$
(A7_8) states that instantaneous change in capital stock $\mathrm{dK}(\mathrm{t}) / \mathrm{dt}$ (i.e. net investment) depends on the difference between saving $\mathrm{sY}(\mathrm{t})$, which is always equal to gross investment in this theory, and depreciation of capital $\delta \mathrm{K}(\mathrm{t})$, where $\delta$ is the exogenous depreciation rate. ${ }^{260}$

Production function exhibits constant returns to scale, so (A7_1) might be divided by the amount of effective labour $\mathrm{A}(\mathrm{t}) \mathrm{L}(\mathrm{t})$ to stationarize the model. Thus:

$$
\begin{equation*}
\frac{Y(t)}{A(t) L(t)}=F\left(\frac{K(t)}{A(t) L(t)}, 1\right) \tag{A7_9}
\end{equation*}
$$

By defining $y=Y / A L, k=K / A L$ and $f(k)=F(k, 1)$, we get an intensive form of the production function (A7_10). Due to the CRS assumption, the size of the economy does not matter.

$$
\begin{equation*}
y(t)=f(k(t)) \tag{A7_10}
\end{equation*}
$$

Furthermore, all assumptions about the extensive form are inherited also by the intensive form. Thus:

$$
\begin{equation*}
f(0)=0 \tag{A7_10b}
\end{equation*}
$$

$$
\begin{equation*}
f^{\prime}(k)>0 \tag{A7_10c}
\end{equation*}
$$

$$
\begin{equation*}
f^{\prime \prime}(k)<0 \tag{A7_10d}
\end{equation*}
$$

[^163]$\lim _{k \rightarrow 0} f^{\prime}(k)=\infty$
$\lim _{k \rightarrow \infty} f^{\prime}(k)=0$
We will obviously assume that capital cannot be negative, hence:
\[

$$
\begin{equation*}
k(t) \geq 0 \tag{A7_10g}
\end{equation*}
$$

\]

The law of motion of capital in the intensive form is derived from:

$$
\begin{aligned}
& \dot{k}(t)=\frac{d\left(\frac{K(t)}{A(t) L(t)}\right)}{d t}=\frac{\dot{K}(t) A(t) L(t)-K(t)[\dot{A}(t) L(t)-A(t) \dot{L}(t)]}{[A(t) L(t)]^{2}}=\frac{s Y(t)-\delta \cdot K(t)}{A(t) L(t)}-\frac{K(t)}{A(t) L(t)} \frac{\dot{A}(t)}{A(t)}- \\
& -\frac{K(t)}{A(t) L(t)} \frac{\dot{L}(t)}{L(t)}=s y(t)-(\delta+g+n) k(t)=s f(k(t))-(\delta+g+n) k(t)=f(k(t))-c(t)-(\delta+g+n) k(t)
\end{aligned}
$$

$\mathrm{c}(\mathrm{t})$ denotes consumption per effective worker $\mathrm{c}=\mathrm{C} / \mathrm{A}$. The last part of (A7_11) uses the fact that saving $\operatorname{sf}(\mathrm{k}(\mathrm{t}))$ is the difference between output $\mathrm{y}(\mathrm{t})$ and consumption $\mathrm{c}(\mathrm{t})$.

The objective of a representative infinitely lived dynasty (household) is to maximize lifetime utility given by:

$$
\begin{equation*}
U=\int_{0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} d t \tag{A7_12}
\end{equation*}
$$

$\mathrm{L}(\mathrm{t}) / \mathrm{H}$ is the number of members of this dynasty at time $t$, since $H$ measures the number of dynasties (households) in the economy, which is fixed by assumption, and the $\mathrm{L}(\mathrm{t})$ stands for the size of the labour force at time $t$. Every member of the household works and offers one unit of labour in-elastically in every period.
(A7_12) might be easily transformed to an intensive form:

$$
\begin{align*}
& U=\int_{0}^{\infty} e^{-\rho \mathrm{t}} \frac{[A(t) c(t)]^{1-\theta}}{1-\theta} \frac{L(t)}{H} d t=\int_{0}^{\infty} e^{-\rho \mathrm{t}} \frac{\left[A(0) e^{g t} c(t)\right]^{1-\theta}}{1-\theta} \frac{L(0) e^{n t}}{H} d t=  \tag{A7_13}\\
& =\frac{A(0)^{1-\theta} L(0)}{H} \int_{0}^{\infty} e^{-[\rho-n-(1-\theta) g] \mathrm{t}} \frac{c(t)^{1-\theta}}{1-\theta} d t
\end{align*}
$$

The convergence of this integral requires that: ${ }^{261}$

[^164]$\rho-n-(1-\theta) g>0$
Suppose, along with Hayek (1941), ${ }^{262}$ that a central planner is trying to maximize lifetime utility of a representative dynasty (A7_13) subject to the resource constraint of the economy (A7_11) and the non-negativity constraint (A7_10g). Let us set up a simple (present value) Hamiltonian: ${ }^{263}$
\[

$$
\begin{equation*}
H=\frac{A(0)^{1-\theta} L(0)}{H} e^{-[\rho-n-(1-\theta) g]_{t}} \frac{c(t)^{1-\theta}}{1-\theta}+\lambda(t)[f(k(t))-c(t)-(\delta+g+n) k(t)] \tag{A7_15}
\end{equation*}
$$

\]

The first order conditions are:

$$
\begin{align*}
& \frac{\partial H}{\partial c(t)}=\frac{A(0)^{1-\theta} L(0)}{H} e^{-[\rho-n-(1-\theta) g]_{t}} c(t)^{-\theta}-\lambda(t)=0  \tag{A7_16}\\
& \frac{\partial H}{\partial k(t)}=\lambda(t)\left[f^{\prime}(k(t))-(n+g+\delta)\right]=-\dot{\lambda}(t) \tag{A7_17}
\end{align*}
$$

And the transversality condition is:

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \lambda(t) k(t)=0 \tag{A7_17b}
\end{equation*}
$$

This condition requires that the shadow price of capital $\lambda(\mathrm{t})$ in the infinite horizon is zero for the social planner or that the capital per effective worker $k(t)$ in infinity is zero. We will see that $\mathrm{k} *>0$ is the steady of this model, thus $\mathrm{k}(\mathrm{t})$ is non-zero in infinity. Hence, we require $\lambda(t)=0$ in infinity.

Condition (A7_17) implies that:

$$
\begin{equation*}
\frac{-\dot{\lambda}(t)}{\lambda(t)}=\left[f^{\prime}(k(t))-(n+g+\delta)\right] \tag{A7_18}
\end{equation*}
$$

At the same time, (A7_16) might be transformed to a similar differential equation by taking logarithm of both sides:

$$
\begin{equation*}
\ln \frac{A(0)^{1-\theta} L(0)}{H}-[\rho-n-(1-\theta) g] \mathrm{t}-\theta \ln c(t)=\ln \lambda(t) \tag{A7_19}
\end{equation*}
$$

And differentiating (A7_19) with respect to time:

$$
\begin{equation*}
-[\rho-n-(1-\theta) g]-\theta \frac{\dot{c}(t)}{c(t)}=\frac{\dot{\lambda}(t)}{\lambda(t)} \tag{A7_20}
\end{equation*}
$$

[^165](A7_20) and (A7_18) then imply an optimum growth rate of consumption:
\[

$$
\begin{align*}
& {[\rho-n-(1-\theta) g]+\theta \frac{\dot{c}(t)}{c(t)}=f^{\prime}(k(t))-(n+g+\delta)}  \tag{A7_21}\\
& \frac{\dot{c}(t)}{c(t)}=\frac{f^{\prime}(k(t))-\delta-\rho-\theta g}{\theta} \tag{A7_22}
\end{align*}
$$
\]

Realizing that $\mathrm{c}=\mathrm{C} / \mathrm{A}$, equation (A7_22) implies that the optimum growth rate of consumption of one single member of the dynasty is as follows:

$$
\begin{equation*}
\frac{\dot{C}(t)}{C(t)}=\frac{\dot{c}(t)}{c(t)}+\frac{\dot{A}(t)}{A(t)}=\frac{f^{\prime}(k(t))-\delta-\rho-\theta g}{\theta}+g=\frac{f^{\prime}(k(t))-\delta-\rho}{\theta} \tag{A7_22b}
\end{equation*}
$$

Equation (A7_22b) (or A7_22) is the Euler equation for this model, known also as the Keynes-Ramsey rule. It states that the optimum growth rate of consumption depends positively on the net return to capital $\left(\mathrm{f}^{\prime}(\mathrm{k}(\mathrm{t}))-\delta\right)$ and negatively on the subjective discount rate $\rho$. The coefficient of the relative risk aversion $\theta$ modifies the optimum response of the growth rate of consumption to the difference between the net return to capital and the subjective discount rate. We will see that at the steady state, $\mathrm{c}(\mathrm{t})$ is constant, thus consumption $\mathrm{C}(\mathrm{t})$ grows at the rate of technological progress $g$.
Now, we can determine the set of parameters that will satisfy the transversality condition (A7_17b). Equation (A7_18) implies that:

$$
\begin{equation*}
\lambda(t)=\lambda(\mathrm{O}) e^{-\left[f^{\prime}(k(t))-(n+g+\delta)\right] \cdot t} \tag{A7_22b}
\end{equation*}
$$

(A7_17b) then results in:

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \lambda(t) k(t)=\lim _{t \rightarrow \infty} \lambda(0) e^{-\left[f^{\prime}(k(t))-(n+g+\delta)\right] . t} k(t)=0 \tag{A7_22c}
\end{equation*}
$$

As we will see, the steady state of capital per effective worker $\mathrm{k}(\mathrm{t})$ in this model is positive ( $\mathrm{k}^{*}>0$ ). Condition (A7_22c) thus requires:
$\mathrm{f}^{\prime}\left(\mathrm{k}^{*}\right)-\delta>\mathrm{n}+\mathrm{g}$

Transversality condition (A7_17b) can be also expressed (using A7_16) as:
$\lim _{t \rightarrow \infty} \frac{A(0)^{1-\theta} L(0)}{H} e^{-[\rho-n-(1-\theta) g] t} c(t)^{-\theta} k(t)=0$
Since at the steady state of this model, $c^{*}$ and $k^{*}$ are positive (i.e. $c(t)=c^{*}>0$ and $k(t)=k^{*}>0$ as time goes to infinity) condition (A7_23e) requires condition (A7_14), i.e. $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$, again to be valid.

A more straightforward economic interpretation of (A7_23e) might be obtained if it is rearranged as:

$$
\begin{align*}
& \lim _{t \rightarrow \infty} \frac{A(0) e^{g \mathrm{t}} A(0)^{-\theta} e^{-\theta g \mathrm{t}} L(0) e^{n \mathrm{t}}}{H} e^{-\rho \mathrm{t}} \frac{C(t)^{-\theta}}{A(t)^{-\theta}} \frac{K(t)}{A(t) L(t)}=0  \tag{A7_23f}\\
& \lim _{t \rightarrow \infty} e^{-\rho \mathrm{t}} C(t)^{-\theta} \frac{K(t)}{H}=0 \tag{A7_23g}
\end{align*}
$$

(A7_23g) states that in the infinite horizon, the discounted marginal utility of consumption $\mathrm{e}^{-p t} \mathrm{C}(\mathrm{t})^{-\theta}$ multiplied by the amount of capital per household must be zero. This might be achieved by several ways. Either capital is consumed at the terminal date, or consumption is infinite in the infinite future depressing marginal utility $\mathrm{C}(\mathrm{t})^{-\theta}$ to zero, or the subjective discount rate is positive $(\rho>0)$.

However, since capital $K(t)$ is not zero (quite the contrary - it is infinite, growing at the rate of $n+g$ ) at the steady state of $k(t)=k^{*}$, the zero value of the shadow price of capital $\lambda(\mathrm{t})$ in infinity is achieved either by the fact that personal discounting sufficiently depresses the importance of capital to zero or that unbounded consumption in the infinite horizon places zero value to the additional unit of capital. Moreover, these two tendencies must more than offset the eternal growth in capital. Thus, condition (A7_14) reflects this requirement since it can be rewritten as:
$\rho+\theta g-n-g>0$
At the steady state of this model (balanced growth path), the growth rate in capital $\mathrm{n}+\mathrm{g}$ must fall short of the sum of the subjective discount rate $\rho$ and the growth rate in consumption $g$ modified by parameter $\theta$, which reflects the rapidity at which the marginal utility diminishes. As can be seen, the higher this rapidity, the higher the chance that condition (A7_23h) will be satisfied provided that the technological progress is positive.

So far, we considered the problem of the benevolent central planner. However, our main task is to explore the behaviour of the interest rate, which is a market economy phenomenon. Nonetheless, the solution for a decentralized economy will be exactly the same. This is a direct proof that the decentralized market economy in the RCK model finds its dynamic equilibrium at the same point as would be chosen by a benevolent social planner.
A thorough discussion in Appendix 5 B gave us a clear idea of the accumulation of assets of a consumer in the discrete time (see A5_3):

$$
\begin{equation*}
Y_{t+1}+r_{t+1} B_{t}-C_{t+1}=B_{t+1}-B_{t}=\Delta B_{t+1} \tag{A7_23}
\end{equation*}
$$

In continuous time (where the difference between period $t$ and $t+1$ is infinitely small), equation (A7_23) might be written as: ${ }^{264}$

[^166]$\dot{B}(t)=W(t)-C(t)+r(t) B(t)$

However, in this section, we consider a problem of the whole dynasty that is growing over time at the rate of $n$. As a result, equation (A7_24) will be slightly modified since we will consider assets (or debt) of the entire household, not just one member. A similar idea as developed in the previous two equations will give us:
$\dot{B}_{H}(t)=W(t) \frac{L(t)}{H}-C(t) \frac{L(t)}{H}+r(t) B_{H}(t)$
$\mathrm{B}_{\mathrm{H}}(\mathrm{t})$ represents the assets accumulated by a representative household till time $t . \mathrm{L}(\mathrm{t}) / \mathrm{H}$ is the number of members of each household, therefore $\mathrm{W}(\mathrm{t}) \mathrm{L}(\mathrm{t}) / \mathrm{H}$ is the entire labour income of the household at time $t$, and $\mathrm{C}(\mathrm{t}) \mathrm{L}(\mathrm{t}) / \mathrm{H}$ is the total consumption of a household in the same period.
Each household is, however, restricted by the credit market. It simply cannot roll over its debt forever (see a thorough discussion in Appendix 5). As a result, we have to impose the No-Ponzi-Game condition even in the continuous time model (see equation A5_14). Again, it requires that the debt of each household cannot asymptotically grow faster than the interest rate (Blanchard and Fischer 1989:49; Romer 2006:52):
$\lim _{t \rightarrow \infty} e^{-R(t)} B_{H}(t) \geq 0$
$R(t)=\int_{0}^{t} r(\tau) d \tau$
$\mathrm{e}^{-\mathrm{R}(\mathrm{t})}$ is a continuous time version of discounting presented in A5_14. It might be derived from the idea that:
$\frac{1}{\left(1+\mathrm{r}_{1}\right)\left(1+\mathrm{r}_{2}\right) \ldots\left(1+\mathrm{r}_{\mathrm{t}}\right)} \approx \exp \left[-\left(r_{1}+r_{2}+\ldots r_{t}\right)\right]=\exp \left(-\sum_{\tau=0}^{t} r_{\tau}\right)=\exp \left(-R_{t}\right)$

Thus, $\mathrm{e}^{-\mathrm{R}(\mathrm{t})}$ is a continuous time version of (A7_25d). As can be clearly seen, $\mathrm{R}(\mathrm{t})$ in equation (A7_25c) adds all instantaneous rates of interest (i.e. "forces" of interest) from time 0 to time $t$. Yet, summing in the continuous time is performed by the integral rather than by the sum. Hence, $\mathrm{e}^{-\mathrm{R}(\mathrm{t})}$ represents the idea of (continuous) discounting, whose discrete time version can be seen in the first part of expression (A7_25d).

Furthermore, it can be easily seen that the amount of assets of the whole household is simply the amount of assets of one member times the number of members of each household:

$$
\begin{equation*}
B_{H}(t)=\frac{L(t)}{H} B(t) \tag{A7_26}
\end{equation*}
$$

Thus, assets of one member are represented by:

$$
\begin{equation*}
B(t)=H \frac{B_{H}(t)}{L(t)} \tag{A7_27}
\end{equation*}
$$

As a result, the law of motion of assets of one member is given by:

$$
\dot{B}(t)=H \frac{\dot{B}_{H}(t) L(t)-B_{H}(t) \dot{L}(t)}{L^{2}(t)}=H\left[\frac{\dot{B}_{H}(t)}{L(t)}-\frac{B_{H}(t)}{L(t)} \frac{\dot{L}(t)}{L(t)}\right]=H\left[\frac{\dot{B}_{H}(t)}{L(t)}-n \frac{B_{H}(t)}{L(t)}\right](\text { A7_28) }
$$

Inserting (A7_25) into (A7_28), we get:
$\dot{B}(t)=H\left[\frac{W(t) \frac{L(t)}{H}-C(t) \frac{L(t)}{H}+r(t) B_{H}(t)}{L(t)}-n \frac{B_{H}(t)}{L(t)}\right]$

Using (A7_26), equation (A7_29) yields:
$\dot{B}(t)=H\left[\frac{W(t) \frac{L(t)}{H}-C(t) \frac{L(t)}{H}+r(t) \frac{L(t)}{H} B(t)}{L(t)}-n \frac{\frac{L(t)}{H} B(t)}{L(t)}\right]$
Thus, we arrive at a slightly modified version of equation (A7_24):
$\dot{B}(t)=W(t)-C(t)+r(t) B(t)-n B(t)$
Because the size of the family is growing at rate $n$, this term negatively affects the increase in assets of one single member.

The No-Ponzi-Game condition for (A7_31) is as follows:
$\lim _{t \rightarrow \infty} e^{-[R(t)-n t]} B(t) \geq 0$
(A7_32) might be derived from (A7_25b) by the following steps:
$\lim _{t \rightarrow \infty} e^{-R(t)} B_{H}(t)=\lim _{t \rightarrow \infty} e^{-R(t)} \frac{L(t)}{H} B(t)=\lim _{t \rightarrow \infty} e^{-R(t)} \frac{L(0) e^{n t}}{H} B(t)=\frac{L(0)}{H} \lim _{t \rightarrow \infty} e^{-R(t)} e^{n t} B(t) \geq 0$
Since L(0)/H is surely positive, (A7_33) results in (A7_32).
Provided that (A7_25b) holds with equality due to monotonically increasing utility function, the intertemporal budget constraint implied by (A7_25) and (A7_25b) might be expressed as (see section B):
$\int_{0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} d t=\frac{K(0)}{H}+\int_{0}^{\infty} e^{-R(t)} W(t) \frac{L(t)}{H} d t$
Equation (A7_34) simply states that the discounted flow of consumption of a representative household might not exceed (is equal to) the discounted flow of labour income plus the initial value of assets $\mathrm{B}_{\mathrm{H}}(0)=\mathrm{K}(0) / \mathrm{H}$. In this model, we assume that each household owns a proportional part of the initial stock of capital in the economy. (A7_34) is a continuous time version of the thoroughly discussed budget constraint (A5_12) from Appendix 5.

The objective of the household is to maximize lifetime utility function (A7_12) with respect to the flow constraint (A7_31) (or A7_25) or alternatively with respect to the intertemporal budget constraint (A7_34). Inserting equation (A7_31) in (A7_12) for $\mathrm{C}(\mathrm{t})=\mathrm{W}(\mathrm{t})+\mathrm{r}(\mathrm{t}) \mathrm{B}(\mathrm{t})-$ $\mathrm{nB}(\mathrm{t})-\mathrm{dB}(\mathrm{t}) / \mathrm{dt}$, and (A7_6) for the evolution of the labour force, the dynamic optimization problem is as follows:
$\max U=\int_{0}^{\infty} e^{-\rho t} \frac{\{W(t)+[r(t)-n] B(t)-\dot{B}(t)\}^{1-\theta}}{1-\theta} \frac{L(0) e^{n t}}{H} d t$
There are two choice variables - the amount of assets at time $t$ (i.e. $B(t)$ ) and the instantaneous change of assets at time $t$ (i.e. $\mathrm{dB}(\mathrm{t}) / \mathrm{dt}$ ). Solution of (A7_35) might be found with the help of the Euler equation (see Kamien and Schwartz 1991):
$\frac{\partial F(\cdot)}{\partial B(t)}=\frac{d \frac{\partial F(\cdot)}{\partial \dot{B}(t)}}{d t}$
where $\mathrm{F}($.$) is the expression under integral in (A7_35). Thus:$
$\frac{\partial F(\cdot)}{\partial B(t)}=e^{-(\rho-n) t} C(t)^{-\theta}[r(t)-n] \frac{L(0)}{H}$
$\frac{\partial F(\cdot)}{\partial \dot{B}(t)}=e^{-(\rho-n) \mathrm{t}} C(t)^{-\theta}(-1) \frac{L(0)}{H}$
$\frac{d \frac{\partial F(\cdot)}{\partial \dot{B}(t)}}{d t}=-\frac{L(0)}{H}\left[e^{-(\rho-n) \mathrm{t}} C(t)^{-\theta}(-\rho+n)+e^{-(\rho-n) \mathrm{t}}(-\theta) C(t)^{-\theta-1} \dot{C}(t)\right]$
(A7_36) gives us:
$e^{-(\rho-n) \mathrm{t}} C(t)^{-\theta}[r(t)-n] \frac{L(0)}{H}=-\frac{L(0)}{H}\left[e^{-(\rho-n) \mathrm{t}} C(t)^{-\theta}(-\rho+n)+e^{-(\rho-n) \mathrm{t}}(-\theta) C(t)^{-\theta-1} \dot{C}(t)\right]$
$[r(t)-n]=-\left[(-\rho+n)+(-\theta) \frac{\dot{C}(t)}{C(t)}\right]$
The optimum growth rate of consumption is:
$\frac{\dot{C}(t)}{C(t)}=\frac{r(t)-\rho}{\theta}$
As we can see, we arrived at a very simple consumption Euler equation known from previous sections. Furthermore, by realizing the fact that $\mathrm{c}(\mathrm{t})=\mathrm{C}(\mathrm{t}) / \mathrm{A}(\mathrm{t})$, the growth rate of $\mathrm{c}(\mathrm{t})$ is:

$$
\begin{equation*}
\frac{\dot{c}(t)}{c(t)}=\frac{\dot{C}(t)}{C(t)}-\frac{\dot{A}(t)}{A(t)}=\frac{r(t)-\rho}{\theta}-g=\frac{r(t)-\rho-\theta g}{\theta} \tag{A7_43}
\end{equation*}
$$

(A7_43) would be virtually the same as the social planner solution (A7_22) provided that $\mathrm{r}(\mathrm{t})=\mathrm{f}^{\prime}(\mathrm{k}(\mathrm{t}))-\delta$.

If we focus on the second part of the capital market, i.e. on firms, (A7_43) might be easily reconciled with (A7_22). The profit-maximizing firm should, according to the neoclassical theory, equalize the marginal product of capital with the marginal cost of capital, which is equal, in a one-good economy, to the sum of the real interest rate and depreciation rate. The proof of this statement runs as follows:
$\pi_{i}=Y_{i}-\left[W L_{i}+(r+\delta) K_{i}\right]$
(A7_44) is the real profit function of a representative firm $i . \mathrm{Y}_{\mathrm{i}}=\mathrm{F}\left(\mathrm{K}_{\mathrm{i}}, \mathrm{AL}_{\mathrm{i}}\right)$ stands for real revenues, the remaining part represents total real costs. Notice that real costs of capital are represented by the real interest for capital rK and the amount of depreciation $\delta \mathrm{K}$.

The FOC in the case of capital is:

$$
\begin{equation*}
\frac{\partial \pi}{\partial K_{i}}=\frac{\partial F}{\partial K_{i}}-(r+\delta)=0 \tag{A7_45}
\end{equation*}
$$

Thus:

$$
\begin{equation*}
\frac{\partial F(\cdot)}{\partial K_{i}}=M P K=r+\delta \tag{A7_45b}
\end{equation*}
$$

The production function is homogenous of degree one, hence its first derivative is homogenous of degree zero. This implies that $\mathrm{f}^{\prime}(\mathrm{k})=\mathrm{MPK}$. The formal proof is simple:
$M P K=\frac{\partial F(\cdot)}{\partial K}=\frac{\partial[A L f(k)]}{\partial K}=A L f^{\prime}(k) \frac{1}{A L}=f^{\prime}(k)$
The real interest rate acts as the coordinating mechanism that will reconcile the behaviour of utility maximizing households (equation (A7_42)) and profit maximizing firms (A7_46). Using (A7_47) and (A7_46), equation (A7_43) for a decentralized economy might be written as (A7_22), which is the optimum solution for a social planner.

## The steady state

Consumption per effective worker reaches its steady state (and consumption and, as we will see, other most important variables in the model (capital, output, real wage) their balanced growth path), if (A7_22) is equal to zero:

$$
\begin{equation*}
\dot{c}(t)=0 \Leftrightarrow f^{\prime}\left(k^{*}(t)\right)-\delta=\rho+\theta g \Leftrightarrow M P K^{*}-\delta=\rho+\theta g \tag{A7_47}
\end{equation*}
$$

Using (A7_43) for a decentralized economy or (A7_22) for a centrally planned economy, the steady state level of the natural rate of interest is as follows:

$$
\begin{equation*}
M P K-\delta=r^{*}=\rho+\theta g \tag{A7_48}
\end{equation*}
$$

Since the marginal product of capital is an endogenous variable, it will always adjust to equal the right-hand side of this equation. Thus, the natural rate of interest in this neoclassical dynamic general equilibrium model is determined by the subjective discount rate (time preference in sense two) and the rate of technological progress, whose influence is modified by the curvature of the utility function. ${ }^{265}$ At the same time, (A7_47) determines the steady state value of capital per effective worker $\mathrm{k}^{*}$, which represents the only level of capital per effective worker, for which consumption per effective worker is constant. This value might be implicitly found using (A7_48):

$$
\begin{equation*}
f^{\prime}\left(k^{*}(t)\right)-\delta=\rho+\theta g \tag{A7_49}
\end{equation*}
$$

Even though there are two crucial endogenous variables, capital and consumption, the size of $\mathrm{k}^{*}$ does not depend on the level of consumption per effective worker (see the vertical line in Figure No.1_A7). To close the model, let us find the steady state value of consumption per effective worker. Using (A7_11), combinations of capital per effective worker and consumption per effective worker for which the capital per effective worker is constant might be written as:

$$
\begin{equation*}
\dot{k}(t)=0 \Leftrightarrow c(t)=f(k(t))-(n+g+\delta) k(t) \tag{A7_50}
\end{equation*}
$$

By substituting $\mathrm{k}^{*}$ from (A7_49), we obtain c*:

$$
\begin{equation*}
c^{*}=f\left(k^{*}\right)-(n+g+\delta) k^{*} \tag{A7_51}
\end{equation*}
$$

$\mathrm{c}^{*}$ can be found at the intersection of the vertical line at point $\mathrm{k}^{*}$ and the concave curve that represents points from equation (A7_50) (See Figure No. 1_A7) Combination $c^{*}$ and $\mathrm{k}^{*}$ then represents the steady state of this model.
Moreover, condition (A7_14) implies that:

$$
\begin{equation*}
\rho+\theta g>\mathrm{n}+\mathrm{g} \tag{A7_52}
\end{equation*}
$$

The left hand side is equal to real interest rate at steady state and the right hand side to the growth rate of GDP. Hence, the economy in the Ramsey model is always dynamically efficient. To prove this, let us determine the growth rate of GDP in the steady state of $\mathrm{k}^{*}$.

First, we can use the fact that:

$$
\begin{equation*}
\dot{y}(t)=f^{\prime}(k(t)) \dot{k}(t) \tag{A7_53}
\end{equation*}
$$

[^167]At the steady state, $\mathrm{k}^{*}$ is constant, thus:

$$
\begin{equation*}
\dot{y}^{*}=f^{\prime}\left(k^{*}\right) \dot{k}^{*}=0 \tag{A7_54}
\end{equation*}
$$

Furthermore, since $y=Y / A L$, the growth rate of $Y$ at the steady state is:

$$
\begin{equation*}
\frac{\dot{Y}(t)}{Y(t)}=\frac{\frac{d[y(t) A(t) L(t)]}{d t}}{y(t) A(t) L(t)}=\frac{\dot{y}(t)}{y(t)}+\frac{\dot{A}(t)}{A(t)}+\frac{\dot{L}(t)}{L(t)}=0+g+n \tag{A7_55}
\end{equation*}
$$

The economy is considered as dynamically efficient if it is not possible to raise consumption of some agents in a given period without reducing consumption of (the same or other) agents in some other periods. Technically it means that an increase in capital (due to higher saving, thus lower present consumption) will result in higher consumption in the future (in the new steady state) and conversely, a reduction in capital (due to lower saving, thus higher present consumption) will result in lower consumption in the future (in the new steady state). As a result, steady state consumption must positively respond to an increase in capital. Using (A7_51), (A7_55), and (A7_46) we can express this positive relationship as:
$\frac{\partial c^{*}}{\partial k^{*}}>0 \Leftrightarrow f^{\prime}\left(k^{*}\right)-\delta>n+g \Leftrightarrow r^{*}>n+g \Leftrightarrow r^{*}>\frac{\dot{Y}(t)}{Y(t)}$
Thus, the condition for dynamic efficiency is that the real interest rate is greater than the growth rate of the economy.

Consumption is then maximized if:

$$
\begin{equation*}
\frac{\partial c^{*}}{\partial k^{*}}=0 \Leftrightarrow f^{\prime}\left(k^{*}\right)-\delta=n+g \Leftrightarrow r^{*}=n+g \Leftrightarrow r^{*}=\frac{\dot{Y}(t)}{Y(t)} \tag{A7_57}
\end{equation*}
$$

The level of capital that satisfies (A7_57) is known as the golden rule level of capital. However, capital at the steady state in the RCK model is always lower than golden rule due to condition (A7_52). $\mathrm{k}^{*}$ in the RCK model is sometimes called a modified golden rule.

At the end of this section, we can easily determine the optimum steady state growth rate of consumption. Using (A7_48) and (A7_42), it can be derived that on the balanced growth path, it grows at the same rate as GDP per head, namely $g$ :

$$
\begin{equation*}
\frac{\dot{C}(t)}{C(t)}=\frac{r^{*}-\rho}{\theta}=\frac{\rho+\theta g-\rho}{\theta}=g \tag{A7_58}
\end{equation*}
$$

Section B. The intertemporal budget constraint in the infinite horizon continuous time model
In section A, it was stated that the flow constraint (A7_25) together with the No-Ponzi-Game condition (A7_25b) imply a continuous time version of the intertemporal budget constraint (A7_34), thoroughly discussed in the discrete time in Appendix 4 B. Let us stress that equation (A7_25) is a simple differential equation which might be rewritten as:

$$
\begin{equation*}
\dot{B}_{H}(t)-r(t) B_{H}(t)=W(t) \frac{L(t)}{H}-C(t) \frac{L(t)}{H} \tag{A7_1B}
\end{equation*}
$$

Suppose that in the initial period, each representative household owns a proportional part of the total initial capital stock. This might be represented as:

$$
\begin{equation*}
B_{H}(0)=\frac{K(0)}{H} \tag{A7_2B}
\end{equation*}
$$

Hence, (A7_2B) is the initial condition of the differential equation (A7_1B), solution of which might be derived with the help of the method of the variation of constants. Let us start with its homogenous representation:
$\dot{B}_{H}(t)-r(t) B_{H}(t)=0$

Solution of (A7_3B) runs as follows:

$$
\begin{align*}
& \frac{d B_{H}(t)}{d t}=r(t) B_{H}(t)  \tag{A7_4B}\\
& \int \frac{d B_{H}(t)}{B_{H}(t)}=\int r(t) d t  \tag{A7_5B}\\
& \ln B_{H}(t)=R(t)+d \tag{A7_6B}
\end{align*}
$$

(A7_6B) uses the fact that $r(t)$ is the first derivative of $R(t)$ (see the discussion below). Furthermore, $d$ is an arbitrary constant of integration. (A7_6B) implies:
$B_{H}(t)=D(t) e^{R(t)}$
$\mathrm{D}(\mathrm{t})$ is simply $\exp (\mathrm{d})$, but in this method it is itself a function of time. Differentiation of (A7_6B) with respect to time yields:

$$
\begin{equation*}
\dot{B}_{H}(t)=\dot{D}(t) e^{R(t)}+D(t) e^{R(t)} r(t) \tag{A7_8B}
\end{equation*}
$$

(A7_8B) uses the Leibnitz rule for differentiation of the integral with respect to the variable in the upper (or lower) limit, generally stated as (see Kamien and Schwartz 1991):

$$
\begin{equation*}
\frac{d V(t)}{d t}=\frac{d \int_{A(t)}^{B(t)} f(\tau, t) d \tau}{d t}=f(B(t), t) B^{\prime}(t)-f(A(t), t) A^{\prime}(t)+\int_{A(t)}^{B(t)} \frac{\partial f(t, \tau)}{\partial t} d \tau \tag{A7_8Bb}
\end{equation*}
$$

For a constant lower limit and a simple form we deal with in (A7_25c), (A7_8Bb) is modified to:

$$
\begin{equation*}
\frac{d V(t)}{d t}=\frac{d \int_{0}^{B(t)} f(\tau) d \tau}{d t}=f(B(t)) B^{\prime}(t)-0+0 \tag{A7_8Bc}
\end{equation*}
$$

Applying (A7_8Bc) to (A7_25c), we get:

$$
\begin{equation*}
\frac{d R(t)}{d t}=\frac{d \int_{0}^{t} r(\tau) d \tau}{d t}=\left.r(\tau)\right|_{\tau=t} \frac{d t}{d t}=r(t) \tag{A7_8Bd}
\end{equation*}
$$

Thus, the time derivative of $R(t)$ is simply $r(t)$. By substituting (A7_8B) and (A7_7B) back to the original differential equation (A7_1B), we may write:

$$
\begin{align*}
& \dot{D}(t) e^{R(t)}+D(t) e^{R(t)} r(t)-r(t) D(t) e^{R(t)}=W(t) \frac{L(t)}{H}-C(t) \frac{L(t)}{H}  \tag{A7_9B}\\
& \dot{D}(t)=e^{-R(t)} W(t) \frac{L(t)}{H}-e^{-R(t)} C(t) \frac{L(t)}{H} \tag{A7_10B}
\end{align*}
$$

By integrating (A7_10B), we get:

$$
\begin{equation*}
D(t)=\int e^{-R(t)} W(t) \frac{L(t)}{H} d t-\int e^{-R(t)} C(t) \frac{L(t)}{H} d t+F \tag{A7_11B}
\end{equation*}
$$

$F$ is an arbitrary constant of integration. Let us insert (A7_11B) into equation (A7_7B):

$$
\begin{equation*}
B_{H}(t) e^{-R(t)}=\int e^{-R(t)} W(t) \frac{L(t)}{H} d t-\int e^{-R(t)} C(t) \frac{L(t)}{H} d t+F \tag{A7_12B}
\end{equation*}
$$

At any time $T$, (A7_12B) can be written as:

$$
\begin{equation*}
B_{H}(T) e^{-R(T)}=\int_{t=0}^{T} e^{-R(t)} W(t) \frac{L(t)}{H} d t-\int_{t=0}^{T} e^{-R(t)} C(t) \frac{L(t)}{H} d t+F \tag{A7_13B}
\end{equation*}
$$

To determine the arbitrary constant $F$, we can use the initial condition (A7_2B) and the fact that $R(0)=0 .{ }^{266}$ Equation (A7_13B) for $T=0$ yields:

$$
\begin{equation*}
B_{H}(0) e^{-R(0)}=\int_{t=0}^{0} e^{-R(t)} W(t) \frac{L(t)}{H} d t-\int_{t=0}^{0} e^{-R(t)} C(t) \frac{L(t)}{H} d t+F \tag{A7_14B}
\end{equation*}
$$

$$
B_{H}(0)=F\left(\mathrm{~A} 7 \_15 \mathrm{~B}\right)
$$

Using this fact and (A7_2B), (A7_13B) is simply:

[^168] value of all integrals in (A7_13B).
$B_{H}(T) e^{-R(T)}=\int_{t=0}^{T} e^{-R(t)} W(t) \frac{L(t)}{H} d t-\int_{t=0}^{T} e^{-R(t)} C(t) \frac{L(t)}{H} d t+\frac{K(0)}{H}$

On the other hand, if $T$ goes to infinity, (A7_16B) yields:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} B_{H}(T) e^{-R(T)}=\lim _{T \rightarrow \infty}\left\{\int_{t=0}^{T} e^{-R(t)} W(t) \frac{L(t)}{H} d t-\int_{0}^{T} e^{-R(t)} C(t) \frac{L(t)}{H} d t+\frac{K(0)}{H}\right\} \tag{A7_17B}
\end{equation*}
$$

However, according to the No-Ponzi-Game condition (A7_25b), the left-hand side of (A7_17B) must be non-negative. Hence, (A7_17B) implies:
$\int_{0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} d t \leq \frac{K(0)}{H}+\int_{t=0}^{\infty} e^{-R(t)} W(t) \frac{L(t)}{H} d t$
(A7_17C) states that the present value of the flow of consumption in the infinite horizon may not exceed the present value of the flow of labour income plus the value of the initial assets.
Moreover, it would be suboptimal for the household not to completely exhaust all lifetime resources, since the lifetime utility has no bliss point. Thus, (A7_17C) is satisfied with equality leading to (A7_34).

## Section C: Transforming the flow constraint in discrete time to continuous time:

In this section, we will show how the flow constraint in discrete time (A7_23) might be converted to a continuous time version (A7_24). Before that, however, we must slightly modify the discrete time equation since our approach from Appendix 5 was not perfectly accurate.

Suppose that at the beginning of period zero, initial assets of our representative agent are represented by $\mathrm{B}_{0}$. At the end of this period, assets (or debt) are increased or decreased by the difference between the flow of the labour income $\mathrm{W}_{0}$ and the flow of consumption $\mathrm{C}_{0}$ in this particular period. It should be stressed again that income and consumption are flow concepts, whereas assets represent a stock concept. Hence, assets at the end of period zero should be denoted as $\mathrm{B}_{1}$ :
$B_{1}=W_{0}-C_{0}+B_{0}$
At the end of period zero (or at the beginning of period one), the interest is paid on the accumulated assets. This serves as an additional source for consumption and saving in the next period together with the labour income. Thus, the next period budget constraint might be represented as:

$$
\begin{equation*}
B_{2}+C_{1}=W_{1}+B_{1}\left(1+r_{1}\right) \tag{A7_2C}
\end{equation*}
$$

(A7_2C) states that the assets at the end of period one, i.e. $\mathrm{B}_{2}$, depend on the difference between the flow of labour income in this particular period, i.e. $\mathrm{W}_{1}$, and the flow of consumption $\mathrm{C}_{1}$. At the same time, assets from the previous period $\mathrm{B}_{1}$ are increased by the interest $\mathrm{r}_{1} \mathrm{~B}_{1}$.

This idea should hold in any time, equation (A7_2C) can be therefore generalized to:

$$
\begin{equation*}
B_{t+1}+C_{t}=W_{t}+B_{t}\left(1+r_{t}\right) \tag{A7_3C}
\end{equation*}
$$

This equation, in turn, may be rearranged to:
$B_{t+1}-B_{t}=W_{t}-C_{t}+r_{t} B_{t}$

In (A7_4C) the time difference between $\mathrm{B}_{\mathrm{t}+1}$ and $\mathrm{B}_{\mathrm{t}}$ is one period. Over that period, assets are either accumulated, partly reduced, or remain the same. The eventual result depends on the variables on the right hand side of the equation. However, if the time period is halved, the accumulation will be also (roughly) halved, i.e. $\mathrm{B}(\mathrm{t}+1 / 2)-\mathrm{B}(\mathrm{t})=1 / 2\left(\mathrm{~B}_{\mathrm{t}+1}-\mathrm{B}_{\mathrm{t}}\right)$. Obviously, halving was chosen arbitrarily since any reduction in the given time period is possible. Thus, (A7_4C) might be expressed as:
$B(t+s)-B(t)=s(W(t)-C(t)+r(t) B(t))$
$s$ might be any number, e.g. one half. Let us divide (A7_5C) by $s$ and assume that $s$ approaches 0 in the limit. Equation (A7_5C) can be then written as:
$\lim _{s \rightarrow 0} \frac{B(t+s)-B(t)}{s}=\lim _{s \rightarrow 0}[W(t)-C(t)+r(t) B(t)]$

The left-hand side of (A7_2C) is the time derivative of $\mathrm{B}(\mathrm{t})$. Thus, we arrive at equation (A7_24):

$$
\begin{equation*}
\dot{B}(t)=W(t)-C(t)+r(t) B(t) \tag{A7_24}
\end{equation*}
$$

The constraint (A7_24) states that the instantaneous change in assets depends on the difference between the instantaneous flow of labour income $\mathrm{W}(\mathrm{t})$ plus the instantaneous flow of interest $\mathrm{r}(\mathrm{t}) \mathrm{B}(\mathrm{t})$ minus the instantaneous flow of consumption $\mathrm{C}(\mathrm{t})$.

## Section D: Behaviour of the economy around the steady state

In this section, we will analyse behaviour of the economy around its steady state. This will help us understand paths of consumption, capital, and natural rate of interest after various shocks.
$\mathrm{dk} / \mathrm{dt}$ can be approximated around the steady state $\mathrm{k}^{*}$, $\mathrm{c}^{*}$ in the system of two differential equations (A7_11) - the law of motion of capital, and (A7_22) - the Euler equation indicating the optimum path of consumption, as follows:

$$
\begin{equation*}
\dot{k}(t)=\left.\dot{k}(t)\right|_{k(t)=k^{*}, c(t)=c^{*}}+\left.\frac{\partial \dot{k}(t)}{\partial k(t)}\right|_{k(t)=k^{*}, c(t)=c^{*}}\left(k(t)-k^{*}\right)+\left.\frac{\partial \dot{k}(t)}{\partial c(t)}\right|_{k(t)=k^{*}, c(t)=c^{*}}\left(c(t)-c^{*}\right) \tag{A7_1D}
\end{equation*}
$$

In the discussion in Figure No. 54 in the main text, we consider constant population and technology and a sudden one-time increase in the level of technologies. Thus, (A7_11) might be written as:

$$
\dot{k}(t)=A f(k(t))-c(t)-\delta . k(t)
$$

$\left(\mathrm{A} 7 \_2 \mathrm{D}\right)^{267}$

Parameter $A$ measures the particular level of technologies that is assumed to be constant. Neglecting time index for simplicity, (A7_1D) can be applied on (A7_2D) as follows:
$\dot{k}(t)=0+\left[A f^{\prime}\left(k^{*}\right)-\delta\right]\left(k-k^{*}\right)+(-1)\left(c-c^{*}\right)$
(A7_3D)
$\mathrm{dc} / \mathrm{dt}$ is approximated around the steady state $\mathrm{k}^{*}, \mathrm{c}^{*}$ in the same system as:

$$
\begin{equation*}
\dot{c}(t)=\left.\dot{c}(t)\right|_{k(t)=k^{*}, c(t)=c^{*}}+\left.\frac{\partial \dot{c}(t)}{\partial k(t)}\right|_{k(t)=k^{*}, c(t)=c^{*}}\left(k(t)-k^{*}\right)+\left.\frac{\partial \dot{c}(t)}{\partial c(t)}\right|_{k(t)=k^{*}, c(t)=c^{*}}\left(c(t)-c^{*}\right) \tag{A7_4D}
\end{equation*}
$$

The Euler equation (A7_22) with constant technology might be written as:

$$
\begin{equation*}
\dot{c}(t)=\frac{A f^{\prime}(k(t))-\delta-\rho}{\theta} c(t) \tag{A7_5D}
\end{equation*}
$$

Applying (A7_4D) on (A7_5D) and neglecting the time index, we get:

$$
\begin{equation*}
\dot{c}=0+\frac{A f^{\prime \prime}\left(k^{*}\right)}{\theta} c^{*}\left(k-k^{*}\right)+\frac{A f^{\prime}\left(k^{*}\right)-\delta-\rho}{\theta}\left(c-c^{*}\right) \tag{A7_6D}
\end{equation*}
$$

However, since $\mathrm{Af}^{\prime}\left(\mathrm{k}^{*}\right)-\delta-\rho=0$ at the steady state, (A7_6D) yields:

$$
\begin{equation*}
\dot{c}=0+\frac{A f^{\prime \prime}\left(k^{*}\right)}{\theta} c^{*}\left(k-k^{*}\right) \tag{A7_7D}
\end{equation*}
$$

At the same time, using $\mathrm{Af}^{\prime}\left(\mathrm{k}^{*}\right)-\delta-\rho=0$ in (A7_3D), we get:

$$
\begin{equation*}
\dot{k}=0+\rho\left(k-k^{*}\right)+(-1)\left(c-c^{*}\right) \tag{A7_8D}
\end{equation*}
$$

[^169](A7_3D) and (A7_8D) constitute a system of two differential equations, representing an (linear) approximation of the economy around its steady state. To solve this system, let us differentiate (A7_8D) with respect to time:
\[

$$
\begin{equation*}
\ddot{k}=\rho \cdot \dot{k}-\dot{c} \tag{A7_9D}
\end{equation*}
$$

\]

Inserting (A7_7D) to (A7_9D) yields:
$\ddot{k}=\rho \cdot \dot{k}-\frac{A f^{\prime \prime}\left(k^{*}\right)}{\theta} c^{*}\left(k-k^{*}\right)$

We may represent (A7_10D) as:

$$
\begin{equation*}
\ddot{k}-\rho \cdot \dot{k}+\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta} k=\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta} k^{*} \tag{A7_11D}
\end{equation*}
$$

Solution of this second-order differential equation can be found as follows:
$\ddot{k}-\rho \cdot \dot{k}+\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta} k=0$
(A7_12D) is the homogenous representation of (A7_11D). The characteristic equation is:
$\lambda^{2}-\rho . \lambda+\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta}=0$

The same equation would be obtained if we designed a characteristic matrix of the system (A7_8D) and (A7_7D) and its determinant:
$\operatorname{det}\left(\begin{array}{cc}\rho-\lambda & -1 \\ A f^{\prime \prime}\left(k^{*}\right) c^{*} / \theta & 0-\lambda\end{array}\right)=\lambda^{2}-\rho \cdot \lambda+\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta}$

The two roots of (A7_13D) are:
$\lambda_{1,2}=\frac{\rho \pm \sqrt{\rho^{2}-4 \frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta}}}{2}$
(A7_15D)
It should be perfectly clear that one root is greater than zero, $\lambda_{1}>0$. The second one is negative, $\lambda_{2}<0$, because $\mathrm{f}^{\prime}\left(\mathrm{k}^{*}\right)$ is negative by assumption (production function is concave, marginal product is diminishing). Thus, the term under the square root is positive and greater than $\rho$. As a result, the solution has a saddle path property.
The particular solution $k_{p}$ (a constant) of (A7_11D) is:

$$
\begin{equation*}
\ddot{k}_{p}-\rho \cdot \dot{k}_{p}+\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta} k_{p}=\frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta} k^{*} \tag{A7_16D}
\end{equation*}
$$

$k_{p}=k^{*}$

Hence, the general solution of (A7_11D) is as follows:
$k(t)=b_{1} \exp \left(\lambda_{1} t\right)+b_{2} \exp \left(\lambda_{2} t\right)+k^{*}$

However, since $\lambda_{1}>0, b_{1}$ must be set equal to zero. $\lambda_{1}$ is associated with the unstable arm in Figure No. $1 \_$A7. On the other hand, $\lambda_{2}<0$ is linked to the stable arm. Hence, from now on, $\lambda_{2}$ will be denoted simply as $\lambda$. The last step is to determine the arbitrary constant $b_{2}$. Using the initial condition $\mathrm{k}(\mathrm{t}=0)=\mathrm{k}(0)$, (A7_18D) yields:
$k(0)=0 \exp \left(\lambda_{1} t\right)+b_{2} \exp \left(\lambda_{2} 0\right)+k^{*}$
$b_{2}=k(0)-k^{*}$

Inserting (A7_20D) into (A7_18D) and putting $\mathrm{b}_{1}=0$, we get:

$$
\begin{equation*}
k(t)=\left[k(0)-k^{*}\right] e^{\lambda t}+k^{*} \tag{A7_21D}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda=\frac{\rho-\sqrt{\rho^{2}-4 \frac{A f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta}}}{2}<0 \tag{A7_22D}
\end{equation*}
$$

As can be seen, the negativity of $\lambda$ leads to the fact that $k(t)$ gradually converges to the steady state $\mathrm{k}^{*}$. Before expressing $\mathrm{k}^{*}$, however, let us first find a similar convergence equation for consumption.
From (A7_2D), we can write:

$$
\begin{equation*}
c(t)=A f(k(t))-\delta \cdot k(t)-\dot{k}(t) \tag{A7_23D}
\end{equation*}
$$

Substituting (A7_21D) and its first time derivative into (A7_23D), we get:

$$
\begin{equation*}
c(t)=A f(k(t))-\delta \cdot k(t)-\lambda\left[k(0)-k^{*}\right] e^{\lambda t} \tag{A7_24D}
\end{equation*}
$$

$$
\begin{equation*}
c(t)=A f\left(\left[k(0)-k^{*}\right] e^{\lambda t}+k^{*}\right)-\delta \cdot\left\{\left[k(0)-k^{*}\right] \cdot e^{\lambda t}+k^{*}\right\}-\lambda\left[k(0)-k^{*}\right] e^{\lambda \mathrm{t}} \tag{A7_25D}
\end{equation*}
$$

$c(t)=A f\left(\left[k(0)-k^{*}\right] \cdot e^{\lambda \mathrm{t}}+k^{*}\right)-(\delta+\lambda) \cdot\left[k(0)-k^{*}\right] e^{\lambda \mathrm{t}}-\delta \cdot k^{*}$
(A7_26D) might be used to determine the optimum consumption at time 0 for some given initial capital stock $\mathrm{k}(0)$ :

$$
\begin{align*}
& c(0)=A f\left(\left[k(0)-k^{*}\right]+k^{*}\right)-(\delta+\lambda) \cdot\left[k(0)-k^{*}\right]-\delta \cdot k^{*}  \tag{A7_27D}\\
& \left.c(0)=A f(k(0))-\delta \cdot k(0)-\lambda \cdot l k(0)-k^{*}\right] \tag{A7_28D}
\end{align*}
$$

Optimum $c(0)$ along with $k(0)$ are depicted in Figure No. 1_A7. Moreover, the equation of the saddle path, relating optimum $\mathrm{c}(\mathrm{t})$ - the control variable, to $\mathrm{k}(\mathrm{t})$ - the state variable, known also as the policy function (Barro 2004:105) might be determined by using (A7_24D) and (A7_21D):

$$
\begin{equation*}
c(t)=A f(k(t))-\delta \cdot k(t)-\lambda\left[k(t)-k^{*}\right] \tag{A7_29D}
\end{equation*}
$$

$c(t)=A f(k(t))-(\delta+\lambda) \cdot k(t)+\lambda \cdot k^{*}$

As we will see, $\lambda$ is lower (in negative value) with higher $\theta$. Thus, the saddle path is closer to the capital locus $\mathrm{dk} / \mathrm{dt}=0$ with higher $\theta$.

Finally, let us determine the steady state value of capital, consumption, and the saving rate. To solve these steady-state values, however, we need to assume a specific form of the production function. Thus, let us consider a simple Cobb-Douglas form:

$$
\begin{equation*}
Y(t)=A K(t)^{\alpha} L^{1-\alpha} \tag{A7_31D}
\end{equation*}
$$

As we assumed before, A and L are constant. The intensive form might be obtained by dividing (A7_31D) by L:

$$
\begin{equation*}
y(t)=A k(t)^{\alpha} \tag{A7_32D}
\end{equation*}
$$

where $\mathrm{y}=\mathrm{Y} / \mathrm{L}$ and $\mathrm{k}=\mathrm{K} / \mathrm{L}$. To find the steady state value of $\mathrm{k}^{*}$, let us use (A7_47) for $\mathrm{g}=0$ :

$$
\begin{equation*}
A \alpha k^{\alpha-1}-\delta=\rho \tag{A7_33D}
\end{equation*}
$$

(A7_33D) states that in the steady state, the natural rate of interest $\mathrm{r}^{*}=\mathrm{MPK}-\delta$ is equal to the subjective discount rate $\rho$. The steady state of $\mathrm{k}(\mathrm{t})$ is thus:

$$
\begin{equation*}
k^{*}=\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \tag{A7_34D}
\end{equation*}
$$

Steady state level of consumption is (using A7_51 for $n=g=0$ ):
$c^{*}=A\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}-\delta\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$
(A7_35D)

We can compare this level with the golden rule that is derived from (A7_57): ${ }^{268}$
$A \alpha k^{\alpha-1}=\delta$
(A7_36D)

The golden rule level of capital is:
$k_{G R}=\left(\frac{A \alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$

Obviously, $\mathrm{k}^{*}$ is lower than $\mathrm{k}_{\mathrm{GR}}$ as long as the subjective discount rate $\rho$ is positive. Hence $\mathrm{c}^{*}$ is also lower than $\mathrm{c}_{\mathrm{GR}}$ :

$$
\begin{equation*}
c_{G R}=A\left(\frac{A \alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}-\delta\left(\frac{A \alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{A7_38D}
\end{equation*}
$$

The optimum saving rate might be also determined from (A7_11). For $\mathrm{n}=\mathrm{g}=0$, we get:

$$
\begin{equation*}
s^{*} f\left(k^{*}\right)=\delta \cdot k^{*} \tag{A7_39D}
\end{equation*}
$$

Which yields:

$$
\begin{align*}
& s^{*}=\delta \cdot \frac{\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}}{A\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}}  \tag{A7_40D}\\
& s^{*}=\frac{\delta}{A} \cdot\left(\frac{A \alpha}{\rho+\delta}\right)  \tag{A7_41D}\\
& s^{*}=\frac{\delta \alpha}{\rho+\delta} \tag{A7_42D}
\end{align*}
$$

As can be seen, the optimum saving rate in the steady state is positively related to the depreciation rate and negatively to the subjective discount rate. Surprisingly, it does not depend on the level of technologies $A$.
Furthermore, golden rule level of saving is simply:
$s_{G R}=\frac{\delta k_{G R}}{y_{G R}}$

[^170]\[

$$
\begin{align*}
& s_{G R}= \frac{\delta \cdot\left(\frac{A \alpha}{\delta}\right)^{\frac{1}{1-\alpha}}}{A\left(\frac{A \alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}}  \tag{A7_44D}\\
& s_{G R}=\frac{\delta \cdot\left(\frac{A \alpha}{\delta}\right)}{A} \tag{A7_45D}
\end{align*}
$$
\]

$s_{G R}=\alpha$

As can be seen, the optimum saving rate is lower than the golden rule. People are too impatient ( $\rho>0$ ) to save such a large part of their incomes. Interestingly enough, if we neglect also depreciation rate (i.e. $\delta=0$ ), the optimum saving rate in the steady state is zero (see A7_42 D). A picture of such an economy that is converging to a seemingly peculiar steady state is given in Figure No. 0a_A7. ${ }^{269}$ At first glance, it might seem optimal to permanently accumulate capital, which will result in ever increasing income and consumption. However, impatient households will eventually choose zero saving (see A7_42 D and Figure No. 0b_A7) and some definite level of consumption (A7_47 D): ${ }^{270}$
$c^{*}=A\left(\frac{A \alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}$

[^171]

Figure No. 0a_A7 RCK model for $\mathrm{n}=\mathrm{g}=\delta=0$


Figure No. 0b_A7 Solow model representation of the RCK model for $\mathbf{n}=\mathbf{g}=\delta=0$
Note: Optimum saving rate gradually falls from $s(0)$ to $s_{R C K, S S}=0$.

And finally, $\mathrm{k}^{*}$ and $\mathrm{c}^{*}$ might be used to determine the speed of convergence $-\lambda$ in (A7_22D). Before that, however, we need to determine $f^{\prime \prime}\left(k^{*}\right)$. Thus, using (A7_32D) we get: ${ }^{271}$

[^172]\[

$$
\begin{equation*}
f^{\prime \prime}(k)=\alpha(\alpha-1) k^{\alpha-2} \tag{A7_48D}
\end{equation*}
$$

\]

At the steady state (see A7_34D), equation (A7_48 D) yields:

$$
\begin{equation*}
f^{\prime \prime}\left(k^{*}\right)=\alpha(\alpha-1)\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{\alpha-2}{1-\alpha}} \tag{A7_49D}
\end{equation*}
$$

Expression $\mathrm{Af}^{\prime \prime}\left(\mathrm{k}^{*}\right) \mathrm{c}^{*}$ in (A7_22D) is thus:

$$
\begin{align*}
& A f^{\prime \prime}\left(k^{*}\right) c^{*}=A \alpha(\alpha-1)\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{\alpha-2}{1-\alpha}}\left[A\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}-\delta\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\right]= \\
& =A^{2} \alpha(\alpha-1)\left(\frac{A \alpha}{\rho+\delta}\right)^{-2}-\delta A \alpha(\alpha-1)\left(\frac{A \alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}=(\alpha-1) \frac{(\rho+\delta)^{2}}{\alpha}-\delta(\alpha-1)(\rho+\delta)= \\
& =(\alpha-1)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha}-\delta\right) \tag{A7_50D}
\end{align*}
$$

Hence, $\lambda$ in (A7_22D) might be expressed as:
$\lambda=\frac{\rho-\sqrt{\rho^{2}-\frac{4}{\theta}(\alpha-1)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha}-\delta\right)}}{2}<0$
$\lambda=\frac{\rho-\sqrt{\rho^{2}+\frac{4}{\theta} \frac{(1-\alpha)}{\alpha}(\rho+\delta)(\rho+\delta-\alpha \delta)}}{2}<0$

As can be seen, the speed of convergence $-\lambda$ depends negatively on parameter $\theta$. Hence the lower the elasticity of substitution (high $\theta$ ), the lower the pace at which the economy moves towards its steady state. The reason is that with high $\theta$, the saving function is rather inelastic. On the other hand, the speed of convergence $-\lambda$ depends positively on the subjective discount rate $\rho$. Thus, higher impatience leads to faster convergence. The reason is that $\rho$ does not affect the slope of the saving function but rather its position in the r-s space.

So far, we have developed enough tools to simulate the behaviour of the economy in response to various shocks. Let us first consider a (permanent) one-time increase in the level of technologies from $\mathrm{A}_{1}$ to $\mathrm{A}_{2}$. The evolution of the real interest rate is depicted in Figure No. 54 in the main text. In Figure No. 1_A7, we plot the impact of an increase in $A$ in the Ramsey model and in the production part of the economy represented by the Solow model (Figure No. 2_A7).


Figure No. 1_A7 Increase in $A$ in the RCK model. Note: $c^{l}(0)$ is the initial optimal consumption for low $\theta, c^{2}(0)$ is the initial optimal consumption for high $\theta$. $k_{1}{ }^{*}$ performs the role of $k(0)$ if the relevant steady state is $k_{2}{ }^{*}$.


Figure No. 2_A7 Increase in $A$ in the RCK model represented in the Solow model. Note: $c_{1}{ }^{*}$ is definitely lower than $c_{2}{ }^{*}$. However, the optimum saving rate is the same in both steady states $\left(s_{R C K}\right)$. The same holds for the steady state natural rate of interest $r^{*}=M P K^{*}-\delta$, which is determined solely by the subjective discount rate $\rho$. Thus, the slope of the production function at the steady state, which represents MPK*, is equal in both steady states. Golden rule might be found at the point where the slope of the production function (MPK) is equal to the slope of the depreciation curve ( $\delta$ ).

As can be seen in Figure No.1_A7, the eventual steady state capital and consumption per worker are greater than in the initial steady state (see A7_47D and A7_34D). The same conclusion obviously holds for the output per worker (see Figure No. 2_A7). However, the steady state natural rate of interest (see A7_33D) is not affected by greater $A$ since it is determined solely by the time preference (in sense two), i.e. by the subjective discount rate $\rho$. The steady state optimum saving rate is not affected by the level of technologies either (see A7_42 D). ${ }^{272}$
The transition period when the economy moves from the old to the new steady state demands several comments as well. As can be seen in Figure No. 54 in the main text, the real natural rate of interest suddenly increases after the rise in the level of technologies. However, it gradually falls to the original level that is solely determined by the time preference (in sense two).

Another important question is the behaviour of the optimum consumption immediately after the shock. Figure No. 1_A7 indicates that the optimum response of consumption depends on the shape of the saddle path. Low elasticity of substitution (high $\theta$ ) is associated with the saddle path that is closer to the capital locus $\mathrm{dk} / \mathrm{dt}=0$. The reason is the strong preference for consumption smoothing resulting in a relatively steep saving curve (or even saving curve with negative slope). ${ }^{273}$ Hence, an increase in the demand for capital and in the investment demand leading to a higher interest rate is followed in this case by a relatively modest response on the part of saving. Since higher level of technologies raises initial income and the impact on saving is relatively low, present consumption rises.
On the other hand, high intertemporal elasticity of substitution (low $\theta$ ) implies a relatively flat saving curve resulting in a considerable increase in saving after the rise in the interest rate. Thus, present optimum consumption drops after the increase in the level of technologies. The economic reason is the low preference for consumption smoothing leading to significant consumption growth over time, once the real interest rate exceeds subjective discount rate (see Euler equation A7_42). Hence, to reach this rapid growth in consumption, present consumption must fall.
Furthermore, the rapidity of convergence of the key variables is also affected by parameter $\theta$. Simulations in figures below clearly demonstrate that higher $\theta$ is associated with less rapid convergence (Figure No. 3_A7 and 4_A7). The simulations therefore verify conclusions discussed above. Let us stress again a considerable increase in the optimum saving rate when $\theta$ is low (Figure No. 6_A7).

[^173]

Figure No. 3_A7 Evolution of capital per worker after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by $10 \%$ ) at time 0 . The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$.


Figure No. 4_A7 Evolution of output per worker after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10\%) at time 0. The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$.


Figure No. 5_A7 Evolution of consumption per worker after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by $10 \%$ ) at time 0 . The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$.


Figure No. 6_A7 Evolution of the saving rate after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10\%) at time 0 . The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$.


Figure No. 7_A7 The growth rate in output per worker after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by $10 \%$ ) at time 0 . The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$.


Figure No. 7_A7b The growth rate in output per worker within one year after the increase in the level of technologies $A$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10\%) at time 0. The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%, \theta=1$.


Figure No. 8_A7 The real interest rate after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10\%) at time 0 . The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$.

The higher saving rate resulting from greater elasticity of substitution also leads to more rapid growth in output in periods that follow after the shock (see Figure No. 7_A7). ${ }^{274}$ Nevertheless, this rapid growth gradually dies out as the growth rate returns to the initial level (which is zero in this case). As can be seen, in more remote periods, growth is lower in the economy populated by consumers with higher intertemporal elasticity of substitution. This specific behaviour of the growth rate is a direct consequence of more rapid convergence of this particular economy. Relatively fast growth in the initial periods is followed by a sluggish growth in GDP in the more remote future.

The evolution of the real interest rate is simulated in Figure No. 8_A7. As can be seen, the initial increase after the technological shock is the same for all values of $\theta$. Nevertheless, the decline to the steady state level is sharper in the economy with higher elasticity of substitution since the accumulation of capital is faster in this case.
Figures No. 7_A7 and 8_A7 also indicate that (apart from the period of the shock) ${ }^{275}$ the real interest rate is always greater than the growth rate of GDP. In other words, the economy is dynamically efficient also in the transition period between the two steady states.
Furthermore, keeping money and velocity constant, the inflation rate is simply the inverse to the growth rate of real output. Since the real interest rate is greater than the growth rate in

[^174]output (and hence the rate of price deflation), nominal interest rate is always positive. ${ }^{276}$ As can be seen in Figure No. 9_A7, the nominal interest rate approaches its steady state level from below if the elasticity of substitution is high (low $\theta$ ). The reason is a rapid growth in GDP and (for constant money) significant price deflation.


Figure No. 9_A7 The nominal interest rate after the increase in $A$ in the RCK model and the role of $\theta$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10\%) at time 0. The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \rho=5 \%$. Money and velocity are constant.

We may conclude that even though the real interest rate is surely higher for all levels of $\theta$, nominal interest rate might either increase or decrease in the periods after the shock. According to our analysis, the initial increase in the nominal interest rate is associated with low intertemporal elasticity of substitution (high $\theta$ ) due to the slow growth in GDP and the resulting modest price deflation. On the other hand, if the elasticity of substitution is high (low $\theta$ ), the nominal interest rate declines owing to the rapid price deflation that reflects fast economic growth.

We also examine the role of the subjective discount rate after the increase in the level of technologies. However, since this parameter affects mainly the position of the saving curve rather than its slope, we do not obtain any interesting observations. $\rho$ affects the levels of the key variables, hence its role in the convergence process is not as interesting as in the case of $\theta$. Yet, there is some role of the subjective discount rate in the speed of convergence. As was already said, lower $\rho$ is associated with less rapid convergence, as is depicted in Figures No. 10_A7 and 11_A7.

[^175]

Figure No. 10_A7 Evolution of capital per worker after the increase in $A$ in the RCK model and the role of $\rho$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10\%) at time 0. The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \theta=1$.


Figure No. 11_A7 The real interest rate after the increase in A in the RCK model and the role of $\rho$.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by $10 \%$ ) at time 0 . The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, \theta=1$.

More interesting is the behaviour of the economy when the subjective discount rate itself changes. Let us consider a fall in the subjective discount rate from $5 \%$ to $4 \%$. The graphical representation of this change in the RCK model is sketched in Figure No. 52 in the main text. The static representation of lower time preference (in sense two) is represented in Figure No.

30 in the main text if we move from point E2 to point E1. As can be seen, this shock will shift the entire saving curve and the natural rate of interest falls.

However, in the dynamic model presented here, the subsequent dynamics of the economy is much more complicated than suggested by a simple static model. An increase in saving and the movement along the investment curve results in the fact that more capital is being accumulated. Thus, next period output is greater along with the next period income. As a result, saving curve shifts outwards, as people earn higher income. This will further decrease the natural rate of interest and increase the amount of invested capital. Hence, more capital might be accumulated anew. Consequently, income will rise along with saving. This is the source for the new accumulation of capital in the next round.

The question is whether the drop in time preference can provoke eternal growth in income and a never-ending decrease in the natural rate of interest. As is obvious, the static model is rather inadequate to account for this complicated dynamics. In the first place, it does not reflect additional shifts in the saving curve that result from the increasing income. In the second place, it cannot answer the question whether the saving curve is being eternally moved to the right.


Figure No. 0c_A7 Decrease in $\rho$ in the RCK model for $\mathrm{n}=\mathrm{g}=\delta=0$.


Figure No. 0d_A7 Solow model representation of the decrease in $\rho$ in RCK model for $\mathrm{n}=\mathrm{g}=\delta=0$ Note: The optimum saving rate after the decrease in $\rho$ gradually falls back to $s_{R C K, S S}=0$.

It is the RCK model presented here that may provide us with the fundamental answers. It clearly demonstrates that the impact of lower subjective discount rate gradually dies out. The increase in income is not eternal due to the diminishing marginal productivity of capital and the impatience of people. Greater capital is subjected to greater depreciation, hence eternal accumulation is not possible under the diminishing marginal productivity. Nonetheless, even if the depreciation rate was zero, positive (though lower) subjective rate of discount would act as a break for further accumulation of capital. The point of optimality in dynamic equilibrium $c^{*}, k^{*}$ in Figure No. 0a,b_A7 would just move to the right (see Figure No. 0c A7 and d_A7). Obviously, there would be a sudden decrease in present consumption after the fall in $\rho$. Yet, even with no depreciation of capital, diminishing marginal productivity of capital and the resulting decrease in the real interest rate together with the presence of the positive subjective discount rate would bring the process of capital accumulation to a halt.

The set of figures below shows the evolution of the key variables after the fall in the subjective discount rate. They clearly demonstrate the transitory impact of the fall in the subjective discount rate. At the same time, the role of the intertemporal elasticity of substitution $1 / \theta$ is also presented.

Figure No. 12_A7 clearly shows that lower $\theta$ is associated with faster convergence. The economic reason is the significant drop of present consumption (see Figure No. 13_A7). As was said before, higher elasticity of substitution is also associated with faster growth of GDP in the subsequent periods after the shock and lower growth in more remote future (Figure No. 14_A7). The reason lies in the considerable increase in the saving rate (see Figure No. 15_A7).


Figure No. 12_A7 Evolution of capital per worker after the decrease in $\rho$ from 5\% to $4 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3$.


Figure No. 13_A7 Evolution of consumption per worker after the decrease in $\rho$ from $5 \%$ to $4 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3$.


Figure No. 14_A7 The growth rate in output per worker after the decrease in $\rho$ from 5\% to $4 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3$.


Figure No. 15_A7 The saving rate after the decrease in $\rho$ from $5 \%$ to $4 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3$.

Figure No. 15_A7 also displays that the eventual level of the saving rate is greater for all values of $\theta$ compared with the initial level (see equation A7_42 D). ${ }^{277}$ This value is also independent of $\theta$. However, lower elasticity of substitution leads to less abrupt increase in saving after the shock due to the high preference for the consumption smoothing. As can be seen in the figure, the saving rate gradually falls to the new steady state level. Nevertheless, it can be shown that the saving rate in the transition period might approach the new steady state level from below if the elasticity of substitution is low enough (Barro 2004: 109, 135-137). ${ }^{278}$
And finally, even though present consumption falls, the eventual level is greater than in the initial steady state. This is a direct consequence of the dynamic efficiency of this economy. As can be seen in Figures No. 16_A7 and 14_A7, the real interest rate is greater than the growth rate of output even in the transition period. Thus, nominal interest rate is also positive, although there might be a temporary period of price deflation (for constant money and velocity). As can be seen in Figure No. 17_A7, the drop in the nominal interest rate after the shock is the largest for low $\theta$, reflecting not only rapid fall in the real interest rate but also a relatively higher price deflation resulting from a faster growth in output.


Figure No. 16_A7 The real interest rate after the decrease in $\rho$ from $5 \%$ to $4 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3$.

[^176]

Figure No. 17_A7 The nominal interest rate after the decrease in $\rho$ from 5\% to $4 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3$. Money and velocity are constant.

In the next section, we relax the assumption of constant population and technology. Yet, we will assume that all changes in technology will reflect only changes in the growth rate in labour-augmenting technological progress $g$. The effects of this change are also discussed in the main text.
Equations of approximations of the economy around its steady state (A7_1D) and (A7_4D) are the same as before. Compared with the previous case, the law of motion of capital is (see A7_11):
$\dot{k}(t)=f(k(t))-c(t)-(n+g+\delta) k(t)$

The Euler equation might be represented as (see A7_22):
$\dot{c}(t)=\frac{f^{\prime}(k(t))-\delta-\rho-\theta g}{\theta} c(t)$

Thus, the linear approximation of (A7_53D) is as follows:
$\dot{k}(t)=0+\left[f^{\prime}\left(k^{*}\right)-(n+g+\delta)\right]\left(k-k^{*}\right)+(-1)\left(c-c^{*}\right)$
(A7_55D)

The linear approximation of (A7_54D) gives us:
$\dot{c}=0+\frac{f^{\prime \prime}\left(k^{*}\right)}{\theta} c^{*}\left(k-k^{*}\right)+\frac{f^{\prime}\left(k^{*}\right)-\delta-\rho-\theta g}{\theta}\left(c-c^{*}\right)$

Since the real interest rate at the steady state is $\mathrm{r}^{*}=\rho+\theta \mathrm{g}$ and because the optimum of profit maximizing firms requires $\mathrm{r}=\mathrm{f}^{\prime}(\mathrm{k})-\delta$, (A7_55D) can be written as:
$\dot{k}(t)=0+[\rho+\theta g-(n+g)]\left(k-k^{*}\right)-\left(c-c^{*}\right)$

Condition (A7_14) implies that the term in the brackets in (A7_57D) is positive. Let us denote this term as $\beta \equiv \rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$. Thus:

$$
\begin{equation*}
\dot{k}(t)=\beta\left(k-k^{*}\right)-\left(c-c^{*}\right) \tag{A7_58D}
\end{equation*}
$$

Furthermore, condition of the steady state (A7_47) implies that the second term in (A7_56D) is zero:
$\dot{c}=\frac{f^{\prime \prime}\left(k^{*}\right)}{\theta} c^{*}\left(k-k^{*}\right)$
(A7_59D)

By comparing the system of equations (A7_58D) and (A7_59D) with (A7_7D) and (A7_8D), we can see that they are almost the same. The only difference is the presence of $\beta$ rather than $\rho$ in equation (A7_58D) and the absence of $A$ in (A7_59D). Hence, applying the same methods as before, we get:
$k(t)=\left[k(0)-k^{*}\right] e^{\lambda t}+k^{*}$
where:

$$
\begin{equation*}
\lambda=\frac{\beta-\sqrt{\beta^{2}-4 \frac{f^{\prime \prime}\left(k^{*}\right) c^{*}}{\theta}}}{2}<0 \tag{A7_61D}
\end{equation*}
$$

The solution for the motion of consumption can be obtained if we use (A7_53D), (A7_60D), and the first time derivative of (A7_60D):

$$
\begin{align*}
& c(t)=f(k(t))-(n+g+\delta) \cdot k(t)-\lambda \cdot\left[k(0)-k^{*}\right] e^{\lambda t}  \tag{A7_62D}\\
& c(t)=f\left(\left[k(0)-k^{*}\right] e^{\lambda t}+k^{*}\right)-(n+g+\delta) \cdot\left[\left\{k(0)-k^{*}\right] e^{\lambda t}+k^{*}\right\}-\lambda\left[k(0)-k^{*}\right] e^{\lambda t}  \tag{A7_63D}\\
& c(t)=f\left(\left[k(0)-k^{*}\right] e^{\lambda t}+k^{*}\right)-(n+g+\delta+\lambda) \cdot\left[k(0)-k^{*}\right] e^{\lambda t}-(n+g+\delta) \cdot k^{*} \tag{A7_64D}
\end{align*}
$$

Optimum initial consumption is:

$$
\begin{equation*}
c(0)=f\left(\left[k(0)-k^{*}\right]+k^{*}\right)-(n+g+\delta+\lambda) \cdot\left[k(0)-k^{*}\right]-(n+g+\delta) \cdot k^{*} \tag{A7_65D}
\end{equation*}
$$

$$
\begin{equation*}
c(0)=f(k(0))-(n+g+\delta) \cdot k(0)-\lambda \cdot\left[k(0)-k^{*}\right] \tag{A7_66D}
\end{equation*}
$$

The equation of the saddle path in this case is (using A7_60D and A7_62D):
$c(t)=f(k(t))-(n+g+\delta) \cdot k(t)-\lambda \cdot\left[k(t)-k^{*}\right]$
$c(t)=f(k(t))-(n+g+\delta+\lambda) \cdot k(t)+\lambda \cdot k^{*}$

The steady state values of capital (per effective worker), consumption (per effective worker), and saving rate might be determined by the same procedure as before. The specific form of the production function will be Cobb-Douglas again. However, now we assume that the population is growing over time and labour-augmenting technological progress is growing as well. Thus, the production function is:

$$
\begin{equation*}
Y(t)=K(t)^{\alpha}[A(t) L(t)]^{1-\alpha} \tag{A7_69D}
\end{equation*}
$$

The intensive form of (A7_69D) can be derived by dividing the whole expression by AL:
$y(t)=k(t)^{\alpha}$
(A7_70D)
where $\mathrm{y}=\mathrm{Y} /(\mathrm{AL})$ and $\mathrm{k}=\mathrm{K} /(\mathrm{AL})$. To find the steady state value of $\mathrm{k}^{*}$, let us use (A7_47) again: $\alpha k^{\alpha-1}-\delta=\rho+\theta . g$

The steady state of $k(t)$ is thus:
$k^{*}=\left(\frac{\alpha}{\rho+\delta+\theta . g}\right)^{\frac{1}{1-\alpha}}$

The steady state level of consumption is (using A7_51):
$c^{*}=\left(\frac{\alpha}{\rho+\delta+\theta . g}\right)^{\frac{\alpha}{1-\alpha}}-(n+g+\delta)\left(\frac{\alpha}{\rho+\delta+\theta . g}\right)^{\frac{1}{1-\alpha}}$

Again, we can compare this level with the golden rule that is derived from (A7_57):
$\alpha k^{\alpha-1}=n+g+\delta$
(A7_74D)

The golden rule level of capital is:
$k_{G R}=\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$

Obviously, $\mathrm{k}^{*}$ is lower than $\mathrm{k}_{\mathrm{GR}}$ as long as $\rho+\delta+\theta \mathrm{g}>\mathrm{n}+\mathrm{g}+\delta$. However, this implies that $\rho-\mathrm{n}-$ $(1-\theta) \mathrm{g}>0$. This condition was arrived at many times before. Hence, $\mathrm{c}^{*}$ is always lower than $\mathrm{C}_{\mathrm{GR}}$ :

$$
\begin{equation*}
c_{G R}=\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}-(n+g+\delta)\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \tag{A7_76D}
\end{equation*}
$$

Let us determine the optimum saving rate at the steady state. Using (A7_11), we get:

$$
\begin{equation*}
s^{*} f\left(k^{*}\right)=(n+g+\delta) \cdot k^{*} \tag{A7_77D}
\end{equation*}
$$

This yields (see A7_72D for $\mathrm{k}^{*}$ and A7_70D for the resulting $\mathrm{y}^{*}=\mathrm{f}\left(\mathrm{k}^{*}\right)$ ):

$$
\begin{equation*}
s^{*}=(n+g+\delta) \frac{\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{\alpha}{1-\alpha}}} \tag{A7_78D}
\end{equation*}
$$

$$
\begin{equation*}
s^{*}=(n+g+\delta) \frac{\alpha}{\rho+\delta+\theta \cdot g} \tag{A7_79D}
\end{equation*}
$$

As before, the optimum saving rate in the steady state is negatively related to the subjective discount rate. Moreover, it positively depends on the rate of population growth. The reason is that the dynastic family is concerned about the well-being of its offspring. Thus, any increase in the rate of the expansion of the household leads to an increase in saving in the effort to secure the optimum growth rate of consumption (see A7_42) for each of its member. More on this will be said in section $E$.
The effect of $\delta$ on $\mathrm{s}^{*}$ might be determined as follows:

$$
\begin{equation*}
\frac{\partial s^{*}}{\partial \delta}=\frac{\alpha(\rho+\delta+\theta . g)-\alpha(n+g+\delta)}{(\rho+\delta+\theta . g)^{2}} \tag{A7_80D}
\end{equation*}
$$

(A7_80D) is definitely positive due to condition (A7_14). Thus, optimum saving in the steady state always increases with higher depreciation rate.

The effect of the growth rate of technological progress on s* might be obtained by a similar procedure:

$$
\begin{equation*}
\frac{\partial s^{*}}{\partial g}=\frac{\alpha(\rho+\delta+\theta . g)-\alpha(n+g+\delta) \theta}{(\rho+\delta+\theta . g)^{2}} \tag{A7_81D}
\end{equation*}
$$

(A7_81D) is positive if:
$(\rho+\delta+\theta . g)>(n+g+\delta) \theta$
(A7_82D)

This yields:
$\rho>\theta . n+(\theta-1) \delta$

Hence, faster technological progress may increase the steady state optimum saving when the elasticity of substitution is high (lower $\theta$ ).

The golden rule level of saving is equal to $\alpha$ even in this case. This outcome can be easily proved generally. Due to condition (A7_14), $s^{*}$ is lower than $\mathrm{S}_{\mathrm{GR}}$ (compare A7_79D and A7_87D):
$s_{G R}=\frac{(n+g+\delta) \cdot k_{G R}}{y_{G R}}$
$s_{G R}=\frac{(n+g+\delta)\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}}$
$s_{G R}=(n+g+\delta) \frac{\alpha}{n+g+\delta}$
$s_{G R}=\alpha$

And finally, $\mathrm{k}^{*}$ and $\mathrm{c}^{*}$ might be used to determine the speed of convergence $-\lambda$ in (A7_61D). Before that, however, we need to determine $f^{\prime \prime}\left(k^{*}\right)$. Thus, using (A7_70D) we get:

$$
\begin{equation*}
f^{\prime \prime}(k)=\alpha(\alpha-1) k^{\alpha-2} \tag{A7_88D}
\end{equation*}
$$

At the steady state (see A7_72D), equation (A7_88D) yields:
$f^{\prime \prime}\left(k^{*}\right)=\alpha(\alpha-1)\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{\alpha-2}{1-\alpha}}$

Expression $\mathrm{f}^{\prime \prime}\left(\mathrm{k}^{*}\right) \mathrm{c}^{*}$ in (A7_61D) is thus:

$$
\begin{align*}
& f^{\prime \prime}\left(k^{*}\right) c^{*}=\alpha(\alpha-1)\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{\alpha-2}{1-\alpha}}\left[\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{\alpha}{1-\alpha}}-(n+g+\delta)\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{1}{1-\alpha}}\right]= \\
& =\alpha(\alpha-1)\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{-2}-\alpha(\alpha-1)(n+g+\delta)\left(\frac{\alpha}{\rho+\delta+\theta \cdot g}\right)^{\frac{\alpha-1}{1-\alpha}}= \\
& =(\alpha-1) \frac{(\rho+\delta+\theta \cdot g)^{2}}{\alpha}-(\alpha-1)(n+g+\delta)(\rho+\delta+\theta \cdot g)= \\
& =(\alpha-1)(\rho+\delta+\theta \cdot g)\left[\frac{\rho+\delta+\theta \cdot g}{\alpha}-(n+g+\delta)\right] \quad \text { (A7_90 D } \tag{A7_90D}
\end{align*}
$$

Hence, $\lambda$ in (A7_61D) can be expressed as:

$$
\begin{align*}
& \lambda=\frac{\beta-\sqrt{\beta^{2}-4 \frac{(\alpha-1)(\rho+\delta+\theta \cdot g)}{\theta}\left[\frac{\rho+\delta+\theta \cdot g}{\alpha}-(n+g+\delta)\right]}}{2}<0  \tag{A7_91D}\\
& \lambda=\frac{\beta-\sqrt{\beta^{2}+4 \frac{1-\alpha}{\alpha} \frac{\rho+\delta+\theta \cdot g}{\theta}[\rho+\delta+\theta . g-\alpha(n+g+\delta)]}}{2}<0 \tag{A7_92D}
\end{align*}
$$

As before, the speed of convergence is positively related to $\rho$ and negatively related to $\theta$.
Let us now simulate the behaviour of the economy after a sudden increase in the growth rate of technological progress $g$. In Figure No. 53 in the main text we show that the real natural rate of interest gradually increases after the rise in $g$. Here, we will discuss this shock in more detail. We focus on the role of the preference for consumption smoothing (parameter $\theta$ ) that reflects curvature of the intertemporal indifference curves.


Figure No. 18_A7 The saving rate after the increase in $g$ from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%$.

First, Figure No. 18_A7 clearly shows that the elasticity of substitution affects the level of the optimum steady state saving rate when the positive technological progress is present in the model. According to equation (A7_79D), the higher the elasticity of substitution (lower $\theta$ ), the higher the steady state saving rate. This can be explained by a flat saving curve associated with low $\theta$ and the expanding investment curve fed by positive technological progress.

Second, the optimum response of saving to an increase in $g$ also depends on $\theta$. As can be seen in Figure No. 18_A7, lower elasticity of substitution (high $\theta$ ) is associated with a sharp drop in the saving rate after the shock. The economic reason is the strong preference for consumption smoothing. An increase in $g$ guarantees higher growth in the (future) income endowment. Thus, this higher future income the consumer shifts closer to the present via reduced saving. ${ }^{279}$

Third, whether the eventual steady state saving rate is lower or higher than the initial one also depends on the elasticity of substitution (see equations from A7_81D to A7_83D). As can be seen in Figure No. 18_A7, higher preference for consumption smoothing results in a lower eventual steady state saving rate, even though the saving rate in the transition period gradually increases from very low levels observed immediately after the shock.

[^177]

Figure No. 19_A7 The growth rate in GDP per worker after the increase in $g$ from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%$.

The behaviour of other fundamental variables then depends on the response of saving discussed above. Figure No. 19_A7 indicates that the most rapid growth in GDP per worker in the initial periods is triggered in the case of high elasticity of substitution. Thus, convergence is the fastest for low $\theta$. However, the eventual growth rate in GDP per worker is dictated solely by g . Thus, the new steady state (BGP) growth will be higher regardless of the size of $\theta$.

The response of saving is mirrored in the optimum behaviour of consumption. Figure No. $20 \_$A7 reports the optimum growth rate of consumption per worker. As can be seen, high elasticity of substitution is associated with only a modest increase in the present consumption after the shock, which enables more rapid growth in consumption and faster convergence to the new steady state growth rate in the following periods.
Furthermore, the elasticity of substitution critically affects the behaviour of the real natural rate of interest not only in the transition period but also its level in the new steady state. Figure No. 21_A7 displays the evolution of the real interest rate for various $\theta$. The most sluggish convergence is seen in the case of high $\theta$. At the same time, the eventual increase in the real rate of interest is the largest in this case as well (see equation A7_48).


Figure No. 20_A7 The growth rate in consumption per worker after the increase in $g$ from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%$.


Figure No. 21_A7 The real interest rate after the increase in $g$ from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%$.

Comparing Figure No. 21_A7 and 19_A7, the economy is always dynamically efficient since the real interest rate always exceeds the growth rate in GDP. This conclusion is reflected (for constant money and velocity) in the evolution of the nominal interest rate (see Figure No. 22_A7). Yet, a very interesting observation might be found in this figure. For $\theta=1$ (i.e. logarithmic utility function discussed many times in the previous text), the eventual nominal
interest rate is not affected by higher technological progress, because higher real natural rate of interest is perfectly offset by more rapid price deflation that is implied by faster growth in GDP. As was discussed in great detail in the previous sections, the nominal interest rate is not affected by productivity only in this very specific case. ${ }^{280}$ Thus, the pure time preference approach is correct as regards the un-importance of productivity even in this complicated dynamic model provided that its theoretical basis is the interest on money (and not the intertemporal exchange of real goods) and the utility function is logarithmic. However, even in this very convenient environment for the Austrian PTPT, interest on money is affected by productivity in the transition period. At the same time, the period for which the nominal interest rate is different from the pure time preference $\rho$ is rather long.


Figure No. 22_A7 The nominal interest rate after the increase in $g$ from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.
Note: The set of exogenous parameters is as follows: $\delta=3 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%$.

It should be stressed that high elasticity of substitution $(\theta<1)$ is associated with a drop in the nominal interest rate after the shock and also with a fall in its steady state level. The reason is that the real interest rate does not grow enough to offset higher price deflation resulting from more rapid economic growth. On the other hand, low elasticity of substitution $(\theta>1)$ leads to a negligible decrease in the nominal rate (it can even increase for very high $\theta$ ) after the shock and, as can be seen in the figure, to an increase in the eventual nominal rate of interest. The reason is obviously a large increase in the natural steady state real rate of interest after the rise in $g$.

## Section E - The role of population growth

In the previous sections, we modelled the lifetime utility of the whole infinitely-lived

[^178]household the size of which was growing at the rate of $n$. This family was concerned also about its future members. Hence, an increase in the growth rate of population $n$ immediately increased saving of this family, leaving the steady-state level of the capital stock per effective worker unaffected (see A7_72D). The same conclusion held for the steady state real interest rate (see A7_48). The economic reason is that the family tries to guarantee the optimum growth rate of consumption for all its members (both present and future) unaltered even after an increase in $n$. Hence, a sudden increase in the rate of expansion of the family leads to an immediate increase in saving (reduction in present consumption) that will perfectly offset this increase in $n$ (see Figures No. 23_A7 and 24_A7). As a result, the optimum growth rate of consumption of each member (see A7_42) is unaffected after the shock to $n$.


Figure No. 23_A7 Increase in $n$ in the RCK model when the family is concerned about its offspring.


Figure No. 24_A7 Increase in $n$ in the RCK model when the family is concerned about its offspring; Solow model representation.
Note: For CDPF, $\alpha=s_{G R}$ is not affected by $n$, so the figure is not accurate at this point.

However, we could have chosen a different modelling strategy (see Blanchard and Fisher 1989). Let us consider an economy of infinitely-lived individuals, each maximizing his or her life-time utility:
$U_{s}=\int_{s}^{\infty} e^{-\rho(t-s)} \frac{C(t)^{1-\theta}}{1-\theta} d t$
The growth rate of population in this economy is $n$. Thus, instead of maximizing from a specific time 0 , we consider a representative agent standing at time $s$. Each agent is concerned only about his or her well-being, not about the others. Hence, he or she does not take into account the growth rate of population as in (A7_12).
The structure of the model and all important equations are (almost) the same as before. Yet, the idea behind the flow budget constraint (A7_25) or (A7_31) must be slightly refined. At the aggregate level the private debt is zero, thus the aggregate holdings of assets $\mathrm{B}^{\text {agr }}(\mathrm{t})$ must be equal to the aggregate stock of capital $\mathrm{K}(\mathrm{t})$. Moreover, aggregate assets might increase only due to aggregate savings in the society:
$\dot{B}^{a g r}(t)=W(t) L(t)-C(t) L(t)+r(t) B^{a g r}(t)$
$\mathrm{W}(\mathrm{t}) \mathrm{L}(\mathrm{t})$ represents total amount of wages in the economy at time $t, \mathrm{C}(\mathrm{t}) \mathrm{L}(\mathrm{t})$ total consumption at time $t$. The aggregate interest at time $t$ is reflected by $\mathrm{r}(\mathrm{t}) \mathrm{B}^{\text {agr }}(\mathrm{t})$.

The law of motion of individual assets might be easily determined from (A7_2 E), once we realize that the level of individual assets $B(t)$ is equal to $B^{\text {agr }}(t) / L(t)$. Hence, the instantaneous change in $\mathrm{B}(\mathrm{t})$ is as follows:

$$
\begin{equation*}
\dot{B}(t)=\frac{\dot{B}^{a g r}(t) L(t)-B^{a g r}(t) \dot{L}(t)}{L^{2}(t)}=\frac{\dot{B}^{a g r}(t)}{L(t)}-\frac{B^{a g r}(t)}{L(t)} \frac{\dot{L}(t)}{L(t)}=\frac{\dot{B}^{a g r}(t)}{L(t)}-n \frac{B^{a g r}(t)}{L(t)} \tag{A7_3E}
\end{equation*}
$$

Inserting (A7_2 E) into (A7_3 E), we get:

$$
\begin{equation*}
\dot{B}(t)=\left[\frac{W(t) L(t)-C(t) L(t)+r(t) B^{a g r}(t)}{L(t)}-n \frac{B^{a g r}(t)}{L(t)}\right] \tag{A7_4E}
\end{equation*}
$$

Since $\mathrm{B}^{\text {agr }}(\mathrm{t})=\mathrm{B}(\mathrm{t}) \mathrm{L}(\mathrm{t})$, equation (A7_4 E) yields:

$$
\begin{equation*}
\dot{B}(t)=W(t) L(t)-C(t) L(t)+r(t) B(t)-n B(t) \tag{A7_5E}
\end{equation*}
$$

Notice that this equation is exactly the same as equation (A7_31).
Furthermore, the objective of our representative individual is to maximize (A7_1 E) with respect to the flow budget constraint (A7_5 E). Using similar methods as before, we will arrive at the Euler equation: ${ }^{281}$
$\frac{\dot{C}(t)}{C(t)}=\frac{r(t)-\rho-n}{\theta}$
As can be seen, the optimum growth rate of consumption per person is negatively affected by the population growth. The economic reason is that the "selfish" individual does not (sufficiently) increase saving immediately after the increase in $n$. This is the key difference compared with the modelling of the entire family that was perfectly altruistic as regards its future members. As was said before, in case of the family, an increase in $n$ did not affect optimum consumption growth (apart from time 0 ), because the entire shock was absorbed by higher saving at time 0 .

With respect to the consumption per effective worker $\mathrm{c}(\mathrm{t})=\mathrm{C}(\mathrm{t}) / \mathrm{A}(\mathrm{t})$, (A7_6 E) might be used to show that:

[^179]\[

$$
\begin{equation*}
\frac{\dot{c}(t)}{c(t)}=\frac{\dot{C}(t)}{C(t)}-\frac{\dot{A}(t)}{A(t)}=\frac{r(t)-\rho-n}{\theta}-g=\frac{r(t)-\rho-n-\theta g}{\theta} \tag{A7_7E}
\end{equation*}
$$

\]

Thus, the steady state real (natural) interest rate is positively affected by the population growth:

$$
\begin{equation*}
r^{*}=\rho+n+\theta g \tag{282}
\end{equation*}
$$

The increase in $n$ is not (fully) reflected by higher saving of existing individuals. As a result, the given capital stock is then split among more individuals. This leads to a lower capital per effective worker $k$ and to a higher interest rate $r$ (see Figure No. 25_A7 and 26_A7). ${ }^{283}$
This conclusion might be easily proved by deriving the steady state level of capital per effective worker. Using the same procedure as before:
$\alpha k^{\alpha-1}-\delta=\rho+n+\theta . g$

We get:
$k^{*}=\left(\frac{\alpha}{\rho+\delta+n+\theta . g}\right)^{\frac{1}{1-\alpha}}$

Hence, steady state level of capital is negatively related to $n$. The optimum saving rate at the steady state is also affected by $n$ :

$$
\begin{equation*}
s^{*}=(n+g+\delta) \frac{\left(\frac{\alpha}{\rho+\delta+n+\theta \cdot g}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha}{\rho+\delta+n+\theta \cdot g}\right)^{\frac{\alpha}{1-\alpha}}} \tag{A7_11E}
\end{equation*}
$$

$s^{*}=(n+g+\delta) \frac{\alpha}{\rho+\delta+n+\theta . g}$

The specific impact of $n$ on $\mathrm{s}^{*}$ is then given by:

[^180]$\frac{\partial s^{*}}{\partial n}=\frac{\alpha(\rho+\delta+n+\theta . g)-\alpha(n+g+\delta)}{(\rho+n+\delta+\theta . g)^{2}}$
$\frac{\partial s^{*}}{\partial n}=\frac{\alpha(\rho+\delta+n+\theta \cdot g-n-g-\delta)}{(\rho+n+\delta+\theta . g)^{2}}$
$\frac{\partial s^{*}}{\partial n}=\frac{\alpha[\rho+(\theta-1) g]}{(\rho+n+\delta+\theta \cdot g)^{2}}$

The impact of $n$ on saving is thus much lower than in our previous model (see A7_79D):

$$
\begin{equation*}
\frac{\partial s^{*}}{\partial n}=\frac{\alpha}{\rho+\delta+\theta \cdot g} \tag{A7_16E}
\end{equation*}
$$

In other words, the increase in $n$ might raise the optimum saving in the new steady state, but not enough to fully compensate for this increase since $\mathrm{k}^{*}$ is negatively related to $n$ (and $\mathrm{r}^{*}$ is positively related to $n$ ). Thus, higher $n$ will result in lower capital but also in lower consumption per worker (see Figure No. 25_A7 and 26_A7) compared with the analysis of dynastic families (see Figure No. 23_A7 and 24_A7). In other words, not enough saving on the part of selfish individuals after the positive shock to $n$ and a relatively higher present consumption is "penalized" by relatively lower steady state (i.e. future) consumption due to lower steady state capital per (effective) worker.


Figure No. 25_A7 Increase in $n$ in the RCK model, the case of "selfish" individuals.
Note: $c^{l}(0)$ is the initial optimal consumption for low $\theta, c^{2}(0)$ is the initial optimal consumption for high $\theta$.


Figure No. 26_A7 Increase in $n$ in the RCK model, the case of "selfish" individuals. Solow model representation.

At the end of this section, let us stress that the RCK economy is dynamically efficient even when we assume selfish infinitely lived individuals. The restriction on convergence of the lifetime utility is as follows:
$\rho-(1-\theta) g>0$
This condition might be derived from (A7_1 E) using the same method as in (A7_13), but neglecting the expansion of the family. Furthermore, (A7_17 E) can be written as:
$\rho+\theta . g>g$
$\rho+\theta . g+n>g+n$

Yet, the left hand side of (A7_19 E) is equal to the steady state real interest rate, and the right hand side is equal to the growth rate of GDP at the steady state (BGP). Thus, the economy is dynamically efficient and the nominal interest rate is positive even for constant money and velocity, growing economy and gradually decreasing price level.

Section F - Deriving the initial level of consumption from the intertemporal budget constraint
In this section, we will demonstrate that the relative strength of the substitution and the income effect from the increase in the real interest rate depends on parameter $\theta$. More specifically, if $\theta<1$, then the substitution effect dominates, and the increase in $r$ leads to lower present consumption (or better to lower MPC - marginal propensity to consume). The opposite conclusion holds for $\theta>1$. If $\theta=1$, both effects compensate each other, and a hypothetical saving curve is vertical.
First, consider the intertemporal budget constraint (A7_34). Since we focus only on the role of the interest rate, let us assume that population and technology are constant, all households have one member, and suppose that both the interest rate and real wage are also time invariant. Thus, the intertemporal budget constraint becomes:
$\int_{0}^{\infty} e^{-r . t} C(t) d t=\frac{K(0)}{H}+\int_{0}^{\infty} e^{-r . t} W d t$

The Euler equation that characterises the optimum path of consumption over time (A7_42) might be written as:

$$
\begin{equation*}
\frac{d \ln C(t)}{d t}=\frac{r-\rho}{\theta} \tag{A7_2F}
\end{equation*}
$$

Integrating both sides with respect to time yields:

$$
\begin{equation*}
\ln C(t)=\frac{r-\rho}{\theta} t+J \tag{A7_3F}
\end{equation*}
$$

$J$ is an arbitrary constant of integration. (A7_3 F) gives us:
$C(t)=\exp \left(\frac{r-\rho}{\theta} t+J\right)$

Since at time 0 consumption is $C(0)$, (A7_4F) implies that $\exp (J)=C(0)$. Thus, the solution of (A7_2 F) is simply:

$$
C(t)=C(0) \exp \left(\frac{r-\rho}{\theta} t\right)
$$

Substitution of (A7_5 F) to (A7_1 F) gives us:

$$
\begin{equation*}
\int_{0}^{\infty} e^{-r . t} C(0) e^{\frac{r-\rho}{\theta} t} d t=\frac{K(0)}{H}+\int_{0}^{\infty} e^{-r . t} W d t \tag{A7_6F}
\end{equation*}
$$

$$
\begin{equation*}
C(0) \int_{0}^{\infty} e^{-\frac{\rho-(1-\theta) r}{\theta} \cdot t} d t=\frac{K(0)}{H}+W \int_{0}^{\infty} e^{-r . t} d t \tag{A7_7F}
\end{equation*}
$$

The integral on the left-hand side converges if $\rho-(1-\theta) \mathrm{r}>0$ :

$$
\begin{equation*}
\int_{0}^{\infty} e^{-\frac{\rho-(1-\theta) r}{\theta} \cdot t} d t=\left[-\frac{\theta}{\rho-(1-\theta) r} e^{-\frac{\rho-(1-\theta) r}{\theta} \cdot t}\right]_{0}^{\infty}=\frac{\theta}{\rho-(1-\theta) r} \tag{A7_8F}
\end{equation*}
$$

The value of the integral on the right-hand side is: ${ }^{284}$
$\int_{0}^{\infty} e^{-r . t} d t=\left[-\frac{1}{r} e^{-r . t}\right]_{0}^{\infty}=\frac{1}{r}$

Hence, (A7_7 F) may be written as follows:

$$
\begin{equation*}
C(0)=\frac{W}{r} \frac{\rho-(1-\theta) r}{\theta}+\frac{\rho-(1-\theta) r}{\theta} \frac{K(0)}{H} \tag{A7_10F}
\end{equation*}
$$

The impact of the interest rate on the initial optimum consumption might be expressed as:

$$
\begin{equation*}
\frac{\partial C(0)}{\partial r}=\frac{W}{\theta} \frac{-(1-\theta) r-\rho+(1-\theta) r}{r^{2}}-\frac{(1-\theta)}{\theta} \frac{K(0)}{H}=-\frac{W}{\theta} \frac{\rho}{r^{2}}-\frac{(1-\theta)}{\theta} \frac{K(0)}{H} \tag{A7_11F}
\end{equation*}
$$

This expression is complicated and may generate a non-linear re-switching relationship between rising interest rate and present consumption (see Figure No. 27_A7 for $\theta=2$ ). This might be explained by the impact of $r$ on the present value of the flow of wages $\mathrm{W} / \mathrm{r}$. On the other hand, $\mathrm{C}(0)$ is easy to determine if $\rho=\mathrm{r}$ :
$C(0)=\frac{W}{r} \frac{r-(1-\theta) r}{\theta}+\frac{r-(1-\theta) r}{\theta} \frac{K(0)}{H}$
(A7_10 Fb)
$C(0)=\frac{W}{r} \frac{r-r+\theta \cdot r}{\theta}+\frac{r-r+\theta \cdot r}{\theta} \frac{K(0)}{H}$
$C(0)=W+r \frac{K(0)}{H}$
$C(0)=W+\rho \frac{K(0)}{H}$

[^181]This level of consumption is then chosen every period onwards because the growth rate of optimum consumption is zero in this case (see A7_2 F and A7_5 F). Furthermore, this particular level is chosen regardless of $\theta$, and may be easily indicated in Figure No. 27_A7 as the intersection of all the curves reported. Another interesting aspect is the size of this consumption. It is equal to the sum of the every-period wage and the permanent dividend obtained from the capital assets. Since the entire labour and capital incomes are consumed every period, saving is zero, and assets are therefore also constant over time being permanently maintained at the initial level of $\mathrm{K}(0) / \mathrm{H}$.


Figure No. 27_A7 The optimum present consumption C(0) for various real interest rates and $\theta$.
Note: The set of exogenous parameters is as follows: $W=100, K(0) / H=1000, \rho=3 \%$.

Furthermore, expression (A7_10 F) is simplified also for $\theta=1$ :

$$
\begin{equation*}
C(0)=\frac{\rho}{r} W+\rho \frac{K(0)}{H} \tag{A7_10Ff}
\end{equation*}
$$

Figure No. 27_A7 clearly indicates that in this case the optimum consumption declines with the increasing real interest rate along a rectangular hyperbola.

Notice that the present consumption is depressed to zero if $\mathrm{r}=6 \%, \theta=3 \%$, and $\rho=3 \%$ (see Figure No. 27_A7). However, this combination is ruled out by condition $\rho-(1-\theta) \mathrm{r}>0$, which guarantees convergence of the "consumption integral" in (A7_7 F). Thus, optimum present consumption cannot be zero.

Furthermore, if we set the real wage to zero, or if we neglect the impact of $r$ on the present value of the flow of wages and denote $\mathrm{W} / \mathrm{r}+\mathrm{K}(0) / \mathrm{H}$ as the "Wealth" of this consumer, we get:
$C(0)=\frac{\rho-(1-\theta) r}{\theta}$ Wealth
$\rho / \theta-\mathrm{r}(1-\theta) / \theta>0$ can be interpreted as the marginal propensity to consume out of wealth $\mathrm{MPC}_{\text {wealth }}$ (Barro 2004:94). As can be seen, present consumption is not depressed to zero even for zero time preference (in sense two, i.e. $\rho=0$ ) and for positive real interest rate provided that the elasticity of substitution is low enough ( $\theta>1$ ). This conclusion was derived many times before, but it applies also for the continuous time model. Hence, we proved again that Mises was not right in this respect.

Furthermore, the sensitivity of $\mathrm{C}(0)$ with respect to the interest rate is:

$$
\begin{equation*}
\frac{\partial C(0)}{\partial r}=-\frac{(1-\theta)}{\theta} \text { Wealth } \tag{A7_13F}
\end{equation*}
$$

The response of present consumption (or better just the $\mathrm{MPC}_{\text {Wealth }}$ ) to the interest rate critically depends on $\theta$ (see A7_13 F). If the preference for consumption smoothing is significant ( $\theta>1$ ), an increase in the interest rate will also raise present consumption. In such a case, we can say that the saving curve is downward sloping. On the other hand, low preference for consumption smoothing $(\theta<1)$ results in a decrease in present consumption (or better $\mathrm{MPC}_{\text {wealth }}$ ) after the rise in the interest rate. In this case, the substitution effect dominates, and the saving curve is upward sloping (keeping wealth constant).

## Section G - Economy gradually approaching zero natural rate of interest

In this section, we briefly discuss an economy that gradually reaches zero real natural rate of interest. Consider an economy with Cobb-Douglas production function with $\alpha=1 / 3$, subjective discount rate of $5 \%$, logarithmic utility $(\theta=1)$, zero population growth, depreciation rate of $6 \%$, and positive technological progress of $2 \%$.

According to (A7_48), the steady state natural rate of interest might be zero if:

$$
\begin{equation*}
r^{*}=0 \Leftrightarrow \rho=-\theta . g \tag{A7_1G}
\end{equation*}
$$

Hence, the necessary technological decay to obtain zero steady state real interest is as follows:

$$
\begin{equation*}
r^{*}=0 \Leftrightarrow g=-\frac{\rho}{\theta} \tag{A7_2G}
\end{equation*}
$$

For our set of parameters this implies $g=-5 \%$. Figure No. 28_A7 depicts the evolution of the real rate of interest after the fall in the technological progress from $+2 \%$ to $-5 \%$. Assuming constant money and velocity, Figure No. 28_A7 also demonstrates that the nominal interest rate never falls to zero or even below zero. Notice that according to equation (51) in the main text, our set of parameters (ZPG and logarithmic utility) implies that the steady state nominal interest rate is equal to the subjective discount rate.

Furthermore, Figure No. 29_A7 and 30_A7 illustrate the behaviour of the optimum saving rate and the growth rate in output per worker. As can be seen, the economy is never dynamically inefficient as the growth rate in GDP is always lower than the real interest rate. Furthermore, because the real interest rate is lower than the subjective discount rate, the
optimum consumption will be decreasing over time; on the BGP at the rate of $g=-5 \%$ (see A7_42 or A7_43, and A7_58).

An interesting behaviour can be observed with respect to the saving rate. First, it increases sharply immediately after the shock. The economic reason lies in the fact that people are preparing for lower future income endowment associated with the expected technological decay. This saving behaviour leads to the fact that the economic decline in the transition period is not as fast as the rate of technological decay. This might be seen in Figure No. 30_A7 in which the growth rate in output is for many years considerably greater than the fall in technological progress.


Figure No. 28_A7 The real interest rate gradually approaching zero if $g$ falls from $+2 \%$ to $-5 \%$.
Note: The set of exogenous parameters is as follows: $\delta=6 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%, \theta=1$ Nominal interest rate is calculated for constant money and velocity.


Figure No. 29_A7 The optimum saving rate if $g$ falls from $+2 \%$ to $-5 \%$.
Note: The set of exogenous parameters is as follows: $\delta=6 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%, \theta=1$.


Figure No. 30_A7 The growth rate of output per worker if $g$ falls from $+2 \%$ to $-5 \%$. Note: The set of exogenous parameters is as follows: $\delta=6 \%, \alpha=1 / 3, n=0 \%, \rho=5 \%, \theta=1$.

However, optimum saving rate gradually falls to much lower levels than we observed before the shock due to relatively low $\theta$ (see A7_82D). This could be explained by an upward sloping saving curve and shrinking investment curve that results from technological decline.
Moreover, according to (A7_79D), we can even find a set of parameters that lead to zero optimum steady state saving rate:

$$
\begin{equation*}
s^{*}=(n+g+\delta) \frac{\alpha}{\rho+\delta+\theta \cdot g}=0 \Leftrightarrow n+g+\delta=0 \tag{A7_3G}
\end{equation*}
$$

For our set of parameters, this could be achieved with ZPG, $\mathrm{g}=-5 \%$, and the depreciation rate of $5 \%$. If we look at Figure No. 26_A7, this implies that the break-even-investment line $(\mathrm{n}+\mathrm{g}+\delta) \mathrm{k}$ is horizontal, and it coincides with the k -axis. Thus, the eventual picture of the economy closely resembles Figure No. 0b_A7. It should be stressed that even though the levels of capital and consumption gradually fall to zero in the infinite horizon, as both variables are decreasing at the rate of $-5 \%$, according to (A7_72D) and (A7_73D), $\mathrm{k}^{*}$ and $\mathrm{c}^{*}$ are still positive.
Furthermore, even a negative steady state saving rate is possible if the depreciation rate is low enough (e.g. $\delta=4 \%$ ). This would mean that it is optimal to consume capital directly. The economic reason behind this peculiar result is as follows. Since the BGP growth rate in capital is $g=-5 \%$, deprecation of capital lower than $5 \%$ requires its direct consumption. In our onegood model, in which capital and consumption goods are represented by the same commodity, this is certainly possible. This also means that gross investment is negative. Yet, in the real world, direct consumption of capital is only a minor phenomenon, and all important (planned or unplanned) decreases in capital take the form of the excess of depreciation over positive gross investment. ${ }^{285}$ Thus, a straightforward restriction that might be immediately imposed on our model is as follows:

[^182]$s^{*}=(n+g+\delta) \frac{\alpha}{\rho+\delta+\theta . g}>0 \Leftrightarrow n+g+\delta>0$
As we will see below, the denominator of (A7_4 G) must be positive by assumption. Assuming ZPG, condition (A7_4 G) simply requires that the technological decay must not be more rapid than the rate of depreciation of capital. This condition in turn implies that the gross investment is always positive and capital is never directly consumed.

Even if we allow for direct consumption of capital, the rate of technological decay in this model has a limit due to the non-negativity constraint imposed on $\mathrm{k}(\mathrm{t})$ (see A7_10g). From the steady state formula of $\mathrm{k}^{*}$ (see A7_72D), we can see that:

$$
\begin{equation*}
k^{*}=\left(\frac{\alpha}{\rho+\delta+\theta . g}\right)^{\frac{1}{1-\alpha}} \geq 0 \Leftrightarrow \rho+\delta+\theta . g \geq 0 \tag{A7_5G}
\end{equation*}
$$

Moreover, since $0<\alpha<1$ due to conditions (A7_10b)-(A7_10f), the denominator must be positive for (A7_72D) to be permissible. Because the steady state real interest rate is $r^{*}=\rho+\theta \mathrm{g}$, condition (A7_5 G) requires:

$$
\begin{equation*}
r^{*}+\delta \geq 0 \tag{A7_6G}
\end{equation*}
$$

(A7_6 G) simply states that the marginal product of capital at the steady state must not be negative (recall that MPK* $=r^{*}+\delta$ ). However, assumptions imposed on the production function (see A7_3 and A7_5 for the extensive form and A7_10c and A7_10f for the intensive form) require that the marginal product of capital is always positive. Hence, (A7_5 G) and (A7_6 G) must hold with strict inequality.

We may conclude that the negative saving rate (and negative real natural rate of interest as well) is technically possible in this model owing to negative technological progress. Yet, this negativity has its limits due to the non-negativity constraint imposed on $k(t)$. Condition (A7_5 G) and the positivity of MPK imply that the maximum rate of technological decay is given by:

$$
\begin{equation*}
g>\frac{-\rho-\delta}{\theta} \tag{A7_7G}
\end{equation*}
$$

For our set of parameters, $\rho=5 \%, \theta=1$, and $\delta=6 \%$, the limiting $g$ is $-11 \% .{ }^{286}$ Furthermore, if we disallow direct consumption of capital, the maximum technological decay is (see A7_4 G):

$$
\begin{equation*}
s^{*}>0 \Leftrightarrow g>-(n+\delta) \tag{A7_8G}
\end{equation*}
$$

Condition (A7_8 G) then requires that the limiting $g$ is $-6 \%$.
Nonetheless, in our benchmark example $\delta=6 \%$ and $g=-5 \%$. Thus, gross investment and the optimum saving rate on the BGP must be positive to reach a year-to-year decline in capital of only $\mathrm{g}=-5 \%$.

[^183]In the main text we focused major investigations on the zero natural real interest rate. Assuming positive subjective discount rate, this might be achieved only with a rapid technological decay. Yet, in the previous paragraphs we found out that other interesting phenomena might emerge as well if the economy is gradually contracting. We also determined necessary limits that must be imposed on this technological decline.

Section H-Behaviour of the natural rate of interest if the technological progress is stochastic
In this section, we assume that the technological level $A$ follows a simple stochastic $\operatorname{AR}(1)$ or AR(2) process:

$$
\begin{equation*}
\ln A_{t}=\beta_{1} \ln A_{t-1}+\beta_{2} \ln A_{t-2}+\varepsilon_{t} \tag{A7_1H}
\end{equation*}
$$

$\beta_{\mathrm{i}}$ 's are autoregressive coefficients that measure the degree of memory of this process. If $\beta_{2}=$ 0 , the process is $\operatorname{AR}(1)$ approaching random walk for $\beta_{1}=1$. $\varepsilon_{\mathrm{t}}$ is the random disturbance having the properties of the white noise.
Figure No. 31_A7 shows the behaviour of the real natural rate of interest if the level of technologies $A$ follows $\operatorname{AR}(1)$ process. Since $\beta_{1}<1$, this process is stationary. The nominal interest reflects a theoretical value when the growth rate in output is fully embodied (with the opposite sign) in the inflation rate and this in turn in the nominal interest rate. This is the reason for its higher volatility compared with the real rate.


Figure No. 31_A7 Real and nominal interest rate for stochastic $A$.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, $\theta=1$. Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.

Such fluctuations in the nominal interest rate would be required to keep money neutral with respect to the real economy. In such a case, ex post and ex ante real interest rate will be equal. However, since shocks are unpredictable, nominal interest rate cannot reflect them at the moment they occur. On the other hand, we depict the behaviour of the economy on a year-toyear basis, but the shock will cause deviation between ex post and ex ante just at that
instantaneous moment. In the subsequent periods within the year, the nominal interest rate can freely adjust. Nevertheless, the resulting fluctuations of the nominal interest rate seem to be rather high. One may wonder whether the real world conditions could deliver this necessary volatility. First, contracts between debtors and creditors are agreed in nominal terms that might be fixed for a considerable period of time. Second, commercial banks usually do not change their interest rates so often and to such a large extent. And finally, modern central banks follow some rule using nominal interest rate as the main policy tool.
The first reason stressing the rigidity of the nominal interest rate implies that constant money cannot deliver consistency between ex post and ex ante real interest rate and may cause disturbances between actual real interest rate and the real natural rate. Thus, it seems at first glance that better policy would be to aim at price stability by manipulating either the money supply or the nominal interest rate, which may deliver greater consistency between the theoretically optimal real interest rate and the actual real interest rate. This problem will be discussed in greater detail in Chapter 4, and it was analysed also in Chapter 2.

However, first we must realise the frequency and magnitude of productivity shocks. Suppose that they occur once a year at the magnitude shown in Figure 39_A7 below. Their maximum impact on the growth rate in output is at one single moment of the shock. Yet, Figure No. 7_A7 clearly shows that the subsequent impact on the real growth rate is much lower. Hence, the influence on the increase or decrease in prices and consequently on the nominal interest rate, assuming constant money and velocity, is far smaller than suggested in Figure 31_A7. Hence, there are self-stabilizing forces in the economy that will significantly dampen the effects of the technological shock in subsequent periods. As a result, the resulting optimum volatility of the nominal interest rate is much lower. Yet, this implication assumes again a relative flexibility in prices and the nominal interest rate so as to reflect the fact that within the same year, the period-to-period conditions that follow the shock are very similar.

On the other hand, we can also ask whether the monetary policy that is aimed at the price stability could deliver optimal doses of money that perfectly adapt to fluctuations in output. Since only with such a policy, fluctuations in the nominal interest rate would accurately accord with the volatility of the real natural rate. Furthermore, monetary policy using the nominal interest rate as the major tool should adapt not only to the evolution of the real natural rate of interest but also to all changes in the inflation rate that are provoked by changes in the growth rate in potential output. One may ask whether any monetary policy conducted by erring human beings is capable to perform such a large degree of sophistication.

Hence, constant money (and constant velocity) may represent the ideal policy that would deliver the greatest possible consistence between actual real interest rate and the natural real interest rate if the nominal rate of interest is allowed to reflect changes in fundamental variables in the economy. ${ }^{287}$

Let us focus again on the behaviour of real variables. Figures No. 32_A7 - 34_A7 demonstrate a cyclical behaviour of the real interest rate, saving rate, and investment in this simple RBC model. ${ }^{288}$ As can be seen, the real interest rate is pro-cyclical in this model. At

[^184]the same time, it is always higher than the growth rate in output, which implies dynamic efficiency. Saving rate is pro-cyclical too due to relatively low $\theta$. Finally, Figure No. 32_A7 clearly shows that investment is much more volatile than GDP in this RBC model.


Figure No. 32_A7 Real interest rate and the growth rate in output for stochastic $A$.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, $\theta=1$. Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.


Figure No. 33_A7 Saving rate and the growth rate in output for stochastic $A$.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, $\theta=1$. Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.


Figure No. 34_A7 Investment growth and the growth rate in output for stochastic $A$.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, $\theta=1$. Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.

It can be added that even though the RBC fluctuations are rather mild, the volatility of desired real investment is rather high. Thus, the real interest rate must be flexible enough to reflect these fluctuations and to convey information about these fluctuations.


Figure No. 35_A7 Volatility of the natural rate of interest for various $\theta$ if A is stochastic.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$,
Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.


Figure No. 35_A7B Correlation between the natural rate of interest and the saving rate for various $\theta$ if $A$ is stochastic.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.


Figure No. 36_A7 Volatility of the saving rate for various $\theta$ if $A$ is stochastic.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, Technological level follows AR(1) process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.


Figure No. 37_A7 Volatility of investment growth for various $\theta$ if $A$ is stochastic.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \rho=5 \%$, Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.

Figures No. 35_A7 - 37_A7 reveal that the key variables fluctuate less with lower intertemporal substitution of consumption (higher $\theta$ ). The key economic reason is that with higher preference for consumption smoothing, the response of consumption is not so significant. In other words, saving does not react much to changes in the interest rate. Figure No. 35_A7B displays the correlation coefficients between the saving rate and the real interest rate for various $\theta$.

Obviously, the only exception is the real natural rate of interest itself. Since higher $\theta$ is associated with less elastic saving curve, real interest fluctuate more in this case. It can be shown that fluctuations in GDP are very similar for various $\theta$. This implies that lower intertemporal substitution is associated with a-cyclical behaviour of the saving rate.

We can even allow the subjective discount rate to follow a stochastic process. Consider the following MA(1) process:

$$
\begin{equation*}
\rho_{t}=0.05+\mu_{t}+\gamma \cdot \mu_{t-1} \tag{A7_2H}
\end{equation*}
$$

Parameter $\gamma$ measures the influence of the previous shock on the present shock. Figure No. 38_A7 demonstrates that the volatility of the natural rate of interest is lower than the volatility of the subjective discount rate.

If both the technological level and the subjective discount rate are stochastic, the picture of such an economy might be represented by Figure No. 39_A7. This figure could reflect a typical behaviour of the natural rate of interest that is affected both by changes in the
subjective discount rate and by changes in productivity. ${ }^{289}$ We obviously assume that shocks to technology and to the time preference are uncorrelated.


Figure No. 38_A7 Volatility of the natural rate of interest if $\rho$ is stochastic.
Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \theta=1$, subjective discount rate follows an $M A(1)$ process, $\gamma=0.1 ; \mu \sim N\left(0 ; 0.003^{2}\right)$.


Figure No. 39_A7 Volatility of the natural rate of interest, if both $\rho$ and $A$ are stochastic. Note: The set of exogenous parameters is as follows: $\delta=4 \%, \alpha=1 / 3, n=g=0 \%, \theta=1$, subjective discount rate follows an $M A(1)$ process, $\gamma=0.1 ; \mu \sim N\left(0 ; 0.003^{2}\right)$. Technological level follows $A R(1)$ process, $\beta_{1}=0.9 ; \varepsilon \sim N\left(0 ; 0.015^{2}\right)$.

[^185]
## REFERENCES

Acemoglu, Daron. 2011. Introduction to Modern Economic Growth: Parts 1-4 Massachusetts Institute of Technology

Barro, Robert J. 2004. Economic Growth., 2nd ed. Cambridge: MIT Press.
Becker, Gary S. and Mulligan, Casey B. 1997. The Endogenous Determination of Time Preference. The Quarterly Journal of Economics 112(3): 729-758.

Becker, Gary S. 2007 [1971]. Economic Theory. New Brunswick, NJ: Transaction Publishers.

Blanchard, Olivier Jean and Fischer, Stanley. 1989. Lectures on Macroeconomics. Cambridge, GB: MIT Press.

Böhm-Bawerk, Eugen von. 1890 [1884]. Capital and Interest. New York: McMillan and Co.

Böhm-Bawerk, Eugen von. 1891 [1888]. Positive Theory of Capital. New York: G. E. Stechert \& Co.

Block, Walter. 1978. The Negative Interest Rate: Toward a Taxonomic Critique. Journal of Libertarian Studies 2(2): 121-124.

Block, Walter. 1990. The DMVP-MVP Controversy: A Note. The Review of Austrian Economics 4(1): 199-207.

Broome, John. 1994. Discounting the Future. Philosophy \& Public Affairs 23(2): 128-156.
Brown, Harry G. 1913. The Marginal Productivity Versus the Impatience Theory of Interest. The Quarterly Journal of Economics 27(4): 630-650.

Carroll, Christopher D. 1997. Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. The Quarterly Journal of Economics 112(1): 1-55.

Cass, David. 1965. Optimum Growth in an Aggregative Model of Capital Accumulation. Review of Economic Studies 37(3): 233-240.

Cwik, Paul. 2004. A Defense of the Traditional Austrian Theory of Interest. mimeo.
Eggertsson, G.B. and Woodford, M. 2003. The Zero Bound on Interest Rates and Optimal Monetary Policy. Brookings Papers on Economic Activity 2003(1): 139-211

Epstein, Larry G. and Hynes, J. Allan. 1983. The Rate of Time Preference and Dynamic Economic Analysis. Journal of Political Economy 91(4): 611-635.

Fetter, Frank A. 1902. The "Roundabout Process" In the Interest Theory. The Quarterly Journal of Economics 17(1): 163-180.

Fetter, Frank A. 1928 [1915]. Economics. New York: The Century Co.

Fishburn, Peter C. and Rubinstein Ariel. 1982. Time Preference. International Economic Review 23(3): 677-694.

Fisher, Irving. 1907. The Rate of Interest. Its Nature, Determination and Relation to Economic Phenomena. New York: The Macmillan Company.

Fisher, Irving. 1913. The Impatience Theory of Interest. The American Economic Review 3(3): 610-618.

Fisher, Irving. 1930. Theory of Interest. New York: The Macmillan Company.
Frederick, Shane. 2003. Time preference and Personal Identity. In Time and Decision: Psychological Perspectives in Intertemporal Choice, ed. Loewenstein G., Read D. and Baumeister R., 89-114. New York. Russel Sage.

Frederick Shane, Loewenstein George and O'Donoghue, Ted. 2002. Time Discounting and Time Preference: A Critical Review. Journal of Economic Literature 40(2): 351401.

Friedman, Milton. 1969. The Optimum Quantity Of Money and Other Essays. Chicago, IL: Aldine.

Garrison, Roger. 1979. In Defense of the Misesian Theory of Interest. Journal of Libertarian Studies 3(2): 141-150.

Garrison, Roger. 2011 [1988]. Professor Rothbard and the Theory of Interest. In The Pure Time-Preference Theory of Interest, ed. Herbener, Jeffrey M., 159-172. Ludwig von Mises Institute.

Ghez, Gilbert and Becker, Gary S. 1975. A Theory of the Allocation of Time and Goods Over the Life Cycle. NBER Chapters, 1-45. National Bureau of Economic Research.

Hayek, Friedrich A. von. 1932. Money and Capital: A Reply. The Economic Journal 42(166): 237-249.

Hayek, Friedrich A. von. 1936. Utility Analysis and Interest. The Economic Journal 46(181): 44-60.

Hayek, Friedrich A. von. 1941. The Pure Theory of Capital. Chicago: The University of Chicago Press.

Hayek, Friedrich A. von. 1945. Time-Preference and Productivity: A Reconsideration. Economica 12 (45): 22-25.

Hayek, Friedrich A. von. 1984 [1927]. On the Problem of the Theory of Interest. In Money, Capital and Fluctuations: Early Essays, ed. McCloughry, R. K., 55-70. University of Chicago Press.

Hayek, Friedrich A. von. 1984b [1932]. Capital Consumption. In Money, Capital and Fluctuations: Early Essays, ed. McCloughry, R. K., 136-158. University of Chicago Press.

Herbener, Jeffrey M. 2011. Introduction. In The Pure Time-Preference Theory of Interest, ed. Herbener, Jeffrey M., 11-58. Ludwig von Mises Institute

Herbener, Jeffrey. 2013. Comment on "A Note on Two Erroneous Ways of Defending the PTPT of Interest". The Quarterly Journal of Austrian Economics 16(3): 317-330.

Hülsmann, Jörg Guido. 2002. A Theory of Interest. The Quarterly Journal of Austrian Economics 5(4): Fall, pp. 77-110.

Jevons, William S. 1957 [1871]. The Theory of Political Economy. New York: Sentry Press,
Jung Taehun, Teranishi Yuki and Watanabe Teutonu. 2005. Optimal Monetary Policy at the Zero-Interest-Rate Bound. Journal of Money, Credit and Banking 37(5): 813-35.

Kamien Morton I., and Schwartz, Nancy L. 1991. Dynamic Optimization: The Calculus Of Variations And Optimal Control in Economics and Management - 2nd ed., NorthHolland, Amsterdam, NY

Knight, Frank H. 1934. Capital, Time, and the Interest Rate. Economica 1(3): 257-286.
Knight, Frank H. 1935a. Professor Hayek and the Theory of Investment. The Economic Journal 45(177): 77-94.

Knight, Frank H. 1935b. The Theory of Investment Once More: Mr. Boulding and the Austrians. The Quarterly Journal of Economics 50(1): 36-67.

Knight, Frank H. 1936a. The Quantity of Capital and the Rate of Interest: I. The Journal of Political Economy 44(4): 433-463.

Knight, Frank H. 1936b. The Quantity of Capital and the Rate of Interest: II. The Journal of Political Economy 44(5): 612-642.

Knight, Frank H. 1941. Professor Mises and the Theory of Capital. Economica 8(32): 409427.

Kirzner, Israel. 2011 [1993]. The Pure Time-Preference Theory of Interest: An Attempt at Clarification. In The Pure Time-Preference Theory of Interest, ed. Herbener, Jeffrey M., 99-126. Ludwig von Mises Institute

Koopmans, Tjalling C. 1960. Stationary Ordinal Utility and Impatience. Econometrica, 28(2): 287-309.

Koopmans, Tjalling C. 1963. On the Concept of Optimal Economic Growth. Cowles Foundation Discussion Papers 163, Cowles Foundation for Research in Economics, Yale University.

Koopmans, Tjalling C., Diamond Peter A. and Williamson Richard E. 1964. Stationary Utility and Time Perspective. Econometrica 32(1/2): 82-100.

Laibson, David. 1997. Golden Eggs and Hyperbolic Discounting. The Quarterly Journal of Economics 112(2): 443-77.

Lancaster, Kelvin. 1963. An Axiomatic Theory of Consumer Time Preference. International Economic Review 4(2): 221-231.

Lewin, Peter. 1997. Murray Rothbard on Interest and Capital: An Exercise in Theoretical Purity. Journal of the History of Economic Thought 19(1): 141-159.

Loewenstein, George. 1992. The Fall and Rise of Psychological Explanations in the Economics of Intertemporal Choice. In Choice over Time, ed. George Loewenstein and Jon Elster, 3-34. New York: Russell Sage.

Loewenstein, George and Prelec, Drazen. 1991. Negative Time Preference, The American Economic Review 81(2): 347-352.

McCallum, Bennett T. 2000. Theoretical Analysis Regarding a Zero Lower Bound on Nominal Interest Rates. Journal of Money, Credit and Banking 32(4): 870-904.

Menger, Carl. 2007 [1871]. Principles of Economics. Auburn, Alabama: Ludwig von Mises Institute.

Mises, Ludwig von. 1996 [1949]. Human Action: A Treatise on Economics, 4th ed. San Francisco: Fox \& Wilkes.

Mises, Ludwig von. 1974 [1940]. A Critique of Böhm-Bawerk's Reasoning in Support of His Time Preference Theory. In Nationalökonomie. Geneva, Switzerland: Editions Union, 439-44. Translated from the German by Bettina Bien Greaves and edited by Percy L. Greaves, Jr.

Murphy, Robert P. 2003. Unanticipated Intertemporal Change in Theories of Interest. Ph.D. Dissertation, New York University.

Olson, Mancur and Bailey, Martin J. 1981. Positive Time Preference. Journal of Political Economy 89(1): 1-25.

Orphanides, Athanasios. 2004. Monetary Policy in Deflation: The Liquidity Trap in History and Practice. The North American Journal of Economics and Finance 15(1): 101-124.

Parfit, Derek. 1971. Personal Identity. The Philosophical Review 80(1): 3-27.
Pellengahr, Ingo. 1996. The Austrian Subjectivist Theory Of Interest: An Investigation Into The History Of Thought. Frankfurt am Main: Peter Lang.

Ramsey, Frank P. 1928. A Mathematical Theory of Saving. Economic Journal 38(152): 543-559.

Rothbard, Murray N. 1956. Toward a Reconstruction of Utility and Welfare Economics. In On Freedom and Free Enterprise: Essays in Honor of Ludwig von Mises, ed. Mary Sennholz. 224-262, Princeton, N.J: D. Van Nostrand.

Rothbard, Murray N. 2004 [1962]. Man, Economy, and State. Ludwig von Mises Institute.
Romer, David. 2006. Advanced Macroeconomics, 3rd edition, McGraw - Hill, New York.
Ryder Harl E., Jr. and Heal, Geoffrey M. 1973. Optimal Growth with Intertemporally Dependent Preference. The Review of Economic Studies 40(1): 1-31.

Samuelson. Paul A. 1937. A Note on Measurement of Utility. The Review of Economic Studies 4(2): 155-161.

Schumpeter, Joseph A. 1961 [1912].The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle. Oxford, NY [US] : Oxford University Press

Seager, Henry R. 1912. The Impatience Theory of Interest. The American Economic Review 2(4): 834-851.

Solow Robert M. 1956. A Contribution to the Theory of Economic Growth. The Quarterly Journal of Economics, Vol. 70 (1): 65-94.

Sraffa, Piero. 1932. Dr. Hayek on Money and Capital. The Economic Journal 42(165): 4253.

Stigler, George J. 1987 [1962]. The Theory of Price, $4^{\text {th }}$ Ed. MacMillan Publishing Company, New York.

Strigl, Richard von. 2000 [1934]. Capital and Production, Ludwig von Mises Institute

Strotz. Robert H. 1956. Myopia and Inconsistency in Dynamic Utility Maximization. The Review of Economic Studies 23(3): 165-180.

Topan, Mihai Vladimir and Păun, Cristian. 2013. A Note on Two Erroneous Ways of Defending the Pure Time Preference Theory of Interest. The Quarterly Journal of Austrian Economics 16(3): 299-316.

Trostel, Philip A. and Taylor, Grant A. 2001. A Theory of Time Preference. Economic Inquiry 39(3): 379-95.

Wicksell, Knut. 1936 [1898]. Interest and Prices. Augustus M Kelley Publishers.
Wolman, Alexander L. 2005. Real Implications of the Zero Bound on Nominal Interest Rates. Journal of Money, Credit and Banking 37(2): 273-96.

Woodford, Michael. 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press.

## Chapter 4

## Business Cycle in a Growing Economy

## 1. INTRODUCTION

In the last chapter of this dissertation, we will introduce an analysis of a growing economy in which the real natural rate of output is initially on a smooth path or on the path that may be easily determined, but that is disturbed by shocks that are coming from the monetary side of the economy. This chapter will therefore build on the findings presented in Chapter 2, where we explored the Austrian theory of economic fluctuations and the dynamics of the interest rate during the business cycle, and in Chapter 3, in which the theory of the natural rate of interest was presented not only in a stationary but also in a growing economy. In other words, Chapter 4 will examine the interaction between the real and the monetary sector of the economy from the Austrian point of view, and it will demonstrate that the impact of the monetary sector on the real part of the economy is never neutral even in the very long run, in which, as is generally believed, the economy is driven by the forces of economic growth and the influence of money is disregarded.

In the first section, we will present an economy on the balanced growth path and a structure of relative prices in various markets that is consistent with this path. We will show that the only structure of relative prices that will preserve the neutral Walrasian equilibrium is achieved by a decline in the general price level. If an attempt is made to stabilize the price level by an increase in the total money supply, the general-equilibrium structure of relative prices must be upset. It will be demonstrated that this idea might be applied not only to the system of relative prices within the given period of time but also to the intertemporal analysis in which the interest rate plays a major role. The first section will also compare the Austrian analysis of a growing economy with the traditional approach represented by the New Keynesian economics. Typically Austrian visions will be translated into the Keynesian language, and the key differences between these two schools of economic thinking will be stressed.

The next section will examine a design that might preserve dynamic equilibrium of a growing economy and protect the smooth path from monetary shocks that lead to the business cycle. This proposal, which first appeared in greater detail in Hayek (1935), will be compared with the modern theory of the nominal income targeting, as it will turn out that these two are almost indistinguishable. However, since the Hayek proposal may lead in a growing economy to secular deflation, the problem of very low nominal interest rates will be examined in this section along with other questions that arise within this specific monetary framework. The most noteworthy will be a similarity with Friedman's theory of the optimum quantity of money and the problem of zero lower bound on nominal interest.
The fourth part will examine the impact of technological shocks. It will combine findings from Chapter 2, in which the endogenous response of the money supply was discussed, and the analysis of Chapter 3, where the dynamics of aggregate output and the natural rate of interest were studied in detail. We will explore whether the path of the economy is closer to its natural, but changing, level within Hayek's framework, or whether the fluctuations of the economy are eliminated in the framework designed by the New Keynesian theory. Simple Keynesian tools will be used to pin down key differences in these two approaches.
The fifth section will integrate the Hayekian approach with a simple monetary policy rule of the Taylor type. It will show that the traditional Taylor rules are too expansionary from the Austrian point of view. However, it will turn out that the Hayek-Taylor rule will operate rather close to the zero lower bound of the nominal interest rate. As a result, the discussion of
this limit will be presented, and the optimality of the rule will be assessed. This section will also briefly discuss traditional Austrian recommendations for the optimal monetary framework. The last part concludes.

## 2. INTRA-TEMPORAL AND INTER-TEMPORAL EQUILIBRIUM IN A GROWING ECONOMY

The majority of real world economies are growing over the very long run. In the economic theory, a wide range of models was developed to describe this complicated, but for the wellbeing of the human race, fundamental, process. Let us consider an economy in which the natural output is growing at some positive rate. Suppose that this growth is produced by technological progress that is, however, not uniform across the economy. In other words, some industries benefit more than others. Such an assumption is quite reasonable since it may be observed that in the real world the productivity in the ICT sector is rising at a more rapid pace than that of hairdressers and other types of personal services.

Suppose that the general Walrasian equilibrium was established in a system of n markets. Three representative markets of this economy are displayed in Figure No. 1. Rapid technological progress is on the way in market A , a modest one in market B , and market C is characterized by stable technologies. Significant advances in technologies in market A are reflected in a positive (i.e. rightward) shift of the supply curve, which lowers the price in this market. The decline in price in market B is not so remarkable, and finally, there is no change in the supply curve and hence the market price in market C.




Figure No. 1, Non-uniform economic growth and the structure of relative prices

Obviously, the change in the individual prices depends on many more factors than presented here. It is an open question of how much the given rate of technological progress is transmitted to the reduction in costs in that particular industry. Secondly, a drop in the price in the given market is surely affected by the elasticity of demand. Furthermore, a wide range of substitution and complementary effects play an important role among various markets, so a change in the price in one market may shift the demand on the other market in one direction or another. However, let us suppose that in the given period of time, the Walrasian equilibrium is consistent with the system of prices presented in Figure No. 1.

As can be seen, the technological progress brings about a change in the structure of relative prices. $\mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{C}}, \mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{B}}$, and $\mathrm{P}_{\mathrm{B}} / \mathrm{P}_{\mathrm{C}}$ are lower than before. Only this new structure is consistent
with the intra-temporal equilibrium in the economy. It should be admitted that it takes some time for the markets to establish such equilibrium, and the continuing technological progress may push the equilibrium itself to a different position. As a result, markets tend to move to equilibrium rather than ever achieving one. Thus, this dynamics might be described as shooting at a moving target. However, if the price system operates reasonably well, sooner or later markets should achieve a position that is very close to a point that was envisioned by the general equilibrium theory as a state of rest. Such a state is characterised by a consistency of plans of various agents. Unless the system of markets is hit by an external shock, it should remain in this state.

The technological progress then brings about an adjustment of relative prices, of the structure of the aggregate output, and the structure of incomes of various factors of production. However, Figure No. 1 clearly suggests that the attainment of the new general equilibrium is consistent only with a decline in the general price level. Even though it is always arbitrary to measure the overall change in prices since each price index is as legitimate (or illegitimate) as the next one, the advance in technologies should be accompanied by a phenomenon that in modern economics we call a price deflation.
The simple graphical apparatus of Figure No. 1 reconstructs key ideas presented in Hayek (1935). Another important question is the evolution of wages. As the growth in productivity is most rapid in market A , the labour force might be attracted to this sector. However, a decline in the price of good A may operate in the opposite direction because it lowers the value of marginal product of labour. Nonetheless, any discrepancy in nominal wages among various labour markets tends to be eliminated by a movement of the labour force. On the other hand, wage differences can never completely disappear if workers in one industry are not perfect substitutes to workers in another industry; in other words, if there is a significant heterogeneity among workers (and jobs).

Even though changes in relative prices might be significant due to non-uniform technological progress, mobility in the labour force tends to equalize nominal wages among various industries. In the extreme case, nominal wages may remain the same. However, since the technological progress should reduce prices in the respective industries and markets, the (consumption) real wages, i.e. the number of real consumption goods that can be purchased with the given nominal wage, should grow. This growth in real wages, which is provoked by positive technological progress, is reached by a decline in the general price level even if the nominal wages are constant.

As a result, the Hayekian picture of a growing economy is as follows: Technological progress, as a major driving force of the economic growth, is not uniform across the economy. Not only the level but also the structure of aggregate output must be changing. This process brings about a change in relative prices. A well-functioning price system should lead the economy to a new general equilibrium. However, such equilibrium is supported by a specific set of relative prices and by a general decline in the overall price level, as is presented in Figure No. 1 above. Figure No. 2 then displays possible equilibrium at the macroeconomic level in a simple AD/AS model. The real wage in the economy should grow due to higher productivity; however, nominal wages may remain constant, and the growth in real wages might be guaranteed by a decrease in the price level.

Figure No. 2 presents a technology-driven economic growth that gradually raises the potential output $\mathrm{Y}^{*}$. Assuming constant money supply and velocity of circulation, the aggregate demand schedule AD may remain constant as well as the system of market demand curves in Figure No. 1. More on this assumption will be said in section 3. Nonetheless, according to the Hayekian theory, for the given time period, the general equilibrium is achieved only at point $E_{2}$, at which the system of relative prices from Figure No. 1 is preserved.


Figure No. 2, Economic growth and a decrease in the general price level

Since Wicksell (1936), it has been generally believed that the state closest to the ideal of general equilibrium is represented by the stability of the price level. Either a decreasing or an increasing level of prices signals that the general equilibrium was not achieved. Similar views can be found in the majority of modern New Keynesian literature in which the key question is whether the optimum of the economy is represented by the stable price level, zero inflation that allows for a drift in the price level, or a slightly positive rate of inflation. ${ }^{290}$

Hayek (1935), however, uncovered that linking the stability of the price level with the general equilibrium and the natural state of the economy might be fallacious. If the price level in a growing economy is to be stabilized, a general increase in the nominal demand is required. Such a general increase might be delivered by an increase in the total money supply. Only a higher money supply can raise the nominal demand on the majority of markets and consequently increase the price level back to its previous level. Although the economic growth may affect not only the supply but also the demand on many markets, the nominal demand cannot be generally raised since, unless people dissolve part of their money balances, there is not enough money to do this job. In other words, with constant money and velocity, total expenditure in the economy must remain constant, and the increase in the total quantity of goods must be reflected in a lower price level. A constant price level then requires either an increase in total money supply or a decrease in the demand for money (increase in velocity of circulation).

As far as the impact of the monetary expansion on the real economy is concerned, Hayek (1935) followed the idea of Ludwig von Mises. According to Mises (1976), the influx of money into the economy never enters all markets at the same time and to the same extent. As a result, market prices are never affected simultaneously. The markets and the set of prices influenced first are obviously those that the money inflow enters first. In these particular markets the nominal demand rises, whereas other markets in the first round may not be affected.

[^186]Figure No. 3 presents this Misesian idea. Market A absorbs the majority of doses of new money, as can be seen by the shift of demand $\mathrm{D}_{\mathrm{A}}$. The minority of the newly created money enters market C , whereas market B is not, in the first round, affected by the increase in the money supply. As we can see, the expansion in the money supply may stabilize the price level if the given price index reflects the specific weights and increases and decreases in prices of the goods in question.




Figure No. 3, Monetary expansion and the structure of relative prices

However, even though the price level is stabilized, the general equilibrium and the system of relative prices are not preserved. The relative prices $\mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{C}}, \mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{B}}$, and $\mathrm{P}_{\mathrm{B}} / \mathrm{P}_{\mathrm{C}}$ differ from the state in the "natural economy" - the first two are higher, the third one is lower. As regards the individual prices, none of them are moved back. $\mathrm{P}_{\mathrm{A}}$ and $\mathrm{P}_{\mathrm{B}}$ are still lower, whereas $\mathrm{P}_{\mathrm{C}}$ is above the initial level. Moreover, along with the change in the relative prices, the structure of the aggregate output is modified as well. More factors of production are attracted to market A and C. It might be suggested that the aggregate output rises as well. Thus, it is an open question whether the total output of the economy will grow due to the increase in the total money supply. The answer will be postponed for a while. Nonetheless, if the economy operates at its potential level $\mathrm{Y}_{2}{ }^{*}$ and if there are no frictions, the output in markets A and C may be increased only at the expense of other markets. It is generally believed that the economy (in the long run) may not produce more only due to higher money supply, so the system of curves in Figure No. 3 is not complete. Owing to the money expansion, the supply curves are affected as well - they shift backwards - due to higher nominal wages. As a result, the final picture can be represented by Figure No. 4. The eventual structure of the relative prices and output is an open question that cannot be answered on an a priori basis.

The implications of the Hayekian theory are straightforward. The technological progress and the increase in the potential output are consistent only with one specific system of "natural relative prices" that do not disturb the "natural equilibrium". This general equilibrium requires a decline in the price level. Any attempt to stabilize the price level via the expansion of the money supply must disturb this "natural equilibrium" owing to the impact on the system of relative prices and consequently on the structure of output. According to Hayek, a stable price level does not mean that money is neutral with respect to the economic system. In the expanding economy the neutrality of money with respect to the real economy is preserved only if prices are allowed to decline.


Figure No. 4, Monetary expansion and the structure of relative prices in the long run.

Another important question is whether the economy tends to return to its previous structure when the money supply ceases to expand. To answer this question from the Hayekian point of view, let us assume that for an extended time period, the money expansion continues at some definite rate. Suppose that in the first round, the injection of money enters market A, and then it spills over to other markets (e.g. market B). Figure No. 5 reconstructs the story that can be found in Hayek (1969). At time $t_{0}$ money expansion begins, and the new money first flows into market A. As a result, the relative price of good A in terms of good B gradually rises. The peak is reached at time $t_{1}$ at which the flow of money to market A continues at the same pace, however, the new money is about to be spent on market $B$ as well. As we can see, from time $t_{1}$ there is a downward correction in the relative price $\mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{B}}$. Nevertheless, as long as the flow of money first enters market A , the relative prices can never return to the position before the monetary expansion (Hayek 1969:278). Thus, the system of relative prices is disturbed even though individual prices may grow at the same rate along with the given rate of the monetary expansion. The interval from $t_{2}$ to $t_{3}$ may represent this situation.
Hayek envisioned a more plastic picture:
The effect we are discussing is rather similar to that which appears when we pour a viscous liquid, such as honey, into a vessel. There will, of course, be a tendency for it to spread to an even surface. But if the stream hits the surface at one point, a little mound will form there from which the additional matter will slowly spread outward. Even after we have stopped pouring in more, it will take some time until the even surface will be fully restored. It will, of course, not reach the height which the top of the mound had reached when the inflow stopped. But as long as we pour at a constant rate, the mound will preserve its height relative to the surrounding pool—providing a very literal illustration of what I called before a fluid equilibrium. (Hayek 1969:281)

Suppose that the monetary expansion is stopped at time $t_{3}$. Market A then loses this extra demand, whereas money is still flowing to market $B$ (from market $A$ ) because market $B$ is at the end of the monetary path in our model. The relative price $\mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{B}}$ tends to overshoot the original level after the cessation of the money expansion. The reason is that within the interval $t_{3}-t_{4}$, the demand might be quite strong in market B. From time $t_{4}$, the inflow of money is gradually losing its power in affecting market B , and the relative price is on the way to the original equilibrium. The enlarged stock of money is absorbed by higher demand for money
that expanded due to a higher general price level, and the impact of the monetary expansion on the economy should fade out. However, because market A benefited from the monetary expansion for an extended period of time, factors of production were attracted there. It may be implied that the structure of the economy was shaped to meet a higher demand on this market. After the halt in the money expansion, the factors of production, the capital apparatus and capacities on this market, and the resulting extended supply of good A may depress the relative price $\mathrm{P}_{\mathrm{A}} / \mathrm{P}_{\mathrm{B}}$ below the level that prevailed before the monetary expansion. Furthermore, much of this production apparatus was profitable only due to the constant flow of money. Once the extra demand is lost, some of these productive structures cannot be maintained without a loss. The economy must be restructured to meet the system of relative demands that prevailed before the monetary expansion, and this process cannot be pursued without a loss in capital and a period of unemployment.


Figure No. 5, Evolution of the relative price during the monetary expansion process.

As can be seen, money might not be neutral to the real economy even in the long run. Although the inter-temporal equilibrium has not been discussed yet, we can conclude that the monetary expansion may result in the loss of some productive capacities. On the other hand, the previous analysis disregarded formation of expectations of individual agents. It can be argued that individuals may anticipate the impact of the monetary expansion on their prices, and the system of relative prices might not be disturbed. However, as was pointed out by Garrison (2001), such an assumption puts too much weight on the abilities of the acting agents. It demands that agents can act without the help of the price system, but according to the information that is, however, delivered only by the price system. We argue that agents cannot learn the effects of the monetary expansion before the prices they are concerned with are changed. Prices are never affected by the money influx at the same moment and to the same extent. Thus, in the market economy, any change, even the change in the money supply, is reflected in the individual prices. In the complex free market economy, there is no other information system agents may use apart from the price system.

### 2.1 INTEREST RATE GAP IN A GROWING ECONOMY

The previous section was mainly concerned with the intra-temporal, i.e. cross-section or spatial, structure of relative prices that was brought about by positive economic growth, and with the distortion of this structure when the monetary expansion was carried out in the effort to stabilize the general price level in the expanding economy. However, since both the economic growth and the business cycle are dynamic phenomena, the intertemporal analysis should be the centre of our investigations.
Chapter 3 examined in great detail the behaviour of the natural rate of interest. It was concluded that if the rate of technological progress, the subjective discount rate, and the elasticity of intertemporal substitution in consumption are constant, the natural interest rate on the balanced growth path should be constant as well. The technological progress was identified not only as the driving force of the growth in potential output, but also as one of the determinants of the natural rate of interest.

The picture of the economy presented in the previous section might be consistent with such a state. The economy is growing at some definite rate, and the natural rate of interest is constant. As we have seen, according to Hayek, the price level should be declining, otherwise the general equilibrium will be disturbed. At this point, Hayek (1939) stressed the adjective "intertemporal" in the general equilibrium analysis. In other words, in the expanding economy, the "natural system" of intertemporal prices is preserved only if the general price level is decreasing. And the key intertemporal price that orchestrates the intertemporal allocation of resources is the rate of interest. As a result, in a growing economy, the actual interest rate is equal to the natural rate of interest only if the price level is allowed to decline.
Thus, in this theory, the technological progress that leads to a permanently increasing potential output is consistent with a stable natural rate of interest. And the actual rate of interest will be equal to the natural rate only if prices are falling over time. However, this conclusion is at odds with the famous Wicksellian theory. As is well known, Wicksell (1936) concluded that either an increasing or a decreasing price level is a signal that the actual rate of interest is not at its natural level. If the price level is increasing, the banking system lowered the actual rate of interest below the natural level, whereas if prices are falling, the actual market rate of interest was raised above the natural level. When the price level is stable, the banking system exactly hits the level of the natural rate of interest.
Hayek (1935) argued that this Wicksellian analysis is valid only in a stationary economy. In a growing economy, the equality of the actual and the natural rate of interest is consistent with a decreasing price level. If an attempt is made to stabilize the price level, an increase in the total money supply must be pursued. However, this monetary expansion should depress the actual rate of interest below the natural level, as was discussed in great detail in Chapter 2. As a result, in the expanding economy, the interest rate will be at its natural level and prices will be falling, or the price level will be constant, but the actual rate of interest must be lowered below the natural level. ${ }^{291}$

This conclusion is very strong, and it will be discussed later on with the help of a simple graphical apparatus. At this point, we stress that it has significant business cycle repercussions. If the interest rate is depressed below the natural level, the Austrian theory predicts that the business cycle should be triggered. However, there is no need to repeat the analysis from Chapter 2. Let us stress the key ideas of this theory. An artificial lowering of the interest rate should lead to an undue lengthening of the capital structure. Longer methods of

[^187]production benefit and attract factors of production from sectors that provide consumption goods in shorter periods. Sooner or later, the interest rate should return to its natural level, but the longer processes might not be fully finished. This reversion process indicates the end of the boom and the path to a recessionary state.

In the present analysis, we consider a growing economy. Yet, an attempt to stabilize the general price level may lead to the business cycle. As can be seen, the theory of economic growth and business cycle are interwoven at this point. Thus, let us stress several important issues. First, the technological progress leads to a continuing growth in potential output. On the other hand, an expansion in the money supply, which attempts to stabilize the price level and which consequently lowers the interest rate below the natural level, usually takes the form of an expansion of credit. This increase in the supply of loanable funds and the corresponding lower interest rate stimulate investment activity. In the initial phase, the economy may grow at a higher rate than can be attributed to the technological advance. Some part of the growth might be regarded as sound as it reflects technological progress; the other part may be termed artificial because it is maintained by credit expansion.
It can be said that the actual output is then growing at a higher rate than the potential output, so a positive output (growth) gap has emerged. However, it might be rather difficult to attribute the specific components of economic expansion to the growth in potential output and to the output gap. The booming economy and the expansion of the investment spending might be reflected in larger capital capacities. Even though this capacity is not sustainable, as was demonstrated in Chapter 2, the usual wisdom may imply that the growth in potential output is supported by a newly created capital. Another aspect is that if the booming economy redistributes wealth to people with lower impatience, the natural rate of interest itself might be partly reduced. Furthermore, various hysteresis effects with the opposite sign may also speed up the growth in potential output. As a result, from the Austrian point of view, the distinction between the actual output and potential output is not as attractive as in modern macroeconomics. This theory would rather stress that the growth is not fully sustainable because an important part is not supported by technological advances or higher voluntary saving, but by the monetary and credit expansion.
Another observation is that the positive output gap, or an unsustainable boom in the Austrian terms, may be accompanied by a stable price level. If the central bank was accurate in the expansion in the money supply, prices might be perfectly stabilized. However, it was exactly this money and credit expansion that triggered the artificial boom. Moreover, nobody would blame the central bank for this boom, because prices are stable, or rising at a very moderate pace, or they are even slightly declining. Yet, the expanding stock exchange in this noninflationary environment may indicate that too much money was injected to the economy, and the last decade could be a fairly good representative of this state of the economy. Indeed, the stability of prices may easily disguise that an artificial boom is present in the data.
Thirdly, since the boom of the economy is not sustainable, only an increasing rate of the money expansion can extend its life. Price stability will then turn to a positive inflation. However, as was demonstrated in Chapter 2, in the next round the booming economy would require additional doses of money that must be increasing at a higher rate. The central bank with its price-stability mandate may stop this process, and the boom will turn to a bust. The economy that was expanding will be on the path to recession. An open question is whether the absolute level of aggregate output will fall or whether it will just be growing at a lower rate. Nonetheless, according to the Austrian theory, the economy must undergo this business-cycle pattern.

Since capital was artificially lengthened in the expansion phase of the business cycle, part of these structures are lost. In the downswing phase, the actual output is (growing) lower than
the potential output. The positive output gap from the booming phase is reversed to a negative output gap. Yet, due to losses in the capital stock, the potential output itself might be lower as well. ${ }^{292}$

As was discussed in Chapter 2, the eventual levels of the potential output and the natural rate of interest are difficult to determine. However, stability of the price level does not mean that money is neutral with respect to the real economy. If the price stability is achieved by an expansion in the money supply, typical business cycle behaviour might be triggered. The boom-bust cycle may then end up with a different level of potential output and the natural rate of interest.
Another observation that might be directly deduced from the Austrian theory is as follows. The more rapid the technological progress is along with the growth in potential output, the stronger pressures on the decline in the price level must be provoked, assuming constant money and velocity. The stabilization of the price level then requires substantial increases in the money supply that may lead to a significant credit expansion. As a result, the higher the growth in natural output, the higher the likelihood that the economy will experience an Austrian-type business cycle due to monetary accommodation.
According to this theory, business cycles are phenomena that should prevail in growing economies rather than in stationary economies. The reason is that the monetary sector does not allow the real economy to reach the natural structure of relative prices both on the intratemporal and on the inter-temporal basis. An interesting fact is that the real forces stand not only behind the economic growth but also behind the business cycle. Yet, the monetary forces represent the true culprit of the economy-wide fluctuations. Hayek (1933) praised Schumpeter (1961) for his theory identifying the expansion of credit in (real) booms, but according to Hayek, Schumpeter failed to recognize that it is exactly this expansion in money supply and credit that is responsible for the unsustainable boom and the resulting bust. Furthermore, the monetary and the real approach to the business cycle theory may be reconciled because the larger the fluctuations in potential output, the higher the instability in the money supply and hence the more volatile the economy is owing to the monetary sector.

A final note we make in this section is of the behaviour of various forms of money in this process. If the monetary expansion, which attempts to stabilize the price level, takes mainly the form of the expansion of credit and deposits, the superstructure of deposits and credit erected on the monetary base during the boom may partly collapse during the recessionary phase. This collapse could provoke deflationary pressures in the economy. So the attempt to overcome "sound deflation", which could accompany technological progress, by the credit expansion may end up in the "unsound deflation" (or secondary deflation) that is provoked by a collapse in the credit superstructure and the resulting collapse in the aggregate demand. However, according to the Austrian theory, without the attempt to stabilize prices at the beginning of the process, there will be no credit expansion, no artificial boom inevitably resulting in recession, no collapse in the money supply (growth), and no deflationary pressures on the side of the aggregate demand.
If we map the ratio of total money supply to the monetary base, which is known as the money multiplier, the expansion of deposits during the boom is most probably faster than the accommodation of the monetary base from the central bank. Hence, the money multiplier should increase during the boom. However, the partial and imminent collapse of the credit and deposit superstructure in the recession may persuade the central bank to expand the monetary base at a very rapid pace. Thus, the money multiplier should decrease. According to

[^188]the Austrian theory, the complicated behaviour of various variables presented in the foregoing paragraphs has one major culprit - the doctrine that prices must be stabilized even in the expanding economy.

### 2.2 THE AUSTRIAN THEORY IN THE KEYNESIAN FRAMEWORK

In this section, we present the Austrian theory of the business cycle in a growing economy in a simple Keynesian model that will allow us to pin down key differences between this theory and the (New) Keynesians. Garrison (2001:54) envisioned an idea of secular growth that maintains the structure of capital as represented by the Hayek triangle, and that keeps the natural rate of interest at a constant level because the growth in investment is the same as the growth in saving. Such an idea is indistinguishable from the balanced growth path in usual neoclassical growth models.


Figure 6, Balanced growth path in the Austrian model.
Source: Garrison (2001:54)

In Figure No. 6, the expanding Hayek triangle, expressed in real terms, maintains its shape because the natural rate of interest is constant. In the neoclassical models, the balanced growth path generates a property that is well documented by the real world data, namely a relatively stable capital-output ratio. In the Hayek triangle, this would mean that the height of the vertical line, representing consumption goods, is expanding at the same rate as the area of the triangle measuring the capital stock in the economy. Forces of technological progress that may create shortening tendencies in the production process are offset by lengthening tendencies that are coming from the accumulation of capital. However, the Hayek triangle, which was mainly designed to illustrate one particular idea in the business cycle theory, is such a simple and stylized model that it may lead sound reasoning astray when used in the economic growth theory.
In the Garrison model, the (potential) output of the economy is growing, and the natural rate of interest is constant. We may sketch these combinations in a space that is different from the loanable funds market. Let us plot the expanding potential output and constant natural interest directly in the space output - interest rate (see panel (a) in Figure No. 7). However, it might be useful not to lose the information from the loanable funds market. Investment and saving equilibrium in the space Y,r is usually represented by the IS curve. The loanable funds representation of the IS curve is rather simple - higher income leads to bigger saving and
hence to a lower interest rate (Mankiw 2003:270). Thus, the second curve that is missing in panel (a) is the IS curve that is depicted in panel (b).


Figure No. 7, Balanced growth path and the natural interest in the (New) Keynesian model

As can be seen, the balanced growth path may be easily represented in this simple IS- $\mathrm{Y}^{*}$ model. Figure No. 7 also uncovers the definition of the natural rate of interest that can be found in the New Keynesian literature. A similar picture as in panel (b) can be found in Williams (2003:1). However, the original definition of Wicksell that was discussed in Chapter 1 is also mentioned in this literature. Amato (2005:3), for example, interpreted the Wicksellian position as follows: the natural interest equilibrates saving and investment, it is consistent with the marginal product of capital, or it is consistent with the price stability.
The genuine New Keynesian vision of the natural interest is, however, as follows: it is the interest rate when the economy operates at the natural, i.e. flexible price, level (Woodford 2003:9). Woodford (ibid:53) also stressed that the natural interest depends on the marginal productivity of capital and the time preference, which is a typical Fisherian approach discussed in detail in Chapter 3. Moreover, Woodford (ibid.) connected this definition to what he called the flexible-price IS curve. Apart from the determinants of the natural interest rate mentioned by Woodford, Amato (2005:3) added the willingness to substitute consumption over time. This determinant was also explored in Chapter 3 under the name of the elasticity of intertemporal substitution in consumption (1/日). Laubach and Williams (2003:64) stressed that the natural rate of interest is affected by the growth rate in potential output, which is also confirmed by the analysis from Chapter 3. Yet, they added that the natural interest is consistent with the natural output and stable inflation (ibid:63). A similar position is held by Orphanides and Williams (2002:64).
Putting all these pieces together, the natural rate of interest in the New Keynesian vision may be found at the intersection of the natural output with the New Keynesian IS-curve, which could be easily derived by a similar procedure as we used in Chapter 3, namely as the Euler equation in a simple intertemporal optimization problem. Panel (b) also uncovers that if the
actual rate of interest is higher than the natural rate ( $r_{1}>r_{n a t}$ ), the difference between the actual output and the natural output is negative, ${ }^{293}$ so there is a negative output gap, whereas the opposite statement holds if the actual interest rate is below the natural level.
In Figure No. 7, we assume that the economy is on the balanced growth path, ${ }^{294}$ so the natural interest rate should be at its steady state level $\mathrm{r}_{\mathrm{ss}}$. In Chapter 3, we derived that in the RCK model, this level is equal to $r_{s s}=r^{*}=\rho+\theta$ g. Hence, it depends on the subjective discount rate (time preference in the second sense) $\rho$, the elasticity of substitution ( $1 / \theta$ ), and the growth rate of technological progress $g$. The IS curve in panel (b) is expanding at the same rate as the potential output $\mathrm{Y}^{*}$ since the positive technological progress (and maybe population growth) is raising the marginal productivity of capital and therefore the investment demand. It should be stressed, however, that the steady state interest rate is only one special case of the natural level, and the natural interest rate may differ from this level if, for example, the economy is converging to its balanced growth path/steady state. More on this will be said in section 4.

The Austrian theory from section 2.1 indicated that in a growing economy, the price level should be declining. An attempt to stabilize the price level by the expansion of the money supply should lead to a boom-bust cycle. To demonstrate this idea in the framework of the ISY* model, the money market must be included. However, this task is easy since the space considered - Y,r - might perfectly fit the LM curve. ${ }^{295}$ This curve requires that the real demand for money is negatively related to the interest rate and positively related to real income. It can be also derived from microeconomic foundations if, for example, money is included in the utility function (Woodford 2003:104), or if money performs the role as a means to facilitate transactions (McCallum 2000:872). In these models, higher real income (or consumption) requires higher real money balances, so the interest rate must increase to equilibrate the money market. However, such an assumption is made for a fixed price level. If the price level is flexible, as we have assumed so far, higher real demand for money must be reflected in lower prices. Accepting this point of view, the IS $-Y^{*}$ model is extended to a flexible price IS-LM-Y* model (Woodford 2003:109).

The working of this model is presented in Figure No. 8. An upward sloping LM curve represents combinations of output and interest rate that are consistent with equilibrium on the money market, assuming fixed price level. Yet, this curve is rather a shadow curve in this model because we assume flexible prices.

Nonetheless, let us present the logic of this model in a simple example. Consider a (temporary) increase in government purchases presented in panel (a). This positive demand shock will shift the IS curve outwards. The natural rate of interest rises from $r_{n a t, 1}$ to $r_{\text {nat, } 2}$ leaving the real output unchanged. However, if prices were fixed, the interest rate would rise only to $r_{2}$. At point $B$, the actual interest rate would be lower than the natural rate of interest $\left(\mathrm{r}_{2}<\mathrm{r}_{\text {nat, }, 2}\right.$ ), and the actual output would exceed the natural output ( $\mathrm{Y}_{2}>\mathrm{Y}^{*}$ ) - there would be a positive output gap.
On the other hand, if prices are flexible, the equilibrium is at point C . This long run equilibrium is delivered by an increase in the price level from $\mathrm{P}_{1}$ to $\mathrm{P}_{2}$. The question is what source raises the prices since we assumed a pure fiscal expansion with no monetary accommodation. The answer lies in the fact that the real demand for money is negatively

[^189]related to the interest rate. Since the eventual rate of interest is higher $\left(r_{n a t, 2}>r_{\text {nat, } 1}\right)$, people will hold less real money balances. Hence, the reduced real money balances represent the source for the increase in the price level. The quantity equation interpretation of this mechanism might be as follows: An upward sloping LM curve indicates that the velocity of circulation is positively related to the interest rate. ${ }^{296}$ With higher interest rate, velocity of circulation increases. Since output is fixed by the amount of available factors of production, price level must increase.
(a)

(b)


Figure No. 8, The Flexible price IS-LM-Y* model
Another important interpretation is the Wicksellian. If we relax for a while the assumption of flexible prices, point B indicates that the actual interest rate is below the natural level $r_{2}<$ $\mathrm{r}_{\text {nat }, 2}$. According to Wicksell (1936), price level should increase. And it is exactly what our model predicts. As a result, the flexible price IS-LM-Y* model might be considered as a simple representation of the Wicksellian theory. ${ }^{297}$ As a result, the New Keynesian and the Wicksellian approaches are interconnected. If the actual rate of interest is below the natural level, output exceeds its natural level (New Keynesian view), and the price level should grow in the future (Wicksellian view).

We may add one more ingredient that is stressed especially in the real business cycle literature. Suppose that the leisure time is included in the (lifetime) utility function. As was demonstrated in Appendix 3B in Chapter 3, this framework may lead to the intertemporal substitution in labour. In such a case, the labour supply is increasing in the interest rate (see equation 19 in Appendix 3B in Chapter 3). Thus, the natural output is an increasing function of the real interest rate as well, and the curve $\mathrm{Y}^{*}$ is no longer vertical in the Y,r space. The flexible price IS-LM- $\mathrm{Y}^{*}$ model is then modified, as can be seen in panel (b) of Figure No. 8. Fiscal expansion leads not only to a higher natural rate of interest, but due to the phenomenon

[^190]of the intertemporal substitution in labour, it also affects the natural output in a positive direction. ${ }^{298}$ Fiscal expansion has a positive impact on output, even though the mechanism is different from the Keynesian system. In the RBC theory, it is the potential output itself that is increasing, whereas in the Keynesian theory, output increased beyond its natural level due to the assumption of rigid prices.
Moreover, the New Keynesian model usually assimilates the RBC framework. As a result, the short-run implication of this theory is that both the potential and the actual output are rising after the (temporary) fiscal expansion. The relative increase depends on the relative slope of the $\mathrm{Y}^{*}$ curve compared with the LM curve. Consider point D and the less elastic $\mathrm{Y}_{\mathrm{A}}{ }^{*}$ curve. In this situation, the fiscal expansion leads to a positive output gap (distance $\mathrm{DH} ; \mathrm{Y}_{2}$ minus the horizontal distance to point H on the $\mathrm{Y}_{\mathrm{A}}{ }^{*}$ curve) because the actual interest rate is lower than the natural rate ( $\mathrm{r}_{2}<\mathrm{r}_{\text {nat,2A }}$ ). However, if the $\mathrm{Y}^{*}$ curve was more elastic ( $\mathrm{Y}_{\mathrm{B}}{ }^{*}$ curve), the increase in potential output would be higher due to a substantial impact of the intertemporal substitution in labour supply. The output gap would be negative (distance DJ; Y 2 minus the horizontal distance to point J on the $\mathrm{Y}_{\mathrm{B}}{ }^{*}$ curve), whereas the interest rate gap would be positive ( $r_{2}>r_{\text {nat, } 2 \mathrm{~B}}$ ).
In the Wicksellian framework with flexible prices, fiscal expansion may lead to an increase in the price level from $\mathrm{P}_{1}$ to $\mathrm{P}_{2 \mathrm{~A}}$ (point F ) if the interest rate was initially below the natural level $\left(r_{2}<r_{\text {nat }, 2 \mathrm{~A}}\right)$, or to a decrease in the price level from $P_{1}$ to $P_{2 B}$ (point $G$ ) if the interest rate was above the natural level ( $\mathrm{r}_{2}>\mathrm{r}_{\text {nat,2B }}$ ). In the second case, which might seem paradoxical, the increase in the potential output is so large compared with the increase in the natural interest rate that the demand for real money balances increases rather than falls. For a constant nominal money supply, the price level must decline. ${ }^{299}$


Figure No. 9, Balanced growth path and the secular decline in the price level

[^191]In the following parts, we will neglect the intertemporal substitution in labour; however, we will use this simple framework to assess the validity of the Austrian theory presented in section 2.1. As was discussed before, Hayek concluded that the increasing potential output should be accompanied by a (secular) decline in the price level. Figure No. 2 displayed this idea. Figure No. 9 represents the same theory in the flexible price IS-LM-Y* model. The natural output is growing at some positive rate, and the natural rate of interest is constant. The secular decline in the price level from $P_{1}$ to $P_{2}$ is reflected by a rightward shift in the LM curve (panel a). Rising incomes bring about higher real demand for money L, which is satisfied, even if the nominal money supply is constant, by a decreasing price level (panel b). As we can see, the actual rate of interest is at its natural level; yet the price level is declining. Thus, this model may easily replicate the idea of F.A. Hayek. In the growing economy, only this dynamics of the interest rate and prices is consistent with the intertemporal general equilibrium. ${ }^{300}$
(a)



Figure No. 10, Boom provoked by the monetary expansion that stabilizes the price level

[^192]If the monetary authority is determined to keep the price level constant, it must inject money into the economy. As we demonstrated before, this may trigger the boom-bust cycle. Figure No. 10 plots this conclusion in the IS-LM-Y ${ }^{*}$ model. Monetary expansion lowers the interest rate below the natural level ( $\mathrm{r}_{2}<\mathrm{r}_{\mathrm{nat}}$ ). If the short-run aggregate supply is non-vertical, the actual output rises above the potential level, and a positive output gap arises in the economy $\left(\mathrm{Y}_{2}>\mathrm{Y}_{2}{ }^{*}\right)$. Panel (b) represents this theory in the AD/AS model (Selgin 1997:38; Beckworth 2008:373). The eventual shift of the LM curve $\left(\mathrm{LM}_{4}\right)$ in panel (a) reflects not only the initial increase in the money supply but also the response of the price level from $\mathrm{P}_{2}$ back to $\mathrm{P}_{1}$.
At point D , the economy is not in the long run equilibrium. If the natural rate hypothesis is at least partly valid, the economy should end up at point E at which the output is back at its potential level, the interest rate returns to its natural level, but the price level is higher than in the initial state. The movement to this point would be represented by a shift in the AS curve in panel (b) backwards, and the LM curve in panel (a) back to position $\mathrm{LM}_{2}$. These shifts are not presented in the figure to keep clarity of the exposition. As can be seen, the action of the central bank was not successful in the long run because it overshot the targeted price level.

However, in the Austrian theory the dynamics of the economy especially in the recessionary state is not as simple as suggested by the natural rate hypothesis. As was discussed before, the Austrian recession is characterised by losses of capital structures due to the misallocation of resources in the boom phase of the business cycle. This will surely affect the potential output in the negative direction.
(a)


Figure No. 11, Economy after the boom-bust cycle (BBC).

Furthermore, if the increase in the money supply took the form of the expansion of deposits (and hence credit), recessionary forces leading to bankruptcies of many firms might reduce the demand for credit and then contract the super-structure of deposits. This would be reflected by a leftward shift in the LM curve. And finally, the collapse of credit and shaken confidence could depress the investment (and consumption) spending, which would move the IS curve to the left. The eventual state is hard to determine. Figure No. 11 suggests one possible position - the natural output is lower, and both the LM curve and the IS curve are depressed. The natural rate of interest is lower as well, owing to depression forces on the side of the IS curve. Moreover, if the misallocation of resources is substantial, scars on the
economy may be permanent. Figure No. 12 suggests three possible paths of the potential output after the recession. If the third path is valid, the natural rate of interest is permanently lowered due to the lower growth rate in potential output.


Figure No. 12, Various paths of potential output after the recovery.
Note: Modified diagram from European Commission (2009:11)

Although the Austrian authors would agree that money is neutral in the sense that it can never permanently increase output and that the only outcome of money expansion is price inflation, monetary manipulation is not neutral in the sense that it leaves the state of real variables unaffected. In the example above, we demonstrated that the long run effect of the money increase, which was directed to stabilize the price level, is the misallocation of real resources and the loss in potential output.
The above exposition is a direct attack on the usual wisdom about beneficial effects of price level stabilization. Woodford (2003:5), for example, stressed that inflation or deflation is a "symptom of systematic imbalances in resource allocation". He also added that "instability of the general level of prices causes substantial real distortions-leading to inefficient variation both in aggregate employment and output and in the sectoral composition of economic activity" (ibid.). With respect to positive inflation, he focused on its harmfulness owing to the existence of price stickiness:

For when prices are not constantly adjusted, instability of the general level of prices creates discrepancies between relative prices owing to the absence of perfect synchronization in the adjustment of the prices of different goods. These relative-price distortions lead in turn to an inefficient sectoral allocation of resources, even when the aggregate level of output is correct. (ibid.:12)

It is rather interesting that this leading New Keynesian put emphasis on the problems of allocation and misallocation of resources, on the role of relative prices, and other topics usually stressed by authors writing in the Austrian tradition. As was indicated in Chapter 3, Woodford (ibid.:5) even mentioned Hayek in the opening chapter as one of the authors that followed, as well as Woodford himself, the Wicksellian tradition. As a result, it might be suggested that the glorious New Synthesis of the Real Business Cycle theory and New Keynesianism so stressed and praised by modern authors could integrate some of the pathbreaking ideas of F.A. Hayek and L. Mises.
However, let us show that this integration is not as easy and straightforward as it might seem at first glance. Woodford (2003:13), for example, explicitly said that the policy of the price level stabilization will eventually eliminate the output gap. This is at odds with our previous analysis. Moreover, in one of his specific models, it is shown that under normal conditions zero inflation target is optimal (Eggertsson and Woodford 2003:167), even though it is suggested that a self-sustained deflation might be also an equilibrium outcome in the model (ibid.:194). The particular speed of deflation in that equilibrium is equal to the rate implied by the Friedman rule. On the other hand, Jung et al. (2005:820) stressed that this specific rate of deflation is inferior to zero inflation target due to the existence of zero lower bound on nominal interest rate. All these topics will be discussed in the next section.
Other New Keynesians prefer a low, yet positive, rate of inflation to zero inflation (Akerlof et al. 1996). There is no need to list arguments that appeared in the literature supporting this proposal. The common practice of the majority of modern central banks targeting a positive rate of inflation is a direct outcome of this theory. Interestingly, Gordon in the discussion section in Akerlof et al. (1996) pointed out that in the 19th century, economic growth was accompanied by a general decline in prices. Nonetheless, Blinder and Reis (2005) in praising the genius of Alan Greenspan explicitly wrote that more rapid growth in potential output requires a loose monetary policy to speed up the movement of actual output on the new potential level.
Let us use the simple IS-LM-Y* model and the AD/AS model to show the core of the New Keynesian argument and the difference against the Austrian analysis presented above. The optimum behaviour of the economy that is growing at some positive rate, and in which the natural rate of interest is at a constant level, is depicted in Figure No. 13. Panel (a) shows that the increase in aggregate demand is as large as the increase in aggregate supply. The economy smoothly moves to the new equilibrium. The actual output is equal to the potential output, and the price level (or inflation in more complex models) is stabilized. There is no disturbance in output or interest; there is no Austrian style boom-bust cycle.

Similar pictures can be found in elementary texts that were designed to introduce basic tenets of the New Keynesian model (Benigno 2009; Mankiw 2009). The economic logic behind the peaceful transition to a new potential level is described within the framework of this model. With a permanently higher level of income, people feel richer and hence they consume more. That is the reason for the shift in the AD curve, which is, however, derived from the New Keynesian IS curve and a simple monetary policy rule of the Taylor type.
As is usual in the New Keynesian literature, there is no reference to money in this process. However, the omission of money is rather disturbing since the total value of aggregate expenditure (PY) cannot increase without expansion in the money supply or the velocity of circulation. Moreover, consumption in this model is endogenous as well as is output, and its increase is hidden in the increase in output. So the endogenous component of Y cannot shift the entire curve. Yet, AD curves in the New Keynesian framework contain one important component - a monetary policy rule. In this rule, the central bank sets the interest rate according to its goals and the state of the economy. The money supply is automatically
adjusted to any demand for money for the given rate of interest - the money supply is endogenous. So the more accurate reasoning should be as follows: with a permanently increasing potential output, the real demand for money is rising. The central bank automatically accommodates this higher demand by the injection of the new money supply, keeping the interest rate at the same level ( $\mathrm{r}_{\mathrm{CB}}$ in panels (b) and (c) in Figure No. 13).


Figure No. 13, Balanced growth path and the price level stabilization in the New Keynesian model.

The money market representation of this process is virtually the same as in panel (b) in Figure No. 9. The only difference is that the vertical real money supply curve is moved to the right due to the expansion in the nominal money supply, not owing to the decrease in the price level, which is held constant. In panel (c) of Figure No. 13, the endogenous character of the money supply on the money market is represented by a horizontal line, which is then reflected in panel (b) in a horizontal "LM" curve. The interest rate is set by the central bank; however, in perfect conformity with the natural level. The expansion in the nominal money supply is
hidden in the model. Nonetheless, this monetary accommodation of the economic growth was admitted, for example, by Frankel (2009:123). Moreover, in the exposition of the New Keynesian model, Carlin and Soskice (2005) suggested an overshooting in the level of output; yet only in the case of a positive supply shock, which cannot be presumably generalized to a permanent increase in output.
As we can see, the increase in the money supply in the New Keynesian (NK) model is perfectly neutral with respect to the growing economy. It is automatically absorbed by higher demand for money that stems from expanding real incomes. Price level is stabilized without any boom-bust pattern envisioned by the Austrian theory. In the New Keynesian terms, there is no output gap, and no better policy the central bank can conduct.


Figure No. 14, Unrealised growth in output due to the fixed price level.
We can also discuss the New Keynesian view on the policy suggested by Hayek. Suppose that the economy is growing, but the money supply is fixed and exogenous, not endogenous as in the NK model. Let us assume for a while that the price level is fixed for one (New Keynesian) reason or another, even though the potential output is growing due to the technological progress. Figure No. 14 depicts this situation in the IS-LM-Y* model and in the AD/AS model. The economic growth leads to a rightward shift in the $\mathrm{Y}^{*}$ curve and the IS curve. However, the (short run) equilibrium of the economy depends on the slope of the LM curve. At this moment, the sensitivity of the demand for money to the interest rate plays a crucial role. It will decide how far the actual output will move from the new potential output. The flatter the LM curve, the lower the difference between the actual and the natural level of interest, and hence the lower the output gap. If both the money supply and the price level are fixed, the higher demand for money can be satisfied only by a movement in the interest rate. If the demand for money is very sensitive to the interest rate $\left(\mathrm{LM}_{3}\right)$, an increase in the rate of interest drastically reduces the required amount of money balances. People dissolve part of their "cash" reserves, which may "finance" the increase in output. In other words, the increase in natural output is accompanied by a significant increase in velocity $\left(\mathrm{V}_{3}\right)$, which is reflected by a considerable shift in the AD curve $\left(\mathrm{AD}_{3}\right)$. The economy ends up very close to the new potential level $\left(\mathrm{Y}_{3}\right)$.

On the other hand, if the demand for money does not (almost) depend on the interest rate $\left(\mathrm{LM}_{1}\right)$, the interest rate rises to very high levels ( $\mathrm{r}_{1}$ ), and the economy is stuck close to the
previous level of output $\left(\mathrm{Y}_{1}\right)$. In this situation, the resulting negative output gap is very large. We observe a surprising paradox - technological progress allows higher output in the economy; yet this potential is not realised due to the fixed price level, fixed money supply, and very stable velocity.
If we relax the assumption of a perfectly fixed price level, the New Keynesian theory may still suggest that not all prices are being instantly adjusted when the potential output is growing over time. Figure No. 15 displays that in the short run, the economy is trapped in quasirecession (point B). The output is growing; however, its growth is below potential due to the non-adjustment of some prices. If prices are flexible in the long run, the economy should end up at the new potential level with a lower price level, as is suggested by the Austrian theory in Figure No. 2 and by point D in Figure No. 15. However, there is an unfortunate transition period of below-potential growth that could have been speeded up if the money supply was allowed to increase.


Figure No. 15, Quasi-recession due to imperfect adjustment in prices

Figure No. 14 may be also used to examine the Wicksellian interpretation of this process. As can be seen in this diagram, the actual rate of interest is above the natural rate (e.g. $r_{2}>r_{n a t}$ ). In the New Keynesian theory, this implies a negative output gap $\left(\mathrm{Y}_{2}<\mathrm{Y}_{2}{ }^{*}\right)$. However, in the long run, this negative output gap should lead to a decrease in the price level, which is in perfect accordance with Wicksell's predictions. As a result, it seems that the Hayek critique of Wicksell was not accurate. Both approaches may be right. Both theories may be consistent. They only refer to a different period of time, they stress a different time horizon, or it can be said they are based on a different set of assumptions. The Hayek theory of the declining price level and of the equality between the actual interest rate and the natural interest rate is designed for a smoothed process in the framework with a perfectly flexible price level in which the growing potential output is immediately reflected in a lower price level. The Wicksell theory, on the other hand, might stress the point at which the actual interest rate is above the natural rate (point W in Figure No. 14), creating pressures on the decline in the price level in the future. It can be deduced that this approach is based on the short-run inflexibility of prices, even though this "short run" may be an infinitely short period of time. As a result, it is rather hard to decide whether the Hayek objection to Wicksell's theory is justifiable. Both economists may describe the same dynamic phenomenon; yet, from a different point of view. If this perspective is correct, then both theories could be easily reconciled.

As we have seen, the New Keynesian recommendation for monetary authorities in the period of a growing economy is exactly the opposite of the Austrians. The money supply should be increasing along with the growth in potential output. The economy (and the commercial banking system) may automatically absorb money it needs from the central bank for the fixed interest rate, and this new money will be neutral with respect to the real economy. Moreover, an insufficient increase in the nominal (and thus real due to partial rigidity of prices) money supply compared with a growing real demand for money may lead to problems in the real economy - real output will grow at a lower rate than it could.

The key dividing point between the Austrian theory and the New Keynesian theory is whether the higher real demand for money, which emerges due to rising potential output and hence real incomes, is to be satisfied by a reduction in the price level or by a higher nominal money supply. Austrians would prefer the former since the latter may lead to the boom-bust cycle. On the other hand, New Keynesians would recommend the latter because the former could result in quasi-recession. In the Austrian vision, money is neutral with respect to the real economy if it is not increased in the situation of expanding output, whereas New Keynesians believe in the rigidity of prices that may cause serious problems if the money supply is not appropriately adjusted.
Let us now present additional arguments that may support the Austrian vision and cast some doubts on the New Keynesian theory. The core of the problem is the demand for money. However, when people demand more money, it is not a one-sided transaction. Money is always demanded in exchange for something else. In the period of growing output, more goods are being offered on the market. But this new supply of goods automatically means a higher demand for money (Rothbard 2004). At the same time, a larger supply of goods can be realized on the markets only for lower prices. As a result, the increase in the supply of goods, a higher demand for money, and a lower price level represent three parts of the same phenomenon. People are able to produce more with better technology. They supply more goods and earn higher real incomes. Real incomes may take the form of constant nominal incomes, but a lower price level, which was reduced due to a higher supply of goods. Higher real incomes bring about a higher demand for money, which is, however, satisfied by the same phenomenon - a lower price level. Hence, we conclude that the smooth process introduced in Figure No. 9 may be automatic in real world. The problems that might emerge are present only in the artificial model, which was designed to analyze short-run fluctuations and not the economic growth issues. In other words, if we believe that higher quantity of goods can be easily sold on the individual market for a lower price, there is no reason to doubt that the same applies at the macroeconomic level - higher aggregate output is sold for lower prices.

Furthermore, New Keynesians underestimate injection effects discussed at the beginning of section 2. Higher supply of money is never directly transmitted to pockets of people who demand more money. In the modern banking system, new money always goes through the financial system. The major part of the money supply takes the form of new deposits that were created in the act of granting new credit. Hence, many prices may be affected before the new money arrives in the wallets of people. If the growth in potential output is really rapid, say $5 \%$, the necessary expansion in the money supply is $5 \%$ as well, even though it is endogenous and hidden in the New Keynesian models (Woodford 2008). In this connection, we may ask: can the banking system find enough profitable, sound, and trustworthy borrowers in order to expand the amount of credit (and hence deposits) by $5 \%$ ? Such a large expansion in credit backed only by artificial creation of new deposits may not only provoke an artificial boom in industries that absorb these credits first, but it can also deteriorate the average quality of portfolios held by commercial banks even before the crisis occurs. If the banking system is
flooded with liquidity from the central bank that is determined to stabilize the price level, new money might be primarily lent to sectors that are supported by the government. The expansion of mortgages and the housing boom before the recent financial crises may serve as an example.
Another question is whether the central bank, when the natural output is expanding, is able to hit the inflation target by the accurate monetary accommodation. A sudden increase in the growth rate in technological progress may create substantial pressures on the decline in (the average growth in) prices. This development then calls for the loosening in monetary conditions. According to Blinder and Reis (2005), the optimal monetary policy in this situation is to allow for a temporary decrease in the unemployment rate below the natural level rather than to allow for lower inflation. Yet, according to the Hayek theory, this monetary easing triggers an artificial boom that will end up in a bitter recessionary hangover.
Dozens of ingenious models were developed by New Keynesians to explain rigidity of prices in various markets (Mankiw and Romer 1991). One of the most popular is the Mankiw (1985) menu cost model. As is well known, this model is based on the idea that the economy is composed of many imperfectly competitive firms. An important macroeconomic implication of this assumption is that the natural level of output that is consistent with price flexibility is lower than the efficient level of output, which would prevail in perfect competition (Woodford 2003). As a result, we may plot one more vertical line in the IS-LM-Y* model, representing efficient level of output, which is always to the right of the original vertical line, representing flexible-price-level of output in imperfect competition. As the economy grows due to the technological progress, both curves are moving to the right. However, the distance between these two is not fixed, since it depends on the degree of monopoly power in the economy. It may fluctuate during the business cycle, but it can be also changing in the very long run. Furthermore, one may redefine the natural rate of interest as the intersection of the IS curve with this second vertical line representing efficient level of output. However, in perfect competition, the IS curve itself might be in a different position. Thus, it is hard to say anything more concrete in this connection. On the other hand, this difference in the two levels of "potential" output may explain how the positive output gap is possible. The economy may operate beyond the natural level because in imperfect competition, real resources are not fully utilized. As a result, it might be deduced that monetary expansion moves the economy closer to the efficient level of output. Yet, these considerations would take us too far afield from our main discussion.

Figure No. 16 shows a simplified version of the Mankiw model (Romer 1993; 2006). It is usually argued that the imperfectly competitive firm may reluctantly reduce its price after a negative demand shock due to the existence of costs associated with this act of price adjustment. However, Selgin (1997:31) in citing Okun (1980:169) suggested that the motivation of firms to adjust prices when they face reduction in costs brought about by technological progress may greatly differ from the situation of contracting demand. It would be rather curious not to sell more goods once they were produced owing to better technology. Reduction in demand might be unexpected; however, the production plans designed by firms should take into account better production possibilities along with the costs of adjusting prices, once larger production is to be sold. As a result, there might be a significant asymmetry in the price adjustment. If the firm faces a reduction in demand, it might be reluctant to lower its price. After all, it will suffer from lower profitability. However, reduction in costs is a different matter that first, should have been taken into account in global decision-making processes of the firm in the past, and secondly, is accompanied by higher profitability.

Figure No. 16 shows the potential loss from non-adjustment of individual price after the reduction in costs (from $\mathrm{MC}_{1}$ to $\mathrm{MC}_{2}$ in panel (a)) resulting from technological advance, which also moves the profit curve outwards (dashed curve in panel (b)). The shaded triangle in panel (a), representing unrealized profits due to the fixity in the individual price (and hence quantity since the demand curve does not move), is to be compared with the "menu costs". If the former exceeds the latter, the firm should reduce its price. In such a case, the firm operates at the profit maximizing $\left(\pi_{2}\right)$ quantity and price ( $\mathrm{p}_{2}{ }^{*}$ in panel (b)). If the firm does not adjust its price, the profit is still larger than before by distance $\left(\pi_{1}-\pi_{0}\right)$. However, it would be rather surprising if the improved technological conditions were not realized in expanded production.


Figure No. 16, Menu cost model and the reduction in costs

Thus, we conclude that it may be much easier for a firm to reduce its price when it faces saving in costs compared with the contraction in demand. The macroeconomic implication is that the shift in the $\mathrm{AS}_{\mathrm{SR}}$ curve downwards is rather smooth in Figure No. 15, so the economy may readily reach its new potential level with a lower general price level. On the other hand, the $\mathrm{AS}_{\mathrm{SR}}$ curve might be rather inflexible and flat as far as the collapse in aggregate demand is concerned.

As regards the importance of the menu costs, Selgin (1997) indicated that the technological progress that arises only in several industries and the resulting price level stabilization may bring about larger menu costs than a reduction in the price level. Rapid technological progress in a handful of industries implies that prices should be adjusted only in this part of the economy. However, if the money expansion is carried out, prices are driven up in the majority of markets. In the rapidly growing industry, prices might be raised, even though the eventual level could be still lower than in the past. However, all other markets face a positive demand shock that creates an upward pressure on prices. As a result, the menu costs considerations must be taken into account in the entire economy when the attempt is made to stabilize the general price level. On the other hand, the menu cost problem would arise only in some industries if the money expansion was not carried out.
We will conclude this section with a brief discussion about the fear from deflation. Figure No. 2 clearly suggests that price level deflation might be a natural response of the economy to an expanding potential output. Many economists would call this process "sound deflation", or "benign deflation" (Beckworth 2008:367). The harmful deflation or "malign deflation" (ibid.) is then attributed to the collapse in aggregate demand. However, as the following set of diagrams shows, the point is not so clear-cut.

Consider a contraction in the aggregate demand in Figure No. 17. As can be seen, the economy in the short run ends up in milder recession the sharper the decline in prices (panel a). ${ }^{301}$ For the given drop in AD, the steeper the AS curve, the more rapid the decline in prices will emerge, and the lower the negative output gap. Panel (b) uncovers that the fear from deflation may stem from the fact that more rapid deflation implies greater contraction in AD for the given shape of the AS curve.


Figure No. 17, Contraction in aggregate demand and deflation.

[^193]Furthermore, if prices of factors of production (or inflation expectations) are readily adjusted, the AS curve itself might move downwards (shift of $\mathrm{AS}_{\mathrm{B}}$ to $\mathrm{AS}_{\mathrm{B}}{ }^{\prime}$ in Figure No. 18). Then, the deflation will be more rapid in such an economy and the recession not as painful. As a result, more flexible expectations may lead to more moderate recession than sluggish adjustment in expectations even with a steeper AS curve.


Figure No. 18, Contraction in aggregate demand and adjustment in expectations.

The last issue we mention is the movement from point B to point D in Figure No. 17, as the economy is approaching the initial potential level. The quicker the decline in the price level, the more rapid the healing process in the economy and the closing of the output gap. As a result, deflation might be a symptom of contraction in the aggregate demand, but it may also indicate that the economy is recovering from the recession owing to its own internal forces. In analyzing the economy in deflation, one must distinguish between the shift of the entire AD curve ("unsound deflation") and the movement along the AD curve ("sound deflation") caused by the increase in the (short run or long run) aggregate supply (i.e. either AS or $\mathrm{Y}^{*}$ ).

## 3. HAYEK MV-RULE AS THE NOMINAL INCOME TARGETING

The previous section demonstrated that according to the Austrian theory, the optimum behaviour of money in the economy with expanding natural output is the constancy in the money supply. Only this "policy" may ensure that the "natural price system" and the intertemporal equilibrium are not disturbed by the monetary part of the economy (Hayek 1928:97). In this section, we will explore this idea in more detail, and we will also discuss its theoretical shortcomings and problems with practical implementation.
Hayek (1933b, 1935) admitted that similar effects on the real economy as changes in the money supply should also have changes in the velocity of circulation. Before we start our investigation, we need to clarify the relationship between the demand for money and the velocity of money. It may seem that these two are interchangeable, but the problem is more difficult. In the previous section, it was indicated that the permanently growing natural output brings about an increase in the (real) demand for money. This higher demand was satisfied either by a reduction in the price level, increase in the nominal money supply, or by the increase in the interest rate. However, such increase in the demand for money does not
necessarily imply a reduction in velocity. The simple Cambridge version of the real demand for money may clarify this point.

$$
\begin{align*}
& M_{d} / P=k(i, \ldots) Y^{(p ?)}  \tag{1}\\
& M_{s}=M_{d}  \tag{2}\\
& M_{s} V(i, \ldots)=P Y \quad \text { where } V=1 / k \tag{3}
\end{align*}
$$

Equation (1) indicates that the real demand for money depends on real income Y, or on real permanent income $\mathrm{Y}^{\mathrm{p}}$ (Friedman 1969), and on the optimum fraction of income people want to hold in the form of money " $k$ ". The famous " $k$ " may then negatively depend on the nominal interest rate and on thousands of other external factors. Equation (2) is the condition of equilibrium on the money market. Equation (3) is the quantity equation (equation of exchange), uncovering the relationship between velocity and " $k$ ".

This simple system of three equations clearly indicates that a pure increase in the natural output, which should be closely related to the permanent income, raises the real demand for money with no impact on velocity (equation (1) and (3)). The equilibrium on the money market then requires that the price level will decrease, the nominal money supply will increase, or the interest rate will go up. Austrians would prefer to keep the nominal money supply constant, and the price level should automatically fall, whereas from the New Keynesian perspective, there should be an automatic flow of nominal money from the central bank, and the price level may be stabilized.
If real money balances ( $\mathrm{M} / \mathrm{P}$ ) are a luxury good, the elasticity of the real demand for money with respect to real (permanent) income is greater than one. In equation (1), this elasticity is exactly one. Hence, to account for higher elasticity in this equation, " $k$ " must be increasing along with $\mathrm{Y}^{\mathrm{P}}$. Velocity will be then decreasing with higher $\mathrm{Y}^{\mathrm{P}}$. However, if we assume unitary income elasticity of the demand for money, rising natural output has no impact on the velocity of circulation, since growing $\mathrm{Y}^{\mathrm{P}}$ is not reflected in the change in " k ".

It can be said that the assumptions of the unitary income elasticity of the demand for money as well as the flexible price level stay behind the Austrian idea of the neutrality of the secular deflation in the expanding economy. Changes in the money supply may disrupt the natural equilibrium that would otherwise prevail in the barter economy, i.e. in the economy that does not use money. One possible explanation may be found in Hayek (1933b:160). An increase in the money supply implies a demand for goods without the previous supply of goods. On the other hand, withdrawing nominal money from the pockets of people and its subsequent destruction leads to a supply of goods without the demand.
The same idea holds for the (autonomous) change in the demand for money. Consider a sudden decrease in the demand for money owing to some external reason apart from income. This change is reflected in the decline in $\mathrm{k}(\ldots)$ in equation (1). The usual flows of expenditures are now supported by money which was previously held in the pockets of people. There is therefore a demand for goods without the supply. ${ }^{322}$ Such a change represents the same type of "shock" as the increase in the supply of money. Furthermore, according to equation (3), reduction in " $k$ " implies an increase in " $V$ ". The money market equilibrium (2)

[^194]then requires that either the price level rises, the interest rate falls, or the income grows (or any combination of these).

A sudden increase in the demand for money due to higher " k " has opposite effects - there is a supply of goods with no corresponding demand. Thus, it may have the same effects as the reduction in the money supply. From system (1) - (3), it can be seen that higher " $k$ " implies lower V. Furthermore, the entire body of the Austrian Business Cycle Theory may be applied to autonomous changes in the demand for money because reduction in the money demand may have the same effects as the increase in the money supply and vice versa. However, there might be significant differences due to specific "injection effects". Some of these were mentioned in Chapter 2, and a more subtle analysis can be found in Potuzak (2007).
As we can see, money represents a loose joint (Hayek 1941) that may lead to the violation of the Say's Law of Markets. According to Hayek (1935), the economy using money will behave as the barter economy only if the term MV remains constant. Only under this condition, the total flow of nominal incomes (PY) will not be affected by the monetary part of the economy, as can be seen on the right hand side of equation (3). Hence, not the constant money supply, but constant MV will preserve the "natural" system of relative prices. Exogenous shocks to velocity may have similar effects as the money supply changes displayed in Figures No. 1, 10, 17 , and 18. A sudden increase in the demand for money due to, for example, higher " $k$ " implies a reduction in velocity with similar effects as depicted in Figures No. 17 and 18.
As far as the economic growth is concerned, the monetary system will be neutral with respect to real economy if MV is constant. In such a case, increasing Y will be perfectly reflected in decreasing P. As a result, Hayek (1935) stressed that this norm would deliver stability to the market economy that is using money. Yet, he also added two more conditions that must be satisfied for money to stay neutral:
The true relationship between the theoretical concept of neutral money, and the practical ideal of monetary policy is, therefore, that the former provides one criterion for judging the latter; the degree to which a concrete system approaches the condition of neutrality is one and perhaps the most important, but not the only criterion by which one has to judge the appropriateness of a given course of policy. It is quite conceivable that a distortion of relative prices and a misdirection of production by monetary influences could only be avoided if, firstly, the total money stream remained constant, and secondly, all prices were completely flexible, and, thirdly, all long term contracts were based on a correct anticipation of future price movements. This would mean that, if the second and third conditions are not given, the ideal could not be realised by any kind of monetary policy. (Hayek 1935:131)

As we can see, constant MV must be, according to Hayek, supported by the flexibility of prices and correct expectations. Furthermore, Hayek did not explicitly recommend this rule for practical monetary policy. Nevertheless, let us further investigate theoretical basis and consequences of this proposal.

We start with the short-run analysis. Consider a sudden increase in the money supply resulting in the shift of the AD curve. Hayek MV-rule implies that the central bank should tighten the policy because MV is larger, whatever the impact of the initial increase in AD on output or the price level is. Figure No. 19 displays that higher MV is reflected in larger area PY $\mathrm{P}_{0} \mathrm{Y}_{1}, \mathrm{P}_{1} \mathrm{Y}_{0}$, or any other combination on the $\mathrm{AD}_{2}$ curve. As can be seen, this rule should
offset positive or negative AD-shocks coming either from the side of the money supply or the velocity of circulation. ${ }^{303}$


Figure No. 19, MV-rule stabilizing aggregate demand.

Thus, if the central bank is determined to hold MV constant, it must keep the area PY constant. But it means that the central bank should hold the nominal GDP - the total value of aggregate expenditure - at some definite level. Even though MV is not observable, data about PY are available, and they may provide the central bank a guideline to conduct this type of monetary policy.

At first glance, it seems that the monetary authority should operate at one particular aggregate demand curve. However, this is not the correct implication of the Hayek MV-rule, as the following example shows. If the economy is hit by a negative supply shock, the eventual value of aggregate income PY depends on the slope of the AD curve. If the AD curve is unit elastic, which may happen for unit-income-elastic and interest-inelastic demand for money, the total value of aggregate expenditure as well as MV are not affected. In such a case, the negative supply shock is evenly split between the reduction in output and increase in the price level. On the other hand, if the AD curve is flatter, the drop in real output will be significant with only a limited impact on the price level. Consequently, the nominal GDP drops as well as MV. To stabilize the nominal output and MV, central authorities ought to increase the money supply. This situation is depicted in Figure No. 20, panel (a). In the new equilibrium with lower nominal income, the central bank expands money supply to reach the MV-rule curve, representing all combinations of real income and the price level leading to identical nominal GDP. And finally, if the AD curve is rather steep (due to, for example, high interestelasticity of the demand for money leading to the instability of the velocity of circulation), the supply shock is mainly absorbed by the higher price level. Yet, the nominal aggregate income increases (because of higher velocity), and central authorities should reduce the

[^195]money supply to keep the economy at the MV-rule curve with constant nominal income PY (panel (b)). ${ }^{304}$
(a)

(b)


Figure No. 20, MV-rule and the negative supply shock

As we can see, the MV-rule has the property as if the $A D$ curve was unit elastic. It offsets $A D$ shocks and allows the supply shocks to fall evenly on output and prices. At first glance, it seems to create a firm theoretical basis for a sound monetary policy. Furthermore, as it is determined to hold the nominal income at a constant level, it is a specific form of policy that is known as the nominal income targeting. This proposal was widely discussed by economists, even the most distinguished ones, in the 1980s and 1990s. It was the era of flexible exchange rates, when the nominal anchor was sought after the collapse of the Bretton Woods system. ${ }^{305}$ It was the period of general dissatisfaction with monetary targeting, yet before the start of the new policy of inflation targeting and the invention of the Taylor rule. Nowadays, big supporters of this monetary policy design are known as "Market Monetarists". Hayek can be thus considered the founder of the idea of nominal income targeting, even though his motivation and theories were rather different from the ideas of proponents mentioned in the footnote above.
The main concern of the previous section was the expanding economy that might be disturbed by monetary forces. The MV-rule was designed as a proposal to secure neutrality of money. Let us now demonstrate how it may operate if the economy is on the balanced growth path. Figure No. 21 shows that the nominal income and thus MV is held constant in the expanding economy only if the AD curve is unit elastic. A more elastic AD curve leads to the expansion in the nominal GDP even when the money supply is held fixed ( $\mathrm{AD}_{\mathrm{A}}$ ). The economic background for this behaviour might be a secular increase in the velocity of circulation owing

[^196]to less-than-one income elasticity of the demand for money. The MV-rule then requires a reduction of the money supply to keep MV and nominal income at the constant level. On the other hand, the permanent increase in the natural income in the economy in which real money balances are a luxury good leads to a decline in nominal income since the drop in the price level is significant (point $B$ at $A D_{B}$ ). In such a case, greater natural output (and permanent income) is reflected in lower velocity, and the MV-rule suggests expanding the money supply to stabilize the nominal GDP and MV, which is depicted as the shift of $A D_{B}$ to $A D_{D}$.


Figure No. 21, MV-rule and the economic growth

In this connection, Hayek (1935) stressed one property of the money supply that is typical for a growing economy. The major part of the money supply is not created by the central bank and is never under perfect control of the central bank, at least in a free market economy and fractional reserve banking system. Hayek pointed out that an expanding economy tends to create more "inside" money even without the action of the central bank. This idea was partly discussed in Chapter 2. Various types of deposits, near-money, quasi-money, and other forms of liquid assets are being issued since the growth in real output and incomes brings about greater demand for money. The financial system itself may meet this demand by the creation of the above-mentioned forms of money. Hayek stressed that this expansion in assets must be checked by the central bank, otherwise the economy may jump on the path of an unsustainable boom. He recommended that the monetary base issued by the central bank should contract in order to offset inflationary tendencies in the economy. If we imagine the monetary system as an inverted pyramid, the forces of the economic growth along with the financial system tend to open the slope of this pyramid erected on the monetary base created by the central bank. Hayek's recommendation was to contract the base in order to stabilize the size of the pyramid.
However, if the financial system in the economy on the balanced growth automatically expands the amount of all forms of money and if the central bank should eliminate this process by the contraction in the monetary base, the ratio of money supply to the base money will be gradually increasing. The money multiplier will increase beyond all limits and the economy might end up as cash-less economy. Cash-less economy was a model used by Wicksell (1936), and one of the rules suggested by Friedman (1984) may converge close to
this state as well. However, it is doubtful whether the MV-rule would be operational in such an economy.

Hayek (1941) also pointed out that in the situation of endogenous expansion of financial assets, it is quite difficult to say whether it is the money supply or the velocity of circulation that is rising. The answer depends on the definition of money. At this point, Hayek followed Wicksell (1977). If money is defined narrowly enough, even the endogenous expansion of deposits may be defined as the increase in the velocity of circulation of pure cash. On the other hand, using a broad and still widening definition of money, one can say that the velocity is constant, but the money supply expands with no control of the central bank.
Hayek (1935) stressed that the easy expansion of all types of quasi-money in modern economies is caused by the general belief that they will be converted to cash if needed. Thus, the question is whether the inability of the central bank to control the money supply is not inbuilt in the framework the central bank itself created. This is also one of the reasons why many Austrian economists, but not only Austrian economists, believe in the virtue of $100 \%$ reserve banking (de Soto 2006). In their opinion, the expansion of the superstructure of various types of deposits could be eliminated in that system. Opponents of this proposal may object that it is always possible to invent a financial product that is not under $100 \%$-reserve regulation, but which is very close in nature to demand deposits. On the other hand, it can be replied that this discussion would require a deep investigation about the nature of money and the substitutes of money, because the core of the problem is whether money represents present goods or future goods and whether the creation of near-money is the exchange of present goods for future goods or an artificial creation of present goods. However, this discussion will not be carried out here.

It should be said that two points are rather disturbing in the theory presented above. According to Hayek, the money supply should not be increased if the natural output of the economy is growing. The increased demand for money resulting from rising incomes ought to be satisfied by a lower price level since the injection of money into the economy may start a boom-bust cycle. On the other hand, if the demand for money changes due to autonomous reasons - in other words, if the economy is hit by a velocity shock - the monetary authority should respond by an offsetting adjustment in the money supply. Otherwise, the business cycle might be triggered.
However, one may ask whether the injection of money, which is determined to satisfy higher demand for money, does not have similar disruptive effects as those presented in the case of the growing economy. Higher demand for money may lead to contractive pressures in the economy that are similar to the reduction in the money supply (Potuzak 2007). However, can the money growth be directed exactly to those parts of the economy that demand more money? What if the money influx is concentrated in sectors in which the demand for money has not increased?

The second inconsistent point might be found by comparing Figure No. 21 and Figure No. 10. The latter diagram suggests that the monetary expansion in a growing economy may provoke an artificial boom. The former illustrates the functioning of the MV-rule. As can be seen in Figure No. 21, the expansion of the aggregate demand from $A D_{B}$ to $A D_{D}$ dictated by the MVrule may lead to similar disturbance as in Figure No. 10 unless the movement from point B to point D is smooth.
These objections were not explored by Hayek, yet they cast serious doubts on the consistency of his theory. On the other hand, changes in the money supply, money demand (panel (b)), or IS-shocks (panel (a)) all affect the MV term. As Figure No. 22 shows, they all lead to the inconsistency between the actual and the natural rate of interest. As was stressed before, it is
this interest-rate gap which is a symptom of the discrepancy in the intertemporal allocation of resources and that is detected by the Austrian theory as the source of the business cycle. The MV-rule may check all these disturbances that move the aggregate demand curve by an offsetting shift in the LM curve that will push the interest rate (and output) back to the natural level.
(a)

(b)


Figure No. 22, IS, Ms, and Md shocks leading to a discrepancy between the actual interest rate and the natural rate.

Nonetheless, suppose that the MV-rule was implemented, business cycles were eliminated, and the economy is on its balanced growth path with constant growth rate of $\mathrm{n}+\mathrm{g}$, and a constant steady state natural rate of real interest $\mathrm{r}^{*}$. The key question is what the rate of secular deflation is in this economy. From equation (3), it can be easily derived that the rate of decline in prices is also $\mathrm{n}+\mathrm{g}$. In other words, the inflation rate in this economy is $\pi=-(\mathrm{n}+\mathrm{g})$. This price deflation should affect expectations of people if it is observed for an extended period of time. Hence, we may write: $\pi^{\mathrm{e}}=\pi=-(\mathrm{n}+\mathrm{g})$.
According to the Fisher equation $\mathrm{i}=\mathrm{r}+\pi^{\mathrm{e}}$, these deflationary expectations should put significant downward pressures on the nominal interest rate i. One may wonder what the resulting level of the nominal interest is. The previous relations imply that $\mathrm{i}=\mathrm{r}-(\mathrm{n}+\mathrm{g})$. In the very long run, the nominal interest rate in this economy will be positive only if the real rate of interest is higher than the growth rate of real natural GDP. As we know from Chapter 3, this is the condition of a dynamically efficient economy.

As a result, the Hayek MV-rule generates a positive nominal interest rate if the economy is dynamically efficient ( $\mathrm{r}^{*}>\mathrm{n}+\mathrm{g}$ ). If the economy operates at the golden rule level of capital accumulation ( $\mathrm{r}^{*}=\mathrm{n}+\mathrm{g}$ ), this rule implies zero nominal rate of interest. However, if the growth in real GDP is too high compared with the real rate of interest, the economy oversaves and is dynamically inefficient ( $\mathrm{r} *<\mathrm{n}+\mathrm{g}$ ) - the MV-rule fails because it implies a negative nominal rate of interest. It is generally believed that zero lower bound on nominal interest is a natural limit. However, if it is costly or dangerous to protect money, even negative nominal interest might be optimal (McCallum 2000:875).

On the other hand, it is also believed that real world economies do not over-accumulate capital (Abel et al. 1989). Hence, the Hayek MV-rule might be neutral with respect to the real economy in the very long run. Assuming reasonable price flexibility, the secular decline in prices will not be so rapid as to depress the nominal interest rate below a zero limit.

We can use one of the neoclassical growth models to exactly determine the nominal interest rate implied by the MV-rule. As was demonstrated in Chapter 3, the steady state real rate of interest in the Ramsey-Cass-Koopmans (RCK) model is $r^{*}=\rho+\theta$ g. Substituting this relationship to the Fisher equation ( $\mathrm{i}=\mathrm{r}+\pi^{\mathrm{e}}$ ) and using the fact that the rate of inflation in the MV-rule is $\pi^{\mathrm{e}}=\pi=-(\mathrm{n}+\mathrm{g})$, we get:

$$
\begin{align*}
& \mathrm{i}^{*}=\rho+\theta \mathrm{g}-(\mathrm{n}+\mathrm{g})  \tag{4a}\\
& \mathrm{i}^{*}=\rho-\mathrm{n}-(1-\theta) \mathrm{g} \tag{4b}
\end{align*}
$$

The right-hand side of equation (4b) must be positive ( $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$ ) in the RCK model, otherwise the lifetime utility will diverge. As we know from Chapter 3, this condition also guarantees normal behaviour of many other economic phenomena. As a result, if condition (4b) holds, the economy is dynamically efficient ( $\mathrm{r}^{*}=\rho+\theta \mathrm{g}>\mathrm{n}+\mathrm{g}$ ), and the nominal interest rate in the Hayek MV-rule is positive. Appendix 7 in Chapter 3 discussed in great detail various combinations of the time preference, real interest rate, nominal interest rate, and the growth in potential output. It was concluded that the time preference might be negative, real interest rate might be negative as well, yet the nominal interest rate under constant MV is always positive in the RCK model. The economic reason for this conclusion is as follows. Consider a rapid growth in technological progress. This should lead to a high rate of deflation in the MV-rule. However, high rate of technological progress also raises the steady state natural rate of interest. Thus, these two tendencies may operate against each other in their impact on the nominal interest rate. On the other hand, if the technological progress is negative, there is a downward pressure on the real natural interest rate. Yet, the inflation under the MV-rule will be positive which will drive up the nominal interest.


Figure No. 23, MV-rule and parameters generating positive nominal interest. ( $\mathrm{n}=0 \%$ )

Figure No. 23 displays various combinations of the relative risk aversion $\theta$, subjective discount rate $\rho$, and the rate of technological progress $g$ that are consistent with the convergence of life-time utility in the RCK model. Equation (4b) implies that if the technological progress is positive, the coefficient of the relative risk aversion requires: $\theta>1+$ $(n-\rho) / g$. If it is negative, then the opposite condition holds: $\theta<1+(n-\rho) / g$.
We will focus on positive technological progress since we are examining a growing economy. As we can see, the higher the time preference $\rho$, the larger the region of positive nominal interest rate. Assuming diminishing marginal utility ( $\theta>0$ ), high impatience (high and positive $\rho$ ) guarantees that the economy may not fall to a negative interest rate environment even for very high rates of technological progress and hence price deflation. Allowing for negative time preference (e.g. $\rho=-3 \%$ ), if the technological progress is sluggish, a very low elasticity of substitution (high $\theta$ ) is needed to preserve a positive nominal interest rate under the MV-rule.

Even though the nominal interest rate in the MV-rule might be positive, it would surely be rather low. As is well known, Milton Friedman (1969) derived that the optimum quantity of money is the point at which the real return to money is equal to the real return to other assets. He concluded that this implies zero nominal interest rate $\mathrm{i}=0 \%$. If the real interest on other assets is positive ( $r>0$ ) due to, for example, positive time preference, ${ }^{306}$ the Fisher equation implies that the rate of (expected) deflation must be equal to this real interest rate ( $\mathrm{r}=-\pi^{\mathrm{e}}$ to obtain $\mathrm{i}=0 \%$ ). If the economy is stationary and the velocity is constant, Friedman rule requires the money contraction at the rate of that particular real rate of interest. The Hayek MV-rule would not be so drastic. In the stationary economy with constant MV, inflation would be zero, so the nominal interest will be equal to real interest.

## State of the Economy

Hayek MV-Rule

| Nominal |
| :---: |
| interest rate |
| $i>0 \%$ |

$i=0 \%$
$i<0 \%$

Growth in
Money Supply
$g_{M}=0 \%$
$g_{M}=0 \%$
$g_{M}=0 \%$

## Friedman Rule

\(\left.$$
\begin{array}{cc}\begin{array}{c}\text { Nominal } \\
\text { interest rate } \\
i=0 \%\end{array} & \begin{array}{c}\text { Growth in } \\
\text { Money Supply } \\
g_{M}=-[r- \\
(n+g)]<0 \%\end{array}
$$ <br>

i=0 \% \& g_{M}=0 \%\end{array}\right]\)| $g_{M}=-[r-$ |
| :---: |
| $i=0 \%$ |
| $(n+g)]>0 \%$ |

| Hayek MV-Rule | Friedman Rule | Compariso <br> "Inflation" Rate <br> $\pi=-(n+g)$"Inflation" Rate <br> of deflation |
| :---: | :---: | :---: |
| $\pi=-(n+g)$ | $\pi=-r$ | $\mathbf{F}>\mathbf{H}$ |
| $\pi=-(n+g)$ | $\pi=-r$ | $\mathbf{F}=\mathbf{H}$ |
| $\pi$ | $\mathbf{H}>\mathbf{F}$ |  |

Dynamic Efficiency
r>n+g
Golden Rule
$\mathbf{r}=\mathbf{n}+\mathbf{g}$
Dynamic Inefficiency $\mathbf{r}<\mathbf{n}+\mathbf{g}$

Dynamic Efficiency
r>n+g
Golden Rule
$\mathbf{r}=\mathbf{n}+\mathbf{g}$
Dynamic Inefficiency $\mathbf{r}<\mathbf{n + g}$

State of the Economy

Figure No. 24, Hayek MV-Rule (H) and Friedman rule (F) for the economy on the BGP and constant velocity of circulation of money.

[^197]If the economy is growing, the Friedman rule does not require such a drastic contraction in the money supply to achieve required deflation and hence zero nominal interest. ${ }^{307}$ If the growth in real income is equal to the real interest rate, and the economy is therefore at the golden rule, Friedman rule will coincide with the MV-rule because, in such a case, it implies constant money supply. And finally, if the economy is dynamically inefficient because growth in real income exceeds the real interest rate, the deflation would be too rapid for the Friedman rule, so it implies positive money expansion to reach zero nominal interest. As we know, in such a case, the Hayek MV-rule fails. Schema in Figure No. 24 summarizes differences between these two rules.

As was argued by Friedman (1969), very low nominal interest rates may significantly increase the real demand for money. This should be the case for the Hayek MV-rule too. Especially in the transition period from a positive (or zero) inflation environment to the economy with secular deflation, the decline in prices might be quicker than implied by the positive growth in GDP on the BGP. If the real demand for money negatively depends on the nominal interest rate and if the nominal money supply is held fixed, the price level must further decrease to equilibrate the money market. People will then hold the required amount of real money balances. As a result, the transition to the MV-rule by stopping the monetary expansion will lead to a reduction in velocity. The MV-rule ought to reflect these changes. However, it is rather hard to prescribe the optimum path of money supply in such a case.
Figure No. 25 is focused on this transition period. At time $t_{0}$, the expansion of the money supply halts (panel (a)). The economy is continuously growing at the rate of $n+g$ (panel (b)). Before the change in monetary policy, the real money balances $\mathrm{M} / \mathrm{P}$ were growing at this particular rate because the money supply M was perfectly accommodating this growth in output, and the price level P was constant; hence, the central bank hit its target of zero inflation. Assuming perfect flexibility of prices in a perfect neoclassical world with neutral money, the sudden cessation of the money supply growth leads to a drop in the inflation rate and then in the nominal interest rate. The interest sensitivity of the real demand for money will lead to the fact that the price level must drop more than is implied by the growth in natural GDP. Thus, the real money balances will be higher, but in the new steady state they will be growing at the same rate $n+g$.


Figure No. 25, Dynamics of the inflation rate in the transition to the MV-rule.

[^198]As can be seen, in the transition period, money is not super-neutral. ${ }^{308}$ The change in the money supply growth affects the real demand for money, i.e. a real variable. The velocity of circulation is temporarily lower, which should be offset by the MV-rule. However, this response is not reflected in the evolution of the money supply displayed in panel (a). The resulting dynamics of the inflation rate is depicted in panel (c). An open question is whether the overshooting of deflation in the transition period will not depress the nominal interest rate below zero. On the other hand, there may operate one specific effect - Mundell-Tobin effect — that could protect the economy from falling into this trap. McCallum (2000:876) argued that a very low nominal interest rate and the resulting increase in the real demand for money implies that people may substitute money for real assets. In Austrian terms, this means that the demand for present goods (money) increases at the expense of the demand for future goods (flow of services from these real assets). As a result, the real interest rate should increase. The drop in the money supply expansion will bring about a lower decrease in nominal interest rate than is the decline in the rate of inflation because the real (natural) rate of interest will increase. ${ }^{309}$ This channel represents another example how the classical dichotomy might be broken. However, McCallum (2000:877) added that the drop in the inflation rate from $2 \%$ to $0 \%$ will increase the steady state real interest by an insignificant amount.

In the above discussion, we assumed the economy being on the BGP. For the constant MV, inflation rate was negative, yet the nominal interest rate did not drop below zero. Another question is the implication of the MV-rule in the economy that is not on the BGP. One may argue that the economy that is converging to its steady state (towards its BGP) grows faster than $(\mathrm{n}+\mathrm{g})$ due to rapid accumulation of capital. Thus, the resulting deflation under the MVrule should be even higher, which may depress the nominal interest rate below zero. Yet, this argument is not valid because the converging economy is also characterised by a higher real rate of interest that is gradually falling to its steady state level. As a result, the deflation in such an economy is faster, but the real rate of interest is higher as well. The nominal interest rate may be even higher than on the BGP. Such a conclusion is supported by the RCK model presented in Appendix 7 in Chapter 3. As can be seen in Figure No. 17_A7 and No. 16_A7, the nominal interest rate in the converging economy is above the steady state level even though its fall is more rapid than the decline in the real interest rate due to the rapid growth in real GDP and hence the high rate of price deflation.

With a rather low nominal interest rate and secular price deflation generated by the MV-rule, another important question is the evolution of nominal wages. A ubiquitous belief may be found in the economic science that nominal wages are rather rigid (Mankiw and Romer 1991a). Under the MV-rule, the price level on the BGP will be falling at the rate of ( $\mathrm{n}+\mathrm{g}$ ). It can be shown that in simple neoclassical growth models, the growth rate of real wages on the BGP is equal to the growth rate of technological progress g . As a result, if nominal wages

[^199]were flexible, the MV-rule would lead to a secular decline in nominal wages at the rate of $n$ - the growth rate of population and labour force. ${ }^{310}$

This rule will therefore require a secular decline in nominal wages. The question is whether the employees would accept this falling pattern. Even though " n " might be very low in modern societies, it may still cause a problem for the MV-rule. Labour market clearing condition would require permanently decreasing nominal wages, yet the sticky wages may prevent this equilibrium to be achieved. Thus, due to the rigidity of wages, the Hayek MVrule may lead to a secular unemployment rate that is higher than the "natural" rate.
On the other hand, employees may accept stability in nominal wage if they got used to the environment of falling prices of consumption goods. Thus, the rigidity of wages is not a problem if the natural result of the monetary policy design is the stability in nominal wages. The Hayek MV-rule should be then adjusted such that the expansion in the money supply must reflect the population growth n . If the growth in population is low (as it was in the last several decades), the growth in money supply should be negligible. Yet, in such a situation the most straightforward and simple policy would be to keep the money supply constant. ${ }^{311}$

In this connection, it should be stressed that Hayek (1933:161) himself concluded that for practical purposes the optimal monetary policy might be to stabilize prices of the factors of production. A similar proposal can be found in Friedman (1969:46). It is quite surprising that both economists suggested the same optimum policy even though they based their recommendations on different theories. Moreover, Woodford (2003:14) argued that the price index in the inflation target (the optimum of which is zero) should mainly comprise of goods with rigid prices, and the sticky wages.
Optimum monetary policy that would allow for a secular decline in prices in the expanding economy was also explored by Selgin (1997). He recommended a similar rule as discussed here that he termed "productivity norm". He argued that the fall in the price level should correspond to the growth in total factor productivity TFP. The second option, according to Selgin, was the rate of deflation that reflects growth in GDP per worker. Under the second option, the deflation will be more rapid.
However, it can be shown that the mathematical analysis presented in Selgin (1997) is not accurate especially as far as the consistency of the growth theory implications is concerned. He did not realise that the technological progress should be considered as labour augmenting, which will modify his growth accounting equation. However, a thorough critique of the Selgin approach will be presented in a different paper. Yet, we can conclude that his proposals would lead to the following results.
In option No. 1 - deflation equal to growth in TFP - it can be shown that the deflation would be $(1-\alpha) \mathrm{g}$, where $\alpha$ is the capital share of output, because $(1-\alpha) \mathrm{g}$ is the growth rate of TFP in a well-behaving neoclassical growth model. Assuming constant velocity, this implies the expansion of the money supply at the rate of: $(\mathrm{n}+\mathrm{g})-(1-\alpha) \mathrm{g}=\mathrm{n}+\alpha \mathrm{g}$. ${ }^{312}$ As can be seen, this rule will be even more "inflationary" than the proposal designed to stabilize nominal wages. In this case, nominal wages will be growing at the rate of $\alpha g$.

[^200]In option No. 2 - deflation equal to growth in output per worker - the deflation should reflect $g$ because the only source of the increasing labour productivity on the BGP is the technological progress. The resulting money expansion is then: $(\mathrm{n}+\mathrm{g})-\mathrm{g}=\mathrm{n}$. Thus, the second Selgin proposal would lead to the stabilization of nominal wages on the BGP. As regards the converging economy, his proposal would lead to a more rapid deflation owing to the fact that the labour-productivity is supported by a rapid accumulation of capital.

### 3.1 THE MOST SERIOUS PROBLEMS OF THE MV-RULE

In the previous section we mentioned several problems of the MV-rule. Let us now extend this list by an additional six. In the first place, V in the MV-rule represented income-velocity of money. However, this is not the only possible option. In Chapters 1 and 2 of this dissertation, it was stressed that the Austrian theory also takes into account processes that are usually hidden in the national income accounting and in macroeconomic theory in general exchanges of goods at various stages of the production process. Moreover, the Austrian theory of capital, which is the core of the Austrian theory of business cycle (ABCT), considers mainly goods in process - circulating capital in modern understanding - as the genuine form of capital. The key economic (calculation) problem is then the allocation of various types of material resources among thousands of possible production opportunities with thousands of possible production techniques. The Hayek triangle itself is a simple graphical representation of this understanding of capital.
As a result, one may extend the amount of goods in our analysis to include various forms of intermediate products. The aggregate expenditures on these goods are much larger than those on GDP. This was stressed by Hayek (1935) in citing the error of A. Smith who thought that the expenditure on final goods must be higher than on intermediate products since the latter is always sold for a lower price than the former. As Hayek pointed out, A. Smith did not take into account that intermediate goods may be purchased several times and in many forms before they end in the hands of consumers.

Although the common practice is to disregard these transactions by referring to the error of double counting, Chapters 1 and 2 tried to show that considerations about goods in process are at the centre of both the theory of capital and the business cycle. As a result, to be consistent within the Austrian framework, the equation of exchange may reflect the entire output of all goods and services, not only the final ones:

$$
\begin{equation*}
\mathrm{M}_{\mathrm{s}} \mathrm{~V}_{\mathrm{Q}}=\mathrm{PQ} \tag{5}
\end{equation*}
$$

PQ stands for the total value of expenditure on all goods and services produced in the economy within a certain period of time, and $\mathrm{V}_{\mathrm{Q}}$ represents "total output" velocity of money. By comparing equations (3) and (5), it is obvious that $\mathrm{V}_{\mathrm{Q}}$ is much higher than V (or $\mathrm{V}_{\mathrm{Y}}$ ), which is known as the income velocity of money. The total amount of money must be transacted more times to finance the given value of total output of all goods - PQ is much larger than PY , hence $\mathrm{V}_{\mathrm{Q}}$ exceeds $\mathrm{V}_{\mathrm{Y}}$.

Hayek (1935) argued that for money to retain its neutrality with respect to the real economy, the total flow of income (i.e. PY in our equations) must remain constant. However, consider a monetary expansion that is directed, as is usually believed in the ABCT, to sectors producing capital goods, i.e. mainly goods that are used in earlier stages of the production process. The Hayek triangle will expand ( $\mathrm{M}_{\mathrm{s}} \mathrm{V}_{\mathrm{Q}}$ grows), as was shown in Chapter 2; however, in the initial periods, the total flows of income may not be affected. In the data, this process would be reflected as follows: the money supply expands, but the total expenditure on final goods and
services, i.e. total income, is almost constant. As a result, the calculated income velocity of circulation declines. The term $\mathrm{MV}_{\mathrm{Y}}$ is constant as well, so the MV-rule may suggest that there is no need to adapt the monetary policy. Yet, the logic of the Hayek MV-rule would require a monetary restriction because the economy is on the path of unsustainable boom. Expansion of the early stages of production took place, and $\mathrm{M}_{\mathrm{s}} \mathrm{V}_{\mathrm{Q}}$ rises; hence, the economy should be cooled off.

The story above did not indicate what the source of the expansion was. It could have been an endogenous increase in deposits that were created to meet the enlarged demand for loanable funds caused by an exogenous increase in the productivity of capital. As was demonstrated in Chapter 2, Hayek blamed the banking sector that it does not increase the interest rate fast enough in response to higher demand for loanable funds, and instead of this it accommodates this demand by the expansion of credit and deposits. The MV-rule was then designed to check this excessive expansion. Yet, as we can see, this rule itself seems to be blind with respect to monetary expansion if it is determined to hold the $M V_{Y}$ (rather than $M V_{Q}$ ) term constant.

As a result, to take into account the ABCT considerations - processes within the structure of production, and the discrepancies in the intertemporal allocation of goods (intermediate and final) caused by the monetary forces - the $\mathrm{MV}_{\mathrm{Y}}$-rule should be redefined as the $\mathrm{MV}_{\mathrm{Q}}$-rule. It should reflect not only total expenditure on final goods and services (e.g. bread), but also the total value of expenditure on all newly produced goods and services (e.g. wheat, flour, and bread). As such, it would cease to be a derivative of the nominal income targeting. It might be redefined as the nominal total output targeting.

The monetary policy conducted according to this newly defined rule would be rather difficult since the modern national income accounting is mainly designed to thoroughly measure processes within the GDP, which is, however, only a fraction of total output of all goods and services. By realizing this new problem, the Hayek design is moving even further away from being considered a good candidate for a practicable monetary policy. Moreover, if the amount of expenditure was extended to comprise all transactions (even with used goods, financial assets, etc.), V could be redefined as the transaction velocity of money $\mathrm{V}_{\mathrm{T}}$, and the third possible rule - $\mathrm{MV}_{\mathrm{T}}$-rule - would be really hard to implement because the total value of all transactions in the economy is impossible to measure.
Three additional problems will be mentioned only in brief. It is not clear whether the MV target is to be absolutely fixed for all times, or if the base was moving over time, the monetary authority should always return the nominal income to its previous level. If the rule operates only with respect to the previous period, it is more flexible, yet it may become less credible. The virtue of flexibility is that from time to time, it might be really hard for the central bank to tighten the policy in the recession and vice versa. Furthermore, the fundamental approach of ever returning the nominal income back to the definite level may destabilise rather than stabilise the economy due to unstable and variable lags between the moment the problem was indicated and the moment the monetary policy affects the economy. As a result, the monetary policy itself may be the key destabilising element of the MV term. An alternative would be to predetermine the path of nominal income or to set a definite growth rate of nominal income.
If the latter is adopted, the MV rule turns into a nominal income targeting (NIT) regime, and all the objections raised against this design might be applied. A fundamental one is that the lag between the policy action and the impact on output differs from the lag between the policy action and the impact on inflation. This asymmetry may lead to a very poor performance of the NIT, as Rudebusch (2002) pointed out.

Another note is that in contrast to the inflation-targeting regime, in which the inflation expectations might be anchored, the MV-rule does not set any inflation or deflation target.

The secular deflation would depend on the growth in potential output, which may be changing over time. As we will see in the next section, technological shocks may lead to instability of the deflation rate under this rule, as well as various nominal AS-shocks. On the other hand, if the policy was credible and successful in the past, people might expect that prices will be falling at some definite rate, and this rate would not be prone to serious fluctuations unless the economy was exposed to significant real and nominal AS-shocks. However, these shocks would deteriorate the performance of the inflation targeting as well.
The two points we consider at the end of this section are closely connected to the discussion in section 5 below. As we demonstrated before, the MV-rule will result in a rather low nominal rate of interest. However, the previous discussion was about a smooth path of the economy on its BGP. Let us now present the short-run analysis. In other words, we will explore the performance of this rule if the economy, which operates close to zero nominal rate of interest, is hit by aggregate demand shocks.


Figure No. 26, BGP and the secular deflation in the IS-LM-Y* model

Figure No. 26 presents an economy under the MV-rule in the IS-LM-Y* model. Since there is a secular deflation in the economy, the nominal interest rate, for which the money market equilibrium - LM - is derived, does not coincide with the real interest rate, for which the IS curve is derived either from the goods market, the loanable funds market, or the consumption Euler equation. As a result, the intersection of the IS curve with the natural output $\mathrm{Y}^{*}$ does not portray the natural real rate of interest (panel (a)); the real rate must be found in a separate model (panel (b)).

Furthermore, as can be seen in panel (a), changes in the expected inflation rate that result in a change in the nominal interest rate do not affect the LM curve, but they move the entire IS curve since it depends on the real interest rate, not on the nominal interest rate. IS IT represents a position of the IS curve in the economy in which zero inflation target (IT) is set, whereas

IS $_{\text {mv }}$ stands for the goods market intertemporal equilibrium in the economy that adopted MVrule with secular price deflation. As can be seen in panel (a), the transition process from IT to MV-rule could have been accompanied by recession due to the fall in inflation expectations that led to the shift in the IS curve and to the reduction in the velocity as the economy moved along the $\mathrm{LM}_{\text {past }}$ curve. ${ }^{313}$ However, Figure No. 26 is designed for the BGP at which all frictions were overcome in the past, and the economy is therefore on the smooth path of the expanding $\mathrm{Y}^{*}$ curve and the IS curve.
Notice that there is not a problem with deflationary expectations. They are perfectly reflected in the lower nominal rate of interest. So the actual real rate of interest is at the natural level. We can show that the "layman" reasoning that the expected deflation should lead to the reduction in present consumption is flawed. According to this popular theory, if people expect lower future prices of consumption goods, they may postpone their purchases, which will depress the economy in the short run. The potential output is growing; however, nobody is buying consumption goods as all people are waiting for the future for even cheaper goods and services.

Let us demonstrate that this argument is absolutely naïve. First of all, in Chapter 3 it was clearly shown that it can never be optimal to postpone all consumption to the future, regardless of the size of the time preference (in sense two, i.e. $\rho$ ) or the real rate of interest $r$. It was clearly shown that the necessary break is performed by the law of diminishing marginal utility. The optimum consumption path is always chosen to obey the Euler consumption equation. It could be optimal to consume more in the present, or in the future. The optimum decision depends on the difference between the real rate of interest, which reflects the impatience of the entire market, and the impatience of that particular individual (i.e. his personal subjective discount rate), and the intertemporal elasticity of substitution in consumption.

Deflation in the price level is a monetary phenomenon that has no impact on the real interest rate (apart from the above-mentioned Mundell-Tobin effect) and hence the optimum flow of consumption over time. The reason is as follows. Firstly, the same idea, as presented above by the supporters of the theory that expected deflation is detrimental to present consumption spending, may be applied to any positive real rate of interest. If the person knows that one unit of real income could purchase more consumption goods in the future if not consumed today, why should this individual consume today at all? This is the same logic as the one used by the fighters against deflation (and unfortunately also by many central bankers) - the banknote should not be spent today because it can purchase more in the future. But what is the necessary break that will overcome this tendency to postpone everything to the future? The answer is the time preference and the law of diminishing marginal utility. No person will postpone everything to the future, even when the real interest rate is positive, since he is impatient. No person will postpone every unit of income to the future, because it is not optimal to starve today and live in great abundance in the future. As we know from Chapter 3, the optimum allocation of consumption over time is orchestrated by the Euler equation, not by considerations about deflation.

The fighters against deflation therefore confuse the fact that if they are talking about the expected reduction in prices, they are actually talking about the real rate of interest. The increase in expected deflation automatically means an increase in the real rate of interest unless the nominal interest rate falls too. But the problem is that according to the Fisher equation, the nominal interest rate must sooner or later be lowered as well. And the relation must be one to one. As a result, the increase in deflationary expectations has no effect on the

[^201]consumption spending if it is perfectly reflected in the fall in the nominal interest rate. The reason is that the real interest rate is not affected.

Moreover, let us assume that the fall in the nominal interest is not one-to-one with deflationary expectations, so the real interest increases. Even in this case, the present consumption need not dramatically decrease. The eventual outcome depends on the Euler equation and the resulting formula for the optimum present consumption. It can even happen that the income effect dominates the substitution effect, and the saver may feel richer with a higher real interest rate. As a result, if the decline in prices is expected, the representative lender with very strong preference for consumption smoothing (high $\theta$ ) may realize that he can consume more in the future, but he will consume more even today due to the preference for a balanced consumption, which results in the dominance of the income effect over the substitution effect.

To take another extreme, suppose that the nominal interest rate falls to the negative region, presumably due to the fact that it is too costly and dangerous to protect money. Suppose that the expected inflation is less negative (deflation is lower) than the negative value of the nominal interest rate. In such a case, the real interest rate is negative and the lender with high elasticity of substitution (low $\theta$ ) or a borrower will consume a lot today. Hence, expected deflation has no impact on his present consumption.
As we have shown, the very low nominal interest rate and secular deflation pose no problem for the real economy, present consumption spending, or investment spending if the actual real rate of interest is equal to the natural level. The only situation that may result in serious problems in the real economy is the zero lower bound on the nominal interest rate. And this phenomenon presents the last potential drawback of the MV-rule we examine.

Consider a drop in the IS curve. This shift could be caused by a fear of a sudden decrease in future income. This assumption is quite inconsistent with the fact that the potential output is growing over time. However, it may clarify the problems of the MV-rule. As can be seen in Figure No. 27, the downward shift of the IS curve decreases the natural real rate of interest from $\mathrm{r}_{\text {nat,Ss }}$ to $\mathrm{r}_{\text {nat,IS-shock, }}$, but not necessarily below zero (see panel b). Nevertheless, the interest rate that is to be below zero is the nominal rate of interest (a drop from " $\mathrm{inatat}, 1 "$ to " $\mathrm{i}_{\text {nat }, 2}$ " in panel a). At this point, the economy has the property of being dynamically inefficient because the real rate of interest is lower than the actual growth rate of the natural output. However, people do expect deterioration in future income, which is why the natural interest rate has fallen down. With respect to the RCK model and the formula of the steady state real interest - $r^{*}=\rho+\theta g$ - there are two versions of $g$ in this situation. The first one is positive and raises the potential income. The second one is negative and is expected by people. Thus, there is an inconsistency between the driving force of the growth in potential output and the expectations of people about their future income. Nevertheless, this error in expectations might cause serious problems, as we presently see.
Before we discuss macroeconomic consequences of this sudden drop in the natural rate of interest, let us present microeconomic reasons that lie behind it. Figure No. 28 presents a typical consumer in this economy. He is impatient ( $\rho>0$ ) because the slope of the indifference curve at the $45^{\circ}$ line is higher than one. Yet, the real interest rate r in this economy is even higher than $\rho$ because the profile of (labour) incomes of the majority of people is increasing ( $\mathrm{W}_{2}>\mathrm{W}_{1}$ ). This property therefore raises impatience of people in the economy. Because the real interest rate (rss) is greater than the subjective discount rate ( $\rho$ ), the time shape of optimum consumption is increasing $\left(\mathrm{C}_{2} *>\mathrm{C}_{1}{ }^{*}\right)$.



Figure No. 28 Representative Fisherian consumer

However, this consumer is a saver $\left(\mathrm{W}_{1}-\mathrm{C}_{1} *\right)$. The saving is used to finance capital accumulation in this economy, and it is larger than dissaving of the old generation. Obviously, our previous discussion was within the infinite horizon RCK model; however, the two-period Fisher model we use at this moment is better designed for an OLG structure. ${ }^{314}$ Since incomes of all generations are growing over time, the distance of dissaving of this generation in the future $\left(\mathrm{C}_{2} *-\mathrm{W}_{2}\right)$ will be lower than saving of the next young generation $\left(\mathrm{W}_{2, \text { young }}-\mathrm{C}_{2, \text { young }}{ }^{*}\right)$, which is obviously not shown in the picture.
Let us now assume that this individual expects a sudden drop in his future (labour) income $\left(W_{2, \text { fear }}<\mathrm{W}_{2}\right)$. Figure No. 29 shows that the budget line shifts inwards, and the individual will save more $\left(\mathrm{W}_{1}-\mathrm{C}_{1}\right.$, fear $\left.*\right)>\left(\mathrm{W}_{1}-\mathrm{C}_{1} *\right)$. If this behaviour can be generalized to the rest of the active population, the aggregate saving in the economy will drastically increase. This effect will be magnified if the elasticity of substitution of people is low (high $\theta$ ). As we already know from Chapter 3, this implies a strong preference for consumption smoothing. Thus, the fear of low future consumption may lead to a reduction in present consumption. Parameter $\theta$ also represents the coefficient of the relative risk aversion. If it was endogenous, the fear of future conditions might make the indifference curves more curved - the consumer would become more risk-averse.


Figure No. 29 Increase in saving resulting from expected drop in future income

Since the aggregate saving increases in the economy, the natural rate of interest is depressed, even below zero. ${ }^{315}$ The fall in the natural interest is positively related to parameter $\theta$ - the higher the preference for consumption smoothing (i.e. the lower the elasticity of substitution),

[^202]the higher the increase in saving. ${ }^{316}$ Figure No. 30 represents the final situation of this individual. The budget line is flatter due to a lower interest rate. Present consumption is partly enhanced. Panel (a) assumes full adjustment of the real interest rate to its natural level, and the nominal interest rate is therefore negative. Panel (b) is constructed for a higher real interest rate when the zero lower bound on nominal interest is binding.


Figure No. 30 Equilibrium of the Fisherian consumer; full recovery (a), ZLB recession (b)

[^203]To be consistent with the macroeconomic level (panel (a) in Figure No. 27), we plotted one more budget constraint that reflects the fact that the present income of people is lower due to the negative demand shock and the resulting negative output gap. As can be seen in panel (b) of Figure No. 30, the ZLB generates the paradox of saving. People saved more and lowered present consumption. The natural rate of interest fell down, but because the actual real rate is higher than the new natural level owing to the ZLB (liquidity trap in the Keynesian system), incomes are depressed, and people end up with even lower saving ( $\mathrm{W}_{1, \mathrm{Y} \text {-gap }}-\mathrm{C}_{1, \mathrm{Y} \text {-gap }}{ }^{*}$ ) $<\left(\mathrm{W}_{1}\right.$ $-\mathrm{C}_{1, \text { fear }}{ }^{*}$ ) and much lower consumption. In panel (a), on the other hand, such a drop in present income is not displayed since the economy is at its potential level owing to the coincidence between the actual and the natural real rate of interest. The increase in saving in panel (a) does not have depressionary implications as in panel (b), because investment spending was supported by a drop of the actual real rate of interest to the natural rate.

There is one critical point that must be stressed. First, the expected drop in future labour income may not be realized. Thus, there might be a difference between the intertemporal budget constraint the consumer trusts and which collapsed in Figure No. 29, and the "true" budget constraint that did not collapse because the technological progress is positive. But what is important is the belief of the consumer because it affects his current behaviour. If he believes in the drop in future income, he expands his saving, which may, if this behaviour is universal in the economy, shift the IS curve inwards and depress the natural rate of interest.
Let us recall Figure No. 27 that represents this specific economy at the macroeconomic level. The natural real rate of interest is so low that for the given deflationary expectations dictated by the MV-rule, the nominal interest rate must be negative. ${ }^{317}$ But suppose that the zero lower bound (ZLB) on the nominal interest rate is binding, and the nominal interest cannot fall below zero to equilibrate intertemporal markets. Neither the $\mathrm{LM}_{\mathrm{BGP}}(\mathrm{P} 1)$ curve that reflects constant price level, nor the $\mathrm{LM}_{\text {adjusted }}$ curve, constructed for a lower price level, is the active one. The money market is frozen at zero nominal interest. So the actual active "LM curve" is the horizontal line at zero. But at this rate, the actual real rate of interest is higher than the natural rate ( $r_{\text {ZLB }}>r_{\text {nat,IS-shock }}$ in panel (b) in Figure No. 27). There is a depressing tendency in the economy that could be healed, if the ZLB was not binding, by a reduction in the price level ( $\mathrm{LM}_{\text {adjusted }}$ ), the consequent decline in the nominal interest, and the resulting fall in the actual real interest rate back to its natural level.

But this process is blocked. Thus, this is the only point at which the deflationary expectations may cause serious problems to the economy. If the price level is expected to fall, the nominal interest rate should fall as well to keep the real rate at the previous level. However, the nominal interest rate hits the ZLB in our case. The real interest rate therefore rises with the reduction in inflation (increase in deflation). This may in turn cause the fall in present consumption and especially in the investment spending. ${ }^{318}$

[^204]Unfortunately, this is not the end of the story. As we can see in Figure No. 27, there is a negative output gap in the economy. As was indicated above, this will create further deflationary pressures. So in the next round, the deflation will be more rapid than is implied by the MV-rule (i.e. $-(\mathrm{n}+\mathrm{g})$ ). If this deflation will convert to expected deflation, the IS curve will be pushed even more to the left. The negative output gap will expand, which will result in even larger deflationary pressures. The New Keynesian recommendation in such an unfortunate state is to expand the IS curve by, for example, radical fiscal expansion, depreciation of the exchange rate (McCallum 2000), or the central bank may try to depress longer term interest rates by purchasing long-term financial assets (Orphanides and Williams 2002; Orphanides 2004). Apart from these, the IS curve might be also enhanced if inflation expectations of people are raised (Krugman et al. 1998). Thus, people must believe that future prices (inflation) will go up. This might be achieved, apart from the above-mentioned policy actions, by a huge expansion of the monetary base, by the announced enduring monetary expansion after the recession is over (Jung et al. 2005), or by a price level target (Wolman 2005) that is gradually adjusted upwards in the period of ZLB (Eggertsson and Woodford 2003). All these might be integrated to the Hayek MV-rule to improve its performance in the ZLB since the MV-term is also collapsing in this specific environment.
As we have seen, the original impulse for the drop to this specific type of liquidity trap was the initial inward shift of the IS curve, i.e. a negative demand shock. This shock was provoked by dismal expectations of people about future incomes, even though the potential output was rising. A surprising outcome is that these gloomy prophecies were realized. Hence, in the ZLB environment, the economy might be trapped in self-fulfilling prophecies. What is more important, the reasons for negative predictions about future incomes were not explicitly presented. They could have been provoked by curious news of the sunspot type. As a result, the economy close to ZLB might be also trapped in sunspot equilibria.
The chance of being trapped in this crazy environment obviously depends on the size of the IS-shock. But due to the fact that the economy operated very close to the zero lower bound (points of equilibria are close to zero in Figure No. 26), there was a much bigger chance that the ZLB would be hit for the given size of the shock. Hence, the existence of the zero lower bound and the danger of self-sustained "unsound" deflation and depression represent a possible problem for the MV-rule with secular "sound" deflation. On the other hand, Beckworth (2008:370) cited historical studies that documented that in the $19^{\text {th }}$ century the economy never hit the zero bound under the sound deflation that accompanied economic growth.

For the Austrian economists, the self-sustained destructive deflation (called secondary deflation in this tradition) is the outcome of the collapse of the unsustainable boom, misallocation of resources, losses in capital structures on a large basis, and the collapse of the superstructure of deposits created in the period of boom. And this boom may be, according to Hayek, caused by the effort of the central bank to stabilize the price level in a growing economy. Thus, the fear from "sound" deflation resulting in the expansion of deposits and the booming economy may lead to a sharp self-sustained "unsound" deflation, when these deposits collapse in the recession, i.e. when the boom bursts. The MV-rule was designed to prevent this boom-bust cycle. However, the potential fragility of the MV-rule should have been mentioned as well.

In the previous exposition, it was derived that the MV-rule may fail if the growth in potential output is higher than the real natural rate of interest. Such an economy is characterized as dynamically inefficient. The MV-rule then indicates a negative nominal interest rate. Let us now use the analysis of the ZLB presented above to demonstrate one theoretical possibility that might be called depression caused by a too strong growth. The possibility of the
expansionary effects of the negative supply shock (e.g. oil shock) under the zero lower bound was analyzed by Wieland (2012). His idea is exactly the opposite presented above - increase in costs may raise inflation expectations, which will reduce the real interest rate and hence boost the aggregate demand.
Consider an economy with a rapid economic growth under the MV-rule. Figure No. 31 presents a model that was designed by Cowen and Tabarrok (2011). As can be seen, it is a dynamic version of the AD/AS model. The DAD curve is derived as follows:

$$
\begin{align*}
\mathrm{M}(\mathrm{t}) \mathrm{V}(\mathrm{t}) & =\mathrm{P}(\mathrm{t}) \mathrm{Y}(\mathrm{t})  \tag{6}\\
\ln \mathrm{M}(\mathrm{t})+\ln \mathrm{V}(\mathrm{t}) & =\ln \mathrm{P}(\mathrm{t})+\ln \mathrm{Y}(\mathrm{t})  \tag{7}\\
{[\mathrm{dM}(\mathrm{t}) / \mathrm{dt}] / \mathrm{M}(\mathrm{t})+[\mathrm{dV}(\mathrm{t}) / \mathrm{dt}] / \mathrm{V}(\mathrm{t}) } & =[\mathrm{dP}(\mathrm{t}) / \mathrm{dt}] / \mathrm{P}(\mathrm{t})+[\mathrm{dY}(\mathrm{t}) / \mathrm{dt}] / \mathrm{Y}(\mathrm{t})  \tag{8}\\
\mathrm{g}_{\mathrm{M}}+\mathrm{g}_{\mathrm{V}} & =\pi+\mathrm{g}_{\mathrm{Y}} \tag{9}
\end{align*}
$$

Equation (7) is a logarithm of (6). If we differentiate (7) with respect to time, we get equation (8). According to the MV-rule, the left hand side of (9) must be zero ( $\mathrm{g}_{\mathrm{M}}+\mathrm{g}_{\mathrm{V}}=0 \%$ ). Then the growth in nominal income will be zero as well, and the growth in real potential GDP (on BGP, for example) will be reflected in the corresponding secular deflation ( $\pi=-\mathrm{g}_{\mathrm{Y}} *=-\mathrm{n}-$ g).


Figure No. 31 Economy on the BGP, and the MV-rule in the DAD/DAS model

Figure No. 31 shows that if the natural output grows faster $\left(\mathrm{g}_{\mathrm{Y} *, 1}>\mathrm{g}_{\mathrm{Y} *, 2}\right)$, the secular deflation is more rapid under the MV-rule ( $\pi_{2}=\mathrm{n}+\mathrm{g}_{2}<\pi_{1}=\mathrm{n}+\mathrm{g}_{1}$ ). The dashed DAD curves represent the inflation targeting regime that set the target to $\pi^{\mathrm{T}}=2 \%$. As can be seen, faster growth in potential output must be accommodated by higher monetary expansion. So the money growth under the inflation targeting must increase from $\mathrm{g}_{\mathrm{M} 1}=\pi^{\mathrm{T}}+\mathrm{n}+\mathrm{g}_{1}$ to $\mathrm{g}_{\mathrm{M} 2}=\pi^{\mathrm{T}}+$ $\mathrm{n}+\mathrm{g}_{2}$.
However, there is a specific rate of deflation (i.e. negative inflation) that results in zero nominal interest rate for some given natural steady state real rate of interest $\left(-\pi=\mathrm{r}_{S S}\right)$. From that level downwards, a reduction in inflation (increase in deflation) does not lead to a higher demanded output, but to a lower demanded output. The reason was presented above - higher deflation leads to a higher real interest rate when the nominal interest rate cannot fall. This contracts aggregate demand and then output. This tendency is self-perpetuating. In the model (6) - (9) above, there is an ever-decreasing reduction in the velocity of circulation along the decreasing part of the AD curve. For the MV-rule, this point is achieved if the economy is at the golden rule level of capital accumulation $(\mathrm{r}=\mathrm{n}+\mathrm{g})$. On the other hand, for the inflationtargeting regime, this critical point would require much larger growth rate in potential output, or maybe it will never occur, because this policy design will always adjust the monetary growth such that the positive inflation target is hit.


Figure No. 32 MV-rule and the dynamically inefficient economy; self-sustained deflationary process

In Figure No. 32 we added the upward sloping DAS curve, which is a dynamic version of the NK Phillips curve. Now suppose that the growth in output is so high (due to $g_{3}$ ) that it overcomes the golden rule point in Figure No. 32. ${ }^{319}$ Unless the DAD is expanded by monetary expansion (dashed line) to accommodate this growth and to raise the inflation (reduce deflation) such that the nominal interest rate is above zero, the dynamic equilibrium E cannot be established. The intersection of the DAD curve and the $\mathrm{DAS}_{3}$ curve is at point B , not at the natural growth rate point E - there is a negative output (growth) gap ( $\mathrm{g}_{\mathrm{Y}, \mathrm{ZLB}}-\mathrm{g}_{\mathrm{Y}, 3^{*}}$ < 0). Not all available factors of production are fully employed, so their prices may be reduced. The DAS curve moves downwards next period (to DAS4). In normal conditions, a negative output gap would be eliminated by the reduction in the inflation rate caused by the favourable shift in the DAS curve. However, if the economy is in the zero lower bound trap, a more rapid deflation raises the real rate of interest which will further depress aggregate demand in the economy. The economy moves to point D with an even larger negative output (growth) gap. As can be seen, the economy is on the path to self-fulfilling deflation with an expanding negative output (growth) gap.
At some moment in the future, the economy will be growing at the same rate as before the growth in potential output accelerated (i.e. at $g_{Y, 2}{ }^{*}=n+g_{2}$ ). And from that particular time onwards, it will be growing at a lower and still lowering rate. At one point in the future, the growth rate will be even negative. As can be seen, the acceleration in potential output led to an ever-increasing negative output (growth) gap - it led to a self-sustained recession and "unsound" deflation. Whether this crazy behaviour is just a theoretical curiosity is an open question. Yet, it might occur in a dynamically inefficient economy described by the model with the New Keynesian properties, where the central monetary authority initially adopted the MV-rule. Nevertheless, even the MV-rule in the ZLB will have to conduct monetary expansion, since along the downward-sloping part of the DAD curve, the MV-term is collapsing.
In the previous sentence, we touched on one last point that should be considered in the discussion of the MV-rule. In the economic depression, both the velocity and the money supply might be endogenously falling. The MV-rule then requires expansion in the money supply to offset these tendencies. However, this influx of money (or better the monetary base) might form a basis for a future monetary expansion, so the "sound" recovery might be easily turned to an unsustainable boom.

## 4. TECHNOLOGICAL SHOCKS - GROWTH AND CYCLES

In section 2, we presented the Hayekian theory, according to which the forces of the secular growth may trigger the business cycle if the monetary forces accommodate this growth by monetary expansion. In Chapter 2 of this dissertation, we argued that the business cycle might be triggered not only by an exogenous increase in the money supply but also by a positive technological shock that increases demand for loanable funds and that is accommodated by the creation of money in the banking system rather than by voluntary savings. In such a case, the natural rate of interest, driven up by a higher marginal productivity of capital, exceeds the actual rate of interest, and more factors of production and material resources are used in early stages of the production process than is justified by voluntary intertemporal decisions of free acting people. The resulting artificial boom is a necessary consequence of this process ending in the recession, when the boosted demand for consumption goods does not allow the lengthened production processes to be finished.

[^205]In Chapter 3, it was indicated that Mises and Rothbard did not recognize marginal productivity of capital as the determinant of the natural rate of interest. They were pure time preference theorists (PTPT). As such, they would not subscribe the Hayekian explanation of economic fluctuations, in which the real productivity shock might initialize the boom-bust cycle. According to Mises and especially Rothbard, the major culprit of the economic cycle was the central bank that pressed the market interest rate below the natural level, which is, according to the PTPT, solely determined by the time preference of people.
Chapter 3 was mainly designed to prove that the pure time preference theory is not accurate. One of the key implications was that the marginal productivity of capital, along with changes in technology, is an important determinant of the natural rate of interest. If that is at least partly correct, then the Hayekian vision of the business cycle that is triggered by a real shock, which is, however, accommodated by the monetary sector of the economy, may be defended against Mises and Rothbard.

In Chapter 3, two major types of technological shocks were presented. First, a permanent shock to the level of technologies, and secondly, a permanent shock to the growth rate in technology. In the present section, we will utilize our investigation from Chapter 2 and Chapter 4, and we will explore the optimum response of the monetary sector to these shocks. Furthermore, the theory of Chapter 3 might be extended by the analysis of the duration of the shock, and by the question of whether the technological advances were anticipated or not.
As regards the level of technologies, it is rather difficult to imagine a situation in the modern economy in which the higher technological level is just temporary. On the other hand, a temporary increase in the growth rate in technologies is rather ubiquitous. As far as the possibility of anticipation is concerned, it is conceivable that both the change in the level and the change in the growth rate of technological progress might be anticipated.
Let us first start with a one-time permanent increase in the level of technology that was not anticipated in advance. The business cycle considerations from the Austrian point of view were presented in Chapter 2, whereas the dynamic approach in the real model was thoroughly explained in Appendix 7 in Chapter 3. In this section we will utilize the IS-LM-Y* model that is a natural extension of the simple loanable funds model used in Chapter 2.
In Chapter 2 we mainly stressed the impact of the technological shock on the investment demand and on the natural rate of interest. However, as we saw in Appendix 7 of Chapter 3, the dynamics is much more complicated. Let us present again the set of graphs that displayed the evolution of the key variables after the positive technological shock in the RCK economy that was stationary before the shock.


Figure No. 1_A7 Increase in A in the RCK model.


Figure No. 2_A7 Increase in A in the RCK model represented in the Solow model.


Figure No. 4_A7 Evolution of output per worker after the increase in A in the RCK model and the role of $\theta$.


Figure No. 7_A7 The growth rate in output per worker after the increase in A in the RCK model and the role of $\theta$.


Figure No. 8_A7 The real interest rate after the increase in A in the RCK model and the role of $\theta$.


Figure No. 9_A7 The nominal interest rate after the increase in A in the RCK model and the role of $\theta$.

In the IS-LM-Y* model, the first-round shock might be represented as follows: Increase in the investment demand caused by the higher marginal productivity shifts the IS-curve to the right. At the same time, the potential output of the economy is affected in the positive direction. However, the natural rate of interest at the time of the shock increases, so the impact on the $\mathrm{Y}^{*}$ curve is much lower, in the first round, than the impact on the IS curve (point B in Figure No. 33). Over time the new impulse for the shift in the investment demand is not arising, apart from the need to replace a higher amount of capital. On the other hand, the income increase is further supported by the accumulation of capital, as can be seen in the diagram of the Solow model in Figure No. 2_A7. Higher income brings about higher savings that leads to a
reduction in the natural rate of interest. This further dynamics was not considered in Chapter 2. Yet, it is very important as it will decide the evolution of the natural rate of interest.


Figure No. 33 Productivity shock in the IS-LM-Y* model.

In this analysis, we have to distinguish between the natural real interest in the steady state ( $r_{\text {nat,ss }}$ ) and on the path towards the steady state (from $r_{\text {nat,shock }}$ to $r_{\text {nat,ss }}$ ). The initial increase in the natural level caused by the productivity shock is followed by a gradual decrease to the original level. This dynamics is due to the shift in the natural output curve in Figure No. 33. The economy is moving, if not disturbed by monetary forces, along the new IS curve to the new steady state at which the income is larger, yet the natural rate of interest is at the initial steady state level (point D).
At this point, let us present the behaviour of the economy that is using money. Figure No. 9_A7 displays the evolution of the nominal interest rate for a fixed MV and for a perfectly flexible price level. In the IS-LM-Y* model, the former (i.e. the fact that velocity does not depend on the interest rate) is reflected by the vertical LM curve, the latter (flexibility of prices) by a smooth shift of this curve along with the potential output curve $\mathrm{Y}^{*}$. As can be seen, the price level is gradually falling to its new steady state level. The simulations above indicate that the real interest rate does not coincide with the nominal interest rate, so the LM analysis should be separated from the IS analysis. However, we will neglect this separation in the following discussion.
Let us now relax both assumptions. First, suppose that the velocity positively depends on the interest rate; that is, the LM curve is increasing. Even if the money supply is fixed, the actual rate of interest may differ from the natural rate, as can be seen in Figure No. 34 ( $r_{2}<r_{\text {nat,shock }}$ ). This idea was first presented in Chapter 2. Higher investment demand, which raises the interest rate, attracts more real savings. Thus, people voluntarily reduce their consumption spending, and the factors of production may be released from stages very close to final consumption to early stages of the production process in which the boosted demand for investment goods is seeking material and human resources. However, a higher interest rate may also lead to the reduction in the demand for money. In other words, people may dissolve
part of their money balances in providing funds to the loanable funds market, which implies savings without the reduction of consumption spending. Using the Hayekian analysis presented above, the increase in the demand for future consumption goods (higher saving) is not fully reflected by a lower demand for real present consumption goods. Part of the higher demand for future goods is financed by the reduction in real money balances. In other words, there is a demand (for future goods) without the corresponding supply. Money is a loose joint that may separate demand and supply of goods that is always equal in the barter economy. This reduction of the real money balances, which may, in addition to "real" saving, finance the demand for investment goods, should have the same effects as the monetary expansionary accommodation from the banking system. The only difference is that instead of M in the MVterm, it is V that is rising. According to the Hayekian theory, this increase in the flow of incomes may trigger the business cycle. Too much investment is initiated compared with the volume of real saving, and factors of production attracted to early stages will be sooner or later demanded back in the consumption stage of the production process.

However, at this point, we cannot say that people involuntarily reduced their consumption as in the case of the monetary expansion that leads to the "forced saving" phenomenon. Here, the problem is that the existence of money and changes in the money demand may break up the connection between the intertemporal demand for goods and the intertemporal supply of goods - in our case, the supply of future goods (investment) exceeds the demand for future goods (saving) due to the reduction in the demand for money (increase in velocity). This analysis suggests that even in the world in which the total money supply consists only of golden coins, the business cycle might be triggered if the money demand is sufficiently sensitive to the interest rate. Such a conclusion might be rather disturbing for the Austrian theorists; yet, it is in perfect accordance with the Hayek (1941) analysis.


Figure No. 34 Productivity shock in the IS-LM-Y* model and the "velocity" accommodation.

Figure No. 34 depicts this theory in the IS-LM-Y* model. Higher natural output and hence permanent income in periods after the shock may generate more savings to finance greater
investment demand. This might be reflected by a rightward shift of the saving curve on the loanable funds market. As a result, the natural interest does not increase to the point we indicated in Chapter 2 ( $\mathrm{r}_{\text {nat, }, \mathrm{Y}^{*} \text { fixed }}$ ), but to a lower level $\mathrm{r}_{\text {nat,shock }}$ owing to higher savings that were generated by the expanded natural output. However, if the demand for money is sensitive enough, and the LM curve is therefore rather flat, the realized volume of investment is even larger. The economy is in the positive output gap $\left(\mathrm{Y}_{2}-\mathrm{Y}_{\text {shock }}{ }^{*}>0\right)$, and the actual interest rate is lower than the natural rate ( $r_{2}<r_{\text {nat,shock }}$ ). Thus, the business cycle in the economy has started, and nobody can be blamed - neither the government, nor the central bank, nor commercial banks with fractional reserves - since we assume that the money supply is fixed. The culprit is the increased velocity of circulation of money, i.e. the voluntary decision of people to hold less money when the interest rate rises.

At this point, we should stress one important terminological issue. This particular economy is characterized by a positive output gap ( $\mathrm{Y}_{2}-\mathrm{Y}_{\text {shock }}{ }^{*}>0$ ), but by a negative convergence gap. The latter measures the difference between the actual natural level of output and the steady state level of natural output ( $\mathrm{Y}_{\text {shock }}{ }^{*}-\mathrm{Y}_{\mathrm{SS}, 2}{ }^{*}<0$ ). However, the movement from one natural output to another one should not be speeded up by the monetary forces, otherwise the business cycle will be triggered. In other words, the convergence gap is a natural state of the economy, whereas the output gap indicates disequilibrium. The central bank should not argue that the economy is far from its steady state level, so the monetary expansion is an appropriate tool to push the economy closer to the steady state natural output. This argument is as fallacious as the argument that all central banks in Eastern Europe should conduct aggressive monetary expansion to reach the level of GDP per capita in Germany, which is believed to be the economy towards which these economies converge.

The same idea applies to the interest rate gap. One cannot argue that the actual interest rate $\left(r_{2}\right)$ is above the steady state level of the natural interest $\left(r_{s s}=\rho\right)$, so the monetary policy is too tight. Exactly the opposite is true - monetary conditions are too loose (due to higher velocity) because the actual natural real rate of interest ( $\mathrm{r}_{\text {nat,shock }}$ ) is higher than the actual market rate of interest $\left(\mathrm{r}_{2}\right)$. With respect to the business cycle theory, it is the difference between these two that should be at the centre of the analysis. As far as the growth theory is concerned, the difference between the actual natural interest rate and the steady state natural level might be analyzed. But the last difference that may emerge in the analysis ( $r_{2}>r_{n a t, S S}$ ) is of zero importance, even though some (political) opportunists may argue that the interest rate in the converging economy is too high with respect to the benchmark economy and must be lowered by monetary expansion. Yet, as our analysis suggests, this recommendation is totally fallacious. The interest convergence gap can be closed, without boom-bust repercussions, only due to higher saving of people and hence accumulation of capital, not by the monetary expansion.
As can be seen from panel (b) in Figure No. 34, the MV-rule is violated since the MV-term is higher. This can be deduced from the larger area PY, which was expanded by the shift of the aggregate demand curve. As was demonstrated in the previous section, the MV-rule is well designed to offset the AD-shocks (in this case the velocity shock). If the CB responds fast enough and reduces the money supply, the economy should move back to the "natural path" presented in Figure No. 33. The MV-rule would not have to respond if the demand for money did not depend on the interest rate. In such a case, the LM curve would be vertical, and the economy would be on the smooth path moving along the given level of the AD curve. If prices were sufficiently flexible, a gradual increase in the potential output would be perfectly reflected in a lower price level.

However, the operation of the MV-rule might not be as smooth as presented above if the demand for money is interest-elastic. The natural output is growing to its steady state, and the
natural rate of interest is gradually falling to its initial level. The demand for money is in turn increasing due to these two reasons. As a result, the velocity of circulation gradually falls with the lower interest rate. Thus, the MV-rule should loosen step by step the monetary conditions. Furthermore, if the real money balances are a luxury good, and the aggregate demand is therefore inelastic, the new steady state value of the nominal income (and MV) might be lower than the initial one.

As a result, the MV-rule might have serious problems in following the changing velocity of circulation after the sudden increase in technological progress. Notice that we assumed that the velocity was varying only due to the changing interest rate, not due to the technological reasons. Hence, this rule might be destabilizing rather than stabilizing. As a result, the best policy in the case of the positive technological shock is presumably to keep the money supply constant and tolerate the business cycle that may arise due to changing velocity of circulation. If prices are flexible enough, which might be true in the case of positive technological progress, the business cycle could be quite imperceptible. However, there should be a general tendency for the decline in the price level, even though temporary positive output gaps caused by higher velocity may create positive inflationary pressures.
It should be stressed that there is certainly a much more powerful source of economic fluctuations after the increase in the technological level than is the reduction in the real money balances. This source is the banking system with the central bank at the top. The banking system can create enormous amounts of money - much greater amounts than people can release from their money balances - and at a more rapid pace.
Suppose that the economy is hit by a positive technological shock. The natural path of the economy is depicted by the system of diagrams presented in Figure No. 1_A7, Figure No. 2_A7, Figure No. 4_A7, Figure No. 7_A7, Figure No. 8_A7, Figure No. 9_A7, and Figure No. 33. The price level should be gradually falling to its new steady state level. As can be seen, the economy is never below the zero lower bound. Furthermore, there should be no business cycle, no unsustainable boom, and no misallocation of resources. The growth in output is not constant since it is driven by the dynamics of the model - at the beginning the growth is the highest, and then it gradually dies out - yet, there is no boom-bust pattern.
Suppose that the central bank in this economy is targeting inflation at some very low level (maybe zero). It is using the short-term nominal interest rate as its main policy instrument. The simple model this central bank adheres to will be presented in section 5; nevertheless, the path of the nominal interest rate that is consistent with the natural equilibrium (and secular deflation) is presented in Figure No. 9_A7. Yet, it would be very surprising if this central bank followed this particular path.
The reasons are as follows. First, this central bank does not allow for price deflation, so its monetary policy should be looser. In other words, it can be deduced that the interest rates set by the central bank will be lower than the natural rate in order to raise the inflation rate back to the inflation target. Yet, according to the Hayekian theory, it is exactly this monetary loosening that starts the unsustainable boom.
Let us show that this boom is even implied by the simple (New) Keynesian IS-LM-Y* model. The technological shock is obviously unobservable. In the free market economy, it is reflected in changes in prices. If it affects intertemporal markets, it should also influence the interest rate - Figure No. 33 suggests a possible path of the natural rate of interest. Individuals in the economy may learn about this shock from a rising interest rate, even though they do not need to know what the source of the change in the interest rate is. They can freely change their optimum consumption plans according to new conditions. Their optimal response is described by the Euler equation. Similar optimization problem could be solved for profit maximizing
firms. As a result, actions of all agents in the economy are coordinated by the price system by the interest rate in this particular case.

However, if the short-term nominal interest rate is controlled by the central bank, part of this information is lost. Even though the central bank usually does not regulate all types of interest rates in the economy, short-term interest rate is an important price in the structure of the intertemporal markets, and it surely affects other interest rates in one way or another. As a result, the interest rate policy of the central bank must significantly influence these markets and the intertemporal allocation of resources. Thus, as the first approximation, we may deduce that the regulation of this price - interest rate - should face similar problems as any kind of regulation or central planning.
Furthermore, it is not a far-fetched assumption to presume that the central bank does not observe the technological shock either, as well as the rest of the society. Suppose that it sets its interest in order to hit the inflation target $\pi^{\mathrm{T}}=0 \%$. Figure No. 35 displays this situation. Initially, the interest rate of the central bank (both real and nominal) is at the natural level, prices are stable, and the economy is at the potential level. The technological shock that moves the "natural economy" to point B is not observable. The central bank should immediately increase the interest rate. Yet, it has no information about this shock. The only medium that may deliver this information is the interest rate. However, this signal is blocked due to the policy regime of the central bank.
The technological shock raises investment demand and consequently the demand for credit. Commercial banks may increase their interest rates to attract more savings and to partly repel the demand, but they can also meet the demand by a simple expansion in deposits. Borrowers just obtain an account in the bank, and they may immediately pay with these new resources. They can afford more material resources to expand production; they can start new investment projects and employ more workers - much more than if the interest rate was increased.


Figure No. 35 Productivity shock in the IS-LM-Y* model and the "response" of the central bank

As can be seen in Figure No. 35, the economy is on the path to boom. The figure indicates a significant positive output gap ( $\mathrm{Y}_{2}>\mathrm{Y}_{\text {shock }}{ }^{*}$ ), and the interest rate that is below the natural level ( $\mathrm{i}_{\mathrm{cb}}<\mathrm{r}_{\text {nat,shock }}$ ). It cannot be argued that the interest rate of the central bank is at the natural steady state level because such a level is the equilibrium in the far distant future. As was shown in the discussion above, the same applies to the argument about the actual level of output and the steady state natural level of output. As a result, keeping the interest rate at the constant level is a policy error.
One can also argue that the ratio of deposits to reserves of the commercial banks is expanding (the reserve-deposit ratio is falling), so the money multiplier increases. The expansion of trade in general and the increased volume of purchases of employees may create pressures on the banking system to provide more cash. This can in turn stop the credit expansion, as the banking system protects its cash reserves. However, the central bank is still fixing the interest rate, as it did not receive a signal that would lead to a different policy action. Thus, the commercial banks can borrow as much reserves as they want from the central bank for the predetermined rate of interest. Thus, the source of money is, in the end, the central bank that is accommodating expanded demand for credit.
As we will see in the next section, the central bank will increase the interest rate if it receives the information either about the (expected) positive output gap or about (projected) higher inflation. The first symptom of the booming process is the expansion in credit; yet, monetary (and credit) aggregates play a minor (or no) role in conducting modern monetary policy. Another signal might be the expansion in output. However, this signal will be received after quite a long time. Moreover, the natural output itself is expanding due to technological progress and the second round capital accumulation (see the Solow model in Figure No. 2_A7), so it is very hard to distinguish what part of the measured increase in real output is to be attributed to natural increase and what part to the artificial boom (output gap).

Another warning that the economy is in an unsustainable boom is the size of the inflation rate. The major objective of the central bank is to ensure price stability. Hence, signals about inflationary pressures should be reflected in a higher interest rate of the central bank. However, the actual data may not deliver this information. The reason is that there is an expansion in natural output that may press the general price level downwards or at least moderate the increase in the inflation rate. As a result, the boom of the economy might proceed in a stable-price level environment.

Thus, the information about the unsustainable boom the central bank itself created or did not check, might be obtained rather late. Moreover, the expansion in natural output and the resulting stability of the price level can totally blur that the economy is on the unsustainable path. It can be therefore argued that the control of the interest rate is a very dangerous policy. Instead of stabilizing the economy, it may lead to destabilization. In other words, the interest rate, as the key information signal, should be free to move in order to coordinate intertemporal allocation of resources. The usual argument about the virtue of the stability of the interest rates in the modern banking system is flawed. It is rather the other way round - the movements in the interest rate are consistent with healthy conditions on the intertemporal markets. Any price to perform its role properly must be moving to reflect changes on the market. The interest rate is not an exception.
Panel (b) in Figure No. 35 also indicates that the specific shape or movement in the AS curve is not important. Even if prices were perfectly flexible (point B), the inflation-targeting regime and the fixed level of the interest rate on the money market would push the economy to the positive output gap (horizontal distance between F and B). However, it can be argued that even though individual entrepreneurs can obtain real resources owing to the new loans, the economy as a whole cannot create real resources from thin air. Nonetheless, many theories
were developed to explain why the economy may operate beyond the natural level. Yet, almost all theories also predict that this situation is not sustainable. Sooner or later, output should return to its potential level. The Austrian theory argues that the end of the artificial boom is accompanied by losses in capital structures. Hence, the potential output might be affected in the negative direction. On the other hand, the New Keynesian theories mainly stress that the recessionary phase can create serious second-round pressures, which may lead to permanent losses in output. Yet, the boom is usually not identified as the major culprit of this situation.


Figure No. 36 Boom-bust cycle after the positive technological shock.

As we have seen, the monetary policy that targets inflation may exaggerate expansionary tendencies if the economy is hit by a positive technological shock. Thus, instead of a smooth movement to a new steady state level of natural output presented in Figure No. 4_A7, the resulting path might rather follow a boom-bust pattern displayed in Figure No. 36.
In the New Keynesian literature, it is argued that the wrong perception of the natural rates from the side of the central bank should lead to a permanent inflation bias and to the poor performance of the monetary policy in general (Orphanides and Williams 2002). However, this policy error is highly underestimated from the ABCT point of view since the NKE does not take into account discrepancies in the intertemporal markets leading to serious mistakes in the allocation of capital. Hayek (1941) frequently stressed that the large volume of capital the modern society may enjoy is not only the corollary of saving and investment decisions of people and entrepreneurs, but to a larger extent, it is the outcome of correct expectations made in the past. Thus, if these expectations are upset due to the false signals that originate in the banking system, it is not surprising that material resources are allocated to projects that turn out to be unprofitable.


Figure No. 19_A7 The growth rate in GDP per worker after the increase in g from 2\% to 3\% in the RCK model and the role of $\theta$.


Figure No. 21_A7 The real interest rate after the increase in g from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.


Figure No. 22_A7 The nominal interest rate after the increase in g from $2 \%$ to $3 \%$ in the RCK model and the role of $\theta$.

The problems the central bank may face in following the evolution of the natural rate of interest may be documented by two other types of shocks. The first one was introduced in Chapter 3 - a permanent increase in the rate of technological progress. Figure No. 21_A7 shows that the natural rate of interest gradually increases. The central bank should be aware of this path. However, this might be quite difficult since the initial gap in the interest rates, which results from the constancy of the central bank's interest rate, may not be reflected in inflationary pressures (or deflationary pressures for low $\theta$ ) due to the gradually accelerating growth in potential output (Figure No. 19_A7). Figure No. 22_A7 displays the neutral paths of the nominal interest rate for constant MV and secular price deflation.
The second case - anticipated increase in the level of technologies - is displayed in the system of diagrams below. Figure No. 37 represents this shock in the RCK model. The time graphs in Figure No. 38 depict the evolution of the potential output (b) and the natural interest rate (a). The IS-LM-Y* model in Figure No. 39 shows that the initial increase in consumption spending is reflected in the outward shift of the IS curve, whereas the gradual decumulation of capital between the time of the anticipation ( $\mathrm{t}_{0}$ ) and the moment the shock occurs ( $\mathrm{t}_{1}$ ) moves the potential output to the left. This process is reverted at time $t_{1}$, at which the IS curve is raised even more due to the increase in investment spending resulting from positive technological shock. From that moment, the natural output is rising. As can be seen, the dynamics of such an economy is rather complicated, and there is a high probability that the natural rates will be misperceived by the central bank.


Figure No. 37 Anticipated increase in technological level in the RCK model (for higher $\theta$ ).


Figure No. 38 Anticipated increase in technological level and the evolution of the natural level of interest (a) and output (b)


Figure No. 39 Anticipated increase in A in the IS-LM-Y* model.

## 5. HAYEK-TAYLOR RULE

The previous discussion with the New Keynesian (NK) theory did not uncover the explicit structure of the NK model. Let us sketch the basic idea of this model in the three-equation system:

$$
\begin{align*}
& y=y^{*}-\alpha\left(r-r_{n a t}\right)  \tag{10}\\
& \pi=\pi^{e}+\beta\left(y-y^{*}\right)  \tag{11}\\
& i_{C B}=r_{n a t}+\pi+\theta_{\pi}\left(\pi-\pi^{\mathrm{T}}\right)+\theta_{\mathrm{y}}\left(\mathrm{y}-\mathrm{y}^{*}\right) \tag{12}
\end{align*}
$$

Equation (10) is the New Keynesian IS curve that simply states that the positive output gap occurs ( $y-y^{*}>0$ ) when the actual real rate of interest is lower than the natural level. Equation (11) is the NK Phillips curve that implies that the actual inflation depends on expected inflation and the output gap. Expression (12) is the Taylor rule that prescribes the interest-rate policy for the central bank. The interest rate of the central bank should increase with higher real natural rate of interest $r_{\text {nat }}$, positive output gap, or increase in inflation. For the stability of the model, it is required that the nominal interest rate of the central bank increases more than one-to-one with the rise in inflation. In such a case, the real interest rate of the central bank grows, which will depress spending in the economy (IS curve) and
consequently inflationary pressures (PC curve). This condition is known as the Taylor principle - the increase in the inflation rate must be reflected in the increase in the real interest rate and vice versa. ${ }^{320}$ It is satisfied if parameter $\theta_{\pi}$ is greater than zero, hence ( $1+\theta_{\pi}$ ) $>1$.

In the case of the ZLB, the Taylor principle was violated. Inflation was falling (deflation was rising); yet, the nominal interest rate was not lowered by a higher amount. Moreover, it was not lowered at all $-\theta_{\pi}$ was effectively equal to -1 , i.e. $\left(1+\theta_{\pi}\right)=0$, and $\theta_{y}$ was equal to 0 . Thus, the decrease in inflation was followed by an increase in the real interest rate, not a decline, as is required by the Taylor principle. The corollary of this violation was that the DAD curve in Figure No. 32 was downward sloping from the ZLB level.

The NK IS curve and the NK PC may be used to re-interpret the Wicksell idea that the price level is rising or falling with an interest rate gap ( $\mathrm{r}-\mathrm{r}_{\mathrm{nat}}$ ). If we substitute (10) to (11), we get:

$$
\begin{equation*}
\pi=\pi^{\mathrm{e}}-\alpha \beta\left(\mathrm{r}-\mathrm{r}_{\mathrm{nat}}\right) \tag{13}
\end{equation*}
$$

Equation (13) states that if the actual real interest rate is higher than the natural rate of interest, the inflation will be below the expected inflation rate. This equation also implies that the equality between the actual and the natural rate of interest is consistent with an increasing or decreasing price level if this increase or decrease was correctly expected. Thus, if the central bank is able to hit its inflation target of $2 \%$, and this rate is anticipated by the general public, the price level is obviously rising, yet there is no interest-rate gap. Accordingly, under the Hayek MV-rule, if the secular deflation is built in people's expectations, the price level might be declining even though the interest is at its natural level. This simple equation therefore extends the analysis from section 2 of this Chapter.

In section 3, we presented the Hayek MV-rule. However, no discussion was provided about the instrument the central bank should use. Let us suppose that it turned out that the most straightforward policy is to control the short-term interest rate. Thus, we may apply the Taylor rule in equation (12). Let us further plug the Hayek idea of the MV-rule into the Taylor rule.

Taylor (1993) in his seminal paper suggested the following combination of parameters: $\theta_{\pi}=$ $0.5 ; \theta_{y}=0.5$. We will follow this suggestion. The Hayek MV-rule implies that the growth in potential output should be reflected one-to-one in the secular deflation. As a result, the implicit inflation target in the MV-rule is equal to the opposite of the growth rate in potential output $\left[\pi^{\mathrm{T}}=-\left(\mathrm{y}_{\mathrm{t}}^{*}-\mathrm{y}_{\mathrm{t}-1} *\right)\right]$. Thus, the Taylor rule in the Hayek framework might be written as follows: ${ }^{321}$

$$
\begin{equation*}
\mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\pi_{\mathrm{t}}+\left(\mathrm{y}_{\mathrm{t}}^{*}-\mathrm{y}_{\mathrm{t}-1} *\right)\right]+0.5\left(\mathrm{y}_{\mathrm{t}}-\mathrm{y}_{\mathrm{t}}{ }^{*}\right) \tag{14}
\end{equation*}
$$

If we compare the parameters from the inflation-targeting (IT) regime and from the Hayek rule, we will see that the former is too expansionary from the point of view of the latter. Consider the natural real rate of interest of $2 \%$, inflation target of $2 \%$, zero output gap, the potential output growth of $1 \%$, and the actual rate of inflation of $1 \%$. The Taylor rule for the IT suggests:

[^206]\[

$$
\begin{equation*}
\mathrm{i}_{\mathrm{CB}, \mathrm{IT}}=2 \%+1 \%+0.5(1 \%-2 \%)+0.5(0)=2.5 \% \tag{15}
\end{equation*}
$$

\]

As can be seen, the implied nominal interest rate of the central bank for the IT is $2.5 \%$, which is 1.5 percentage points below the neutral rate of $i=4 \%=r_{\text {nat }}+\pi^{\mathrm{T}}=2 \%+2 \%$. In this case, the real interest rate of the central bank $\mathrm{r}_{\mathrm{CB}}=2.5 \%-1 \%=1.5 \%$ will be below the natural rate of $\mathrm{r}_{\text {nat }}=2 \%$. A one percent inflation rate is considered to be too low in the IT regime, so the central bank must loosen the monetary policy by the reduction of the real interest rate. The monetary policy will be therefore expansionary, even though there is no output gap.
On the other hand, the Hayek-Taylor rule (HTR) implies:

$$
\begin{equation*}
\mathrm{i}_{\mathrm{CB}, \mathrm{HTR}}=2 \%+1 \%+0.5(1 \%+1 \%)+0.5(0)=4 \% \tag{16}
\end{equation*}
$$

According to the HTR, the actual inflation rate is too high, and the central bank should tighten the policy. The real interest rate of the central bank should be $4 \%-1 \%=3 \%$, which is above the natural rate $\mathrm{r}_{\text {nat }}=2 \%$.
We can also determine the neutral nominal interest rate of the central bank under the HTR. Suppose that the inflation is on the target and there is no output gap. According to (14):

$$
\begin{equation*}
\mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\text {nat }}-\left(\mathrm{yt}_{\mathrm{t}} *-\mathrm{y}_{\mathrm{t}-1} *\right) \tag{17}
\end{equation*}
$$

Thus, the neutral nominal interest rate will be positive only if the economy is dynamically efficient (real interest rate exceeds the growth rate of natural output). As a result, the HTR rule is operational with respect to the economy that is on its BGP, only if the economy does not over-accumulate capital.
Let us now show that the HTR leads to a simple form of the NIT (nominal income targeting) with the interest rate of the central bank as the main instrument. The inflation rate might be rewritten as $\pi_{\mathrm{t}}=\mathrm{p}_{\mathrm{t}}-\mathrm{p}_{\mathrm{t}-1}$. Substituting this expression to (14), we get:

$$
\begin{align*}
& \mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\mathrm{p}_{\mathrm{t}}-\mathrm{p}_{\mathrm{t}-1}+\left(\mathrm{yt}_{\mathrm{t}} *-\mathrm{y}_{\mathrm{t}-1} *\right)\right]+0.5\left(\mathrm{y}_{\mathrm{t}}-\mathrm{y}_{\mathrm{t}} *\right)  \tag{18}\\
& \mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\mathrm{p}_{\mathrm{t}}-\mathrm{p}_{\mathrm{t}-1}+\mathrm{y}_{\mathrm{t}} *-\mathrm{y}_{\mathrm{t}-1} *+\mathrm{y}_{\mathrm{t}}-\mathrm{y}_{\mathrm{t}} *\right]  \tag{19}\\
& \mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\mathrm{y}_{\mathrm{t}}+\mathrm{p}_{\mathrm{t}}-\mathrm{y}_{\mathrm{t}-1} *-\mathrm{p}_{\mathrm{t}-1}\right]  \tag{20}\\
& \mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\mathrm{y}_{\mathrm{t}}+\mathrm{p}_{\mathrm{t}}-\left(\mathrm{y}_{\mathrm{t}-1} *+\mathrm{p}_{\mathrm{t}-1}\right)\right] \tag{21}
\end{align*}
$$

$\left(y_{t}+p_{t}\right)$ is the nominal GDP in the present period $P_{t} Y_{t}$. As a result, one version of the HayekTaylor rule requires that the central bank should respond to a difference between this period nominal income and the previous period nominal potential income $\left(\mathrm{y}_{\mathrm{t}-1} *+\mathrm{p}_{\mathrm{t}-1}\right)$. This might be
superior to the usual Taylor rule since only this period natural rate of interest is required to be known. This period potential output is not required. Only the previous period natural output is to be known.

Furthermore, (21) might be rewritten as:

$$
\begin{equation*}
\mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\left(\mathrm{y}_{\mathrm{t}}+\mathrm{p}_{\mathrm{t}}\right)-\left(\mathrm{y}_{\mathrm{t}-1}+\mathrm{p}_{\mathrm{t}-1}\right)-\left(\mathrm{y}_{\mathrm{t}-1} *-\mathrm{y}_{\mathrm{t}-1}\right)\right] \tag{22}
\end{equation*}
$$

As a result, the central bank should respond to the growth in nominal income and to the previous period output gap. It can be also shown that if the Taylor rule is redefined for the growth gap rather than the output (level) gap, the Hayek-Taylor rule implies that the central bank should respond only to the growth in nominal GDP.
If we extend the analysis a little bit more, the Taylor rule could be rewritten as:

$$
\begin{equation*}
\mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\mathrm{p}_{\mathrm{t}}-\mathrm{p}_{\mathrm{t}-1}-\left(\mathrm{p}_{\mathrm{t}}^{\mathrm{T}}-\mathrm{p}_{\mathrm{t}-1}\right)\right]+0.5\left(\mathrm{y}_{\mathrm{t}}-\mathrm{y}_{\mathrm{t}}{ }^{*}\right) \tag{23}
\end{equation*}
$$

where $\pi^{\mathrm{T}}=\mathrm{p}_{\mathrm{t}}{ }^{\mathrm{T}}-\mathrm{p}_{\mathrm{t}-1}$

$$
\begin{equation*}
\mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\mathrm{y}_{\mathrm{t}}+\mathrm{p}_{\mathrm{t}}-\left(\mathrm{p}_{\mathrm{t}}^{\mathrm{T}}+\mathrm{y}_{\mathrm{t}}^{*}\right)\right] \tag{24}
\end{equation*}
$$

Expression $\left(\mathrm{p}_{\mathrm{t}}{ }^{\mathrm{T}}+\mathrm{yt}^{*}\right)$ might be termed the target of nominal aggregate income. If the MVrule requires that this period nominal income is equal to the previous period nominal income, (24) yields:

$$
\begin{equation*}
\mathrm{i}_{\mathrm{t}, \mathrm{CB}}=\mathrm{r}_{\mathrm{nat}}+\pi_{\mathrm{t}}+0.5\left[\left(\mathrm{y}_{\mathrm{t}}+\mathrm{p}_{\mathrm{t}}\right)-\left(\mathrm{y}_{\mathrm{t}-1}+\mathrm{p}_{\mathrm{t}-1}\right)\right] \tag{25}
\end{equation*}
$$

As we can see, the MV-rule might be transformed such that it prescribes a simple monetary policy rule that we called the Hayek-Taylor rule. This period output gap need not be known. What is required, apart from the information about the neutral nominal interest rate $r_{\text {nat }}+\pi_{t}$, is the information about this period and the last period aggregate nominal income. To be more precise, it requires only the knowledge about the growth rate in aggregate nominal income. As such, it might be more operational than the traditional Taylor rule.

## 6. CONCLUSIONS

The key message of this chapter is obvious - it might be an insurmountable problem to find the optimal monetary policy rule in the free market economy. According to the Austrian theory, the popular inflation-targeting regime may be the critical source of the business cycle in the economy that is either on the path of growth in potential output or that is exposed to technological shocks of various types. The culprit is the monetary accommodation that is inherently present in this monetary policy regime.

An alternative framework was suggested - the Hayek MV-rule. However, as it turned out, it suffers from similar problems as the policy mentioned above, even though it is not as expansionary. Thus, the boom-bust cycle might be more moderate under this rule. On the other hand, the problem of the zero lower bound on the nominal interest rate was examined, and it was demonstrated that much research must be done until we better understand this specific environment in real economies.

One possible implication of this chapter is as follows. Since it seems to be too difficult for a limited human mind to optimally conduct monetary policy in the free market economy, it might be suggested to withdraw this activity from the realm of central planning. Market forces might be better in providing sound currency for a free society. Yet, the optimality of such an alternative would require an investigation of similar length as was presented here.

## REFERENCES

Abel, Andrew B.; Mankiw, N. Gregory; Summers, Lawrence H.; Zeckhauser, Richard<br>J. 1989. Assessing Dynamic Efficiency: Theory and Evidence. The Review of Economic Studies 56(1): 1-19.

Akerlof George A., Dickens William R.; Perry George L.; Robert J. Gordon; N. Gregory<br>Mankiw 1996. The Macroeconomics of Low Inflation. Brookings Papers on Economic Activity 27(1): 1-76.

Amato Jeffery D. 2005. The Role of The Natural Rate of Interest in Monetary Policy. BIS Working Papers 171, Bank for International Settlements.

Barro, Robert J. 1997. Macroeconomics - 5th ed., Cambridge, MIT Press
Bean, Charles R. 1983. Targeting Nominal Income: An Appraisal. Economic Journal 93(372): 806-19.

Beckworth, David 2008. Aggregate Supply-Driven Deflation and Its Implications for Macroeconomic Stability. Cato Journal 28(3): 363-384.

Benigno Pierpaolo 2009. New-Keynesian Economics: An AS-AD View. NBER Working Papers 14824, National Bureau of Economic Research, Inc.

Blinder, Alan S. and Reis, Ricardo 2005. Understanding the Greenspan Standard. Proceedings, Federal Reserve Bank of Kansas City, issue Aug: 11-96.

Bradley, Michael D. and Jansen, Dennis W. 1989. The Optimality of Nominal Income Targeting When Wages Are Indexed to Price. Southern Economic Journal 56(1): 1323.

Carlin, Wendy and Soskice David 2005. The 3-Equation New Keynesian Model - A Graphical Exposition, The B.E. Journal of Macroeconomics 5(1): 1-38.

Clark, Todd E. 1994. Nominal GDP Targeting Rules: Can They Stabilize the Economy?. Economic Review, Federal Reserve Bank of Kansas City, issue Q III: 11-25.

Cowen, Tyler and Tabarrok, Alex 2011 Modern Principles: Macroeconomics. $2^{\text {nd }}$ Ed. Worth Publishers.

Eggertsson, G.B. and Woodford, M. 2003. The Zero Bound on Interest Rates and Optimal Monetary Policy. Brookings Papers on Economic Activity 2003(1): 139-211.

European Commission 2009. "Impact of the Current Economic and Financial Crisis on Potential Output". European Economy Occasional Papers 49, June, Brussels.

Frankel Jeffrey 2009. Comment on "The Simple Geometry of Transmission and Stabilization in Closed and Open Economies." NBER Chapters, in: NBER International Seminar on Macroeconomics 2007: 119-129.

Friedman, Milton 1969. The Optimum Quantity Of Money and Other Essays, Chicago, IL, Aldine

Friedman, Milton 1984. Monetary Policy for the 1980's. In To Promote Prosperity: U.S. Domestic Policy in the Mid-1980s Moore, John H., ed. Hoover Institution Press, Stanford, CA.
Furceri, Davide and Mourougane, Annabelle 2009. The Effect of Financial Crises on Potential Output: New Empirical Evidence from OECD Countries. OECD Economics Department Working Papers, No. 699, OECD publishing.

Garrison, Roger W. 2001. Time and Money, The Macroeconomics of Capital Structure, Routledge.

Haberler, Gottfried 1946 [1937]. Prosperity and Depression, $3^{\text {rd }}$ Edition. United Nations Lake Success, New York.

Hall, Robert E. and Mankiw, N. Gregory 1994. Nominal Income Targeting, NBER Working Papers 4439, National Bureau of Economic Research, Inc.

Hayek, Friedrich A. von 1928 Intertemporal Price Equilibrium and Movements in the Value of Money. In F. A. Hayek. Money, Capital, and Fluctuations: Early Essays, edited by Roy McCloughry, Chicago, University of Chicago Press, 1984.

Hayek, Friedrich A. von 1933 [1929]. Monetary Theory and the Trade Cycle, Jonathan Cape, London.

Hayek, Friedrich A. von 1933 b in German. On 'Neutral Money', in F. A. Hayek. Money, Capital, and Fluctuations: Early Essays, edited by Roy McCloughry, Chicago, University of Chicago Press, 1984.

Hayek, Friedrich A. von 1935 [1931]. Prices and Production, 2nd edition, Augustus M. Kelly, Publishers New York.

Hayek, Friedrich A. von 1969. Three Elucidations of the Ricardo Effect. The Journal of Political Economy 77(2): 274-285.

Hayek, Friedrich A. von. 1939. Profits, Interest and Investment. Augustus M. Kelley Publishers.

Jung Taehun, Teranishi Yuki and Watanabe Teutonu. 2005. Optimal Monetary Policy at the Zero-Interest-Rate Bound. Journal of Money, Credit and Banking 37(5): 813-35.

Krugman, Paul R., Dominquez, Kathryn M. and Rogoff, Kenneth 1998. It's Baaack: Japan's Slump and the Return of the Liquidity Trap. Brookings Papers on Economic Activity, 1998(2): 137-205.

Laubach, Thomas and Williams, John C. 2003. Measuring the Natural Rate of Interest. The Review of Economics and Statistics 85(4): 1063-1070.

Mankiw, N. Gregory 1985. Small Menu Costs and Large Business Cycles: A Macroeconomic Model of Monopoly. The Quarterly Journal of Economics 100(2): 529-537.

Mankiw, N. Gregory and Romer, David, ed. 1991a. New Keynesian Economics: Volume 1. MIT Press.

Mankiw, N. Gregory and Romer, David, ed. 1991b. New Keynesian Economics: Volume 2. MIT Press.

Mankiw, N. Gregory 2003. Macroeconomics, 5th edition, Worth Publishers, New York.
Mankiw, N. Gregory 2009. Macroeconomics, 7th edition, Worth Publishers, New York.
McCallum, Bennett T. 2000. Theoretical Analysis Regarding a Zero Lower Bound on Nominal Interest Rates. Journal of Money, Credit and Banking 32(4): 870-904.

Mises, Ludwig von 1976 [1912]. Theory of Money and Credit. The Foundation for Economic Education.

Okun, Arthur 1980. Prices and Quantities: A Macroeconomics Analysis, Washington: Brookings Institution.

Orphanides, Athanasios 2004. Monetary Policy in Deflation: The Liquidity Trap in History and Practice. The North American Journal of Economics and Finance 15(1): 101-124.

Orphanides, Athanasios and Williams, John C. 2002. Robust Monetary Policy Rules with Unknown Natural Rates. Brookings Papers on Economic Activity, Economic Studies Program, The Brookings Institution 33(2): 63-146.

Potužák, Pavel 2007. Rakouská teorie hospodářského cyklu - pohled současné makroekonomie, diplomová práce, VŠE Praha.

Romer David 1993. The New Keynesian Synthesis. The Journal of Economic Perspectives 7(1): 5-22.

Romer, David 2006. Advanced Macroeconomics, 3rd edition, McGraw - Hill, New York.
Rothbard, Murray N. 2004 [1962]. Man, Economy, and State. Ludwig von Mises Institute.
Rudebusch, Glenn D. 2002. Assessing Nominal Income Rules for Monetary Policy with Model and Data Uncertainty. The Economic Journal 112(479): 402-432.

Schumpeter, Joseph A. 1961 [1912].The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle. Oxford, NY [US] : Oxford University Press.

Selgin, George 1997. Less Than Zero: The Case for a Falling Price Level in a Growing Economy. The Institute of Economic Affairs.
de Soto, Jesús Huerta 2006 [1998]. Money, Bank Credit, and Economic Cycles. Ludwig von Mises Institute.

Taylor, John B. 1985. What Would Nominal GDP Targeting Do to the Business Cycle? Carnegie-Rochester Conference Series on Public Policy, Amsterdam: NorthHolland 22: 61-84.

Taylor, John B. 1993. Discretion Versus Policy Rules in Practice Carnegie-Rochester Conference Series on Public Policy 39: 195-214.

West, Kenneth D. 1986. Targeting Nominal Income: A Note," Economic Journal 96(384):1077-83.

Wicksell, Knut. 1936 [1898]. Interest and Prices. Augustus M Kelley Publishers.
Wicksell, Knut 1977 [1906]. Lectures on Political Economy, Volume 2. Augustus M Kelley Publishers.

Wieland, Johannes 2012. Are Negative Supply Shocks Expansionary at the Zero Lower Bound? Inflation Expectations and Financial Frictions in Sticky-Price Models. job market paper. UC Berkeley. http://crei.eu/files/filesActivity/48/wieland.pdf

Williams, John C. 2003. The Natural Rate of Interest, FRBSF Economic Letter, Federal Reserve Bank of San Francisco, Issue Oct 31.

Wolman, Alexander L. 2005. Real Implications of the Zero Bound on Nominal Interest Rates. Journal of Money, Credit and Banking 37(2): 273-96.

Woodford, Michael 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press.

Woodford, Michael 2008. How Important Is Money in the Conduct of Monetary Policy?. Journal of Money, Credit and Banking 40(8): 1561-1598.

## CONCLUDING REMARKS

There is no need to repeat concluding words and recommendations presented in the four Chapters. Yet, let me highlight major findings of this dissertation.

Chapter 1 developed a new set of tools used to elucidate the Austrian theory of capital. These tools identified time as the key element in this theory along with the fact that capital cannot be considered a homogeneous mass without structure, which is automatically maintained. These properties were best illustrated within a transition from a shorter method of production to a longer one caused by a general decline in time preference. The new diagrammatical apparatus showed the essence of saving and investment in this process: new forms of capital are being created owing to reallocation of factors of production, the output of present consumption goods temporarily drops, and the process eventually ends with a larger amount of consumption goods. However, the analysis demonstrated that a higher growth rate in the output of consumption goods cannot be permanent if the roundabout methods exhibit diminishing marginal productivity. At this point, the analysis corrected some recent visions of Garrison.

Chapter 2 utilized the theory of capital from Chapter 1 by showing that the consistent intertemporal allocation of resources can be disturbed by a monetary shock. This section focused on the U-shaped behaviour of the interest rate during the business cycle. It defended the Austrian theory against Kaldor, who detected seeming inconsistency, by pointing out that the increase in the interest rate at the beginning of the downswing phase of the cycle is caused by a drop in real saving that has a genetic code in the boom phase of the cycle. Chapter 2 also showed how the Hayekian triangles can be used to map the business cycle caused by a real shock that is consequently accommodated by the banking system. Specifically, business cycle can be caused not only by active banking system but also by its passive behaviour if the banking system does not raise the market rate of interest to a natural level, which was increased by a positive technological shock. The same apparatus was then utilized to thoroughly elucidate disruptions in the demand for money, which has not been analyzed in the Austrian literature in this case. A seeming paradox of not reverting behaviour of the interest rate during recent economic crisis was resolved within the Austrian theory of capital. The answer resided in the fact that the natural rate of interest itself might be affected, suggesting long-run non-neutrality of money in the Austrian business cycle theory. These findings opened up new ways for a deep investigation of the natural rate of interest in Chapter 3 since the Mises-Rothbard branch in the Austrian school would not accept the idea that the natural rate of interest can be permanently affected by the productivity of capital.
Chapter 3 demonstrated that the Mises-Rothbard pure time preference theory (PTPT) of interest is flawed. Especially the second author confused two meanings of time preference, as this dissertation tried to highlight. As a result, productivity of capital can co-determine the natural rate of interest. Moreover, under some conditions it can be its sole determinant. Above all, the analysis showed that the natural rate of interest can be negative, which is unthinkable in the Mises-Rothbard system, even if people exhibit a priori positive time preference. This may happen if the time shape of the income stream is decreasing and the intertemporal elasticity of substitution in consumption ( $1 / \theta$ ) is sufficiently low. This dissertation showed that the Mises conclusion of the superiority of the pure time preference $(\rho)$ in determining the real natural rate of interest ( r ) is valid only for the case of non-diminishing (constant) marginal utility from consumption, resulting in linear indifference curves with infinite intertemporal elasticity of substitution in consumption ( $1 / \theta \rightarrow \infty$, hence $\theta \rightarrow 0$ ). Other Austrian economists argued in favour of the pure time preference theory such that this theory explains the value difference between expended inputs and the resulting output, or even explicitly the PTPT is well designed to explain the interest on money. Chapter 3 proved that
pure time preference $(\rho)$ coincides with the nominal interest (i) in the world with constant money only if the intertemporal elasticity of substitution in consumption is equal to one ( $1 / \theta=$ 1). The Mises very strong statement that zero time preference would lead to eternal postponement of the act of consumption was disproved as well. Chapter 3 generalised findings of modern authors and showed that positive interest rate might coexist with zero time preference and positive present (and future) consumption if the intertemporal elasticity of substitution in consumption is low enough $(1 / \theta<1$, thus $\theta>1)$, which was derived from a more general and ubiquitous condition $\rho-(1-\theta) \mathrm{r}>0$.

Chapter 4 presented an analysis of the business cycle in a growing economy. It built on the findings from the previous Chapters. This section introduced arguments that questioned modern New Keynesian inflation targeting regime, stressing injection effects of the newly created money supply aimed at price level stabilization in an economy with expanding natural output. A gradually falling price level was identified as a natural response of the price system in the expanding economy. It was also shown that the key dividing point between the Austrian and the New Keynesian theory is whether the higher real demand for money, which emerges due to rising potential output and real incomes, is to be satisfied by a reduction in the price level or by a higher nominal money supply. Austrians prefer the former since the latter may lead to the boom-bust cycle. New Keynesians would recommend the latter because the former could result in quasi-recession. In the Austrian vision, money is neutral with respect to the real economy if it is not increased in the situation of expanding output, whereas New Keynesians believe in the rigidity of prices that may cause serious problems when the money supply is not appropriately adjusted. The core of Chapter 4 examined the Hayek proposal of constant MV. Shifts in aggregate demand caused by the LM shocks and the IS shocks tested robustness of the MV-rule along with the AS-shocks. It was demonstrated that in the economy with expanding natural output, the MV rule is passive only for unit incomeelasticity of the demand for money. Hayek recommendations for a stable money supply in a growing economy might be therefore inconsistent with stable MV. In the very long run, secular deflation provoked by this rule may also depress the nominal interest rate to very low levels. A close resemblance to the Friedman rule of the optimum quantity of money was studied. It was shown that the two rules coincide if the economy is at the golden rule level of capital accumulation, and both imply zero nominal interest. The Hayek rule is less deflationary compared with the Friedman rule if the economy is dynamically efficient, and it leads to a positive nominal interest rate. If the economy over-saves and is dynamically inefficient, the MV is blocked by the zero lower bound on the nominal interest rate, and selffulfilling accelerating deflation may be triggered. At the end of Chapter 4, the Hayek MV-rule was integrated with a simple monetary policy rule of the Taylor type. The Hayek-Taylor rule was created. It was shown that the traditional Taylor rules are too expansionary from the Austrian point of view. Furthermore, compared with the Taylor rule, the present period output gap need not be known. The Hayek-Taylor rule only requires the estimation of the natural rate of interest along with the growth rate of the aggregate nominal income.

At the final stage, let me also briefly indicate the most pressing problems in the theory here examined and in the economic theory in general. First, it should be stressed that the theory of capital is disregarded in modern economics. Modern economics textbooks spend only a handful of pages on the explanation of the theory of capital. Even more surprising, some basic microeconomics textbooks do not mention capital at all, apart from the reference to variable K on the vertical axis in the analysis of isoquants in the theory of firm. As a result, since generations of economists are not trained in the ideas of the Austrian authors who formed the basis of modern theory of capital, and no alternative theory is being taught as a substitute, students of economics leave universities with serious gaps in the understanding of the intertemporal allocation of resources.

Secondly, there is a rising tendency to disregard money in modern macroeconomic theory. The major concern is about the interest rate; yet, no satisfactory theory of interest is presented. Authors of macroeconomics textbooks often work with the natural rate of interest. However, as regards the proper explanation of this key variable in their models, they refer to microeconomics textbooks since this variable is, in the end, determined by real forces. The problem is, however, that the microeconomics textbooks do not provide sound explanation of this phenomenon, because they lack satisfactory theory of capital.
As a result, if one compares chapters in modern textbooks about interest and capital with the writings of Böhm-Bawerk, it is immediately obvious that much more insights and knowledge about the problem can be found in the latter rather than in the former. It is fascinating that, with the exceptions of Hayek or Knight, economic science made such little progress in this field. Even more disturbing is that the most brilliant minds in economic science are only marginally interested in this field of the theory that demands so much progress.

The same objection holds to the theory of money. Pre-war economic theorists had deep understanding about the functioning of the banking system since these scholars were also economic professionals and sometimes bankers, but modern monetary theory textbooks, on the other hand, explore this topic so little. The entire discussion of money is concentrated in the definition of M . No reference is made to the nature of money, to its role in the intertemporal exchange of goods. No discussion is carried out regarding whether money is a present good or a future good, and the same applies to money substitutes. It can be argued that this lack of interest in these questions is complementary to the ignorance of the problems in the theory of capital.

As a final word, let me present my personal recommendations as regards the future development of the Austrian theory of capital and business cycle. In my opinion, this theory has not developed much, since Hayek abandoned the topic after the WWII. All brilliant minds in economic science are engaged in research that is outside the field of Austrian economics. Thus, this economic school will not survive as a vital part of the economic theory unless it absorbs the great body of knowledge accumulated in mainstream economics. There are serious gaps in mainstream economics especially in the theory of capital, and the Austrian theory may fill these gaps if it learns better the language and the key ideas of modern neoclassical theory. This study was a modest contribution in this specific direction.


[^0]:    ${ }^{1}$ At this point, Lachmann roughly accepted Irving Fisher's approach. For Fisher (1930), capital (or capital value) was just the discounted future income, the source of which could be any wealth (including human beings).
    ${ }^{2}$ It may be interesting to add the definition of capital offered by William S. Jevons. Curiously enough, Jevons (1957:223) defined capital as "commodities which are required for sustaining labourers of any kind or class engaged in work." In this connection, he even added that: "The capital is not the railway, but the food of those who made the railway (Jevons 1957:243). Böhm-Bawerk (1890:57) rejected this approach to capital by saying that a primitive tribe endowed with free food given by nature would be the most capitalist nation in the world. However, another Austrian theorist, R. von Strigl (2000), partly adopted the Jevonian approach by stressing the significance of the subsistence fund in the roundabout methods, detracting the role played by particular capital goods.

[^1]:    ${ }^{3}$ This fundamental property of human action will be embedded in the utility function of a representative agent presented in Chapter 3 of this dissertation.
    ${ }^{4}$ Very similar ideas can be found also in Jevons (1957) who talked about a lower intensity of future anticipated feelings (1957:34-35) or about the fact that future utilities must be weighed (i.e. scaled down) by the probability of their realisation (1957:72).
    ${ }^{5}$ However, traces of this idea can be also found in Menger (2007:154): Setting aside the irregularities of economic activity, we can conclude that economizing men generally endeavor to ensure the satisfaction of needs of the immediate future first, and that only after this has been done, do they attempt to ensure the satisfaction of needs of more distant periods, in accordance with their remoteness in time.
    ${ }^{6}$ In Chapter 3, we will discuss whether this reasoning is strong enough to guarantee the existence of the natural (or originary) rate of interest under all circumstances, as was postulated not only by Mises but also by Rothbard (2004) and other proponents of the pure time preference theory.
    ${ }^{7}$ Fisher (1930) used the term "impatience" rather than "time preference" because it seemed to him less ambiguous since the time preference could also mean a preference for future goods over present goods.

[^2]:    ${ }^{8}$ Lachmann (1956:79ff) called this phenomenon a "division of capital" or an "increasing specialization of the processing function", and a "vertical disintegration of the capital structure."
    ${ }^{9}$ Interestingly, the original example of Adam Smith (2001) about the pin factory is more in line with vertical division rather than with horizontal division of labour. This topic is also mentioned in Wicksell (1954:116).
    ${ }^{10} \mathrm{We}$ should qualify this statement by saying that only the method that provides the highest output up to the given length will be chosen.
    ${ }^{11}$ For example, making a stick and knocking down apples can take less time than climbing the tree directly (Murphy 2003).

[^3]:    ${ }^{12}$ It would be more consistent with Hayek's approach to talk about the length of the period for which the given input is invested (Hayek 1941:69-70). Hence only the methods, for which the given input matures into the highest possible output for the given period of investment of this input, will be chosen.

[^4]:    ${ }^{13}$ This idea is partly elaborated in Hayek (1941:73), Mises (1996:529), and Rothbard (2004). It should be stressed that the unused knowledge is not a symptom of inefficiency in the economy. Using latent technologies may be too costly in terms of "waiting" that must be undergone to fully utilise their potential.

[^5]:    ${ }^{14}$ The analysis is even more complicated once we realise that the inventions themselves do not fall from heaven. As was demonstrated by the new growth theory, to create new knowledge or invention, investments are also necessary as in other branches of production. In Austrian terms, to create new invention a long roundabout process must be started.
    ${ }^{15}$ Wicksell (1977a) demonstrated that usual considerations about marginal productivity of capital may lead to curious results.
    ${ }^{16}$ We can imagine a fourth process - a very long process - that will mature in 10 periods, providing 24 units of output. As before, the prolongation leads to a higher output, yet the marginal increase falls down. After 10 periods, the output of the methods in question is as follows: 10,20 , and 24.

[^6]:    ${ }^{17}$ In Chapter 3, we identify the second reason with the subjective discount rate that may depend on the average level of income.
    ${ }^{18}$ We will return to this problem when we develop the simple model of loanable funds.
    ${ }^{19}$ Hayek (1935a:38) himself accepted Marschak's suggestion to call this figure the Jevonian investment figure since the triangle first appeared in Jevons (1957).

[^7]:    ${ }^{20}$ This objection was especially raised by Knight (1934; 1935a; 1935b; 1936a; 1936b).

[^8]:    ${ }^{21}$ In this connection, the discussion between Hayek and Knight about the essence of capital is especially illuminating. See, for example, Hayek (1935b; 1936a) and Knight (1935a).
    ${ }^{22}$ For instance, 12 workers are always more than 10 workers regardless of the wage rate. However, as capital can be measured only in value terms, the change in prices may reverse which process represents a higher quantity of capital. Hence, the Austrians demonstrated the problems connected with the capital theory much earlier than the famous Cambridge capital controversy (Samuelson 1966).

[^9]:    ${ }^{23}$ It would surely be possible to reproduce the entire three-dimensional model. Although our figure lacks one dimension and hence one feature of the production process, it seems that it also picks up the most important characteristics of the production process from the point of view of the Austrian capital theory.
    ${ }^{24}$ In the analysis of the business cycle, we will be mainly interested in triangles (or various types of triangles) that will mature in the future (or at different moments in the future depending on the time preference, which will decide the roundaboutness and hence productivity). Our approach will be again forward-looking.

[^10]:    ${ }^{25}$ In this connection, it should be perfectly clear that if, for one reason or another, the capitalistic processes are not initiated anew, the processes underway may for some time (even for a considerable time) provide an undisturbed flow of consumption goods. The lack of consumption goods will thus arise after some period of time, and their flow will only gradually, not abruptly, decline since the stock of unfinished goods will only continuously diminish (Rothbard 2004).
    ${ }^{26}$ The critique of Garrison's approach can be found in Hülsmann (2001) and Fillieule (2005), the critique of Hayekian triangles in Barnett and Block (2006).

[^11]:    ${ }^{27}$ This property is in line with one of the Inada conditions presented in Appendix 7 of Chapter 3.
    ${ }^{28}$ The pure time preference approach of the Austrian authors may be found in Herbener (2011). On the other hand, Murphy (2003) showed that Böhm-Bawerk is perfectly consistent, and his theory was misinterpreted by the above-mentioned authors.

[^12]:    ${ }^{29}$ Fisher (1930) himself attributed this explanation to Böhm-Bawerk, although in his point of view BöhmBawerk's inclusion of productivity was rather confused and misguided.
    ${ }^{30}$ As will be seen in Chapter 3, this sentence should be slightly reformulated to make the theory more consistent. The interest rate is to be compared with the difference between the value of present factors of production and the (future) value of consumption goods that will be produced by these present factors of production in the future. However, in the stationary economy, which has been analysed so far, the value of consumption goods is the same at every moment.

[^13]:    ${ }^{31}$ In Chapter 2, this assumption will be relaxed and we will analyse the behaviour of the economy if part of the saved income is not invested but hoarded - retained in money balances.

[^14]:    ${ }^{32}$ Hayek (1932: 27) in a discussion with J.M. Keynes added that if entrepreneurs in the consumption stage continue with their production at an unaltered level even when the demand for their products diminished, the losses they suffer and the resulting decline in their capital can be offset only by a reduction of their own consumption. However, this peculiar behaviour would block the factors of production in the consumption stage from being released to the early stages, and the production process could never be lengthened.
    ${ }^{33}$ An interesting question is what will happen to consumption goods that are not sold due to a decrease in the consumption demand. Hayek (1941:275) argued (quite in line with a simple story about Robinson Crusoe) that unless the goods in question easily spoil, they can wait in the stock of firms till the moment when the supply of consumption goods is diminished due to the transfer of resources to early stages of the production process and when the new longer processes still do not provide consumption goods. At this particular moment, the stored-up consumption goods may be released from the stocks and mitigate their lack on the market. At the same time, the opportunity costs of holding this reserve of consumption goods are reduced due to the decrease in the interest rate. Hence, the image of the economy presented above may not be far from the real world.
    ${ }^{34}$ At this stage, we discuss the behaviour of the real part of the economy that responded to changes in the data transmitted through the price system. In Chapter 2 and Chapter 4, we will show the reaction of the economy if the decline in prices of consumption goods is not allowed (or not wanted) by the central bank.
    ${ }^{35}$ As can be seen in Figure No. 14, there is a point in the structure of production where the demand neither rises nor falls. It is exactly the point at which the so-called effect of derived demand and the time discount effect offset each other (Garrison 2001).

[^15]:    ${ }^{36}$ The following calculations and graphs of the labour markets can be found in Potuzak (2007).

[^16]:    ${ }^{37}$ All features about the process just analyzed hold for any factor of production and can be easily generalized for the circulating capital.

[^17]:    ${ }^{38}$ It can be argued that the given worker will be specialized only in the production of one specific intermediate good, e.g. III. This specialization and division of labour may further enhance productivity of the longer process. This vertical division of labour was already discussed in this Chapter.

[^18]:    ${ }^{39}$ It might be argued that the approach of R. Garrison is quite inconsistent since he did not distinguish between Hayekian triangles in nominal terms and in real terms. When it is inevitable for the clarity of the exposition, we will explicitly discuss whether the figure is in nominal terms or in real terms, as in Figure No. 21.

[^19]:    ${ }^{40}$ In this schema, when four processes are roundabout and one is direct, the sequence of output is as follows: $1,11,11,11,11 ; 1,11,11,11,11 ; 1,11,11,11,11$
    ${ }^{41}$ It can be demonstrated again that if some part of the labour force is withdrawn from the market, the decline in output of consumption goods will occur much later. Such a phenomenon will be discussed in more detail in the business cycle analysis in Chapter 2.
    ${ }^{42}$ This model and the RCK model below will be developed in Chapter 3 in which we utilise their structure to find the key determinants of the natural rate of interest in the neoclassical framework. We will also explore whether the three causes of interest given by Böhm-Bawerk are hidden in their solution or not.
    ${ }^{43}$ In the Solow model, output represents the total amount of final goods (total amount of one single good produced in the economy), either devoted to consumption or to investment. It is also equal to total income that can be used either for consumption or saving. Saving and investment is virtually the same thing in this model.
    ${ }^{44}$ Consumption will be higher in the end only if the economy is dynamically efficient (Phelps 1961; 1965). The discussion of the dynamic efficiency will be introduced in Chapter 3.

[^20]:    ${ }^{45}$ This system of graphs can be found in Romer (2006:19, 66). This model will be developed in Appendix 7 in Chapter 3.
    ${ }^{46}$ Ramsey (1928) did not accept the idea of the underestimation of future wants. Hence, he did not include the subjective discount rate to his model. To avoid the divergence of the utility integral, he posited the satiation of

[^21]:    wants at some time in the future at the so called "bliss point". It was Cass (1965) and Koopmans (1963) who introduced the subjective discount rate to the model.

[^22]:    ${ }^{47}$ The interest rate in the Solow model is given by the difference between the marginal product of capital (MPK) and the rate of depreciation $\delta$. Furthermore, the MPK is represented by the slope of the tangent line to the production function at any given point. As can be seen in Figure No. 22, the MPK (together with the interest rate) gradually declines, as the economy moves to the new steady state.

[^23]:    ${ }^{48}$ Hayek (1935:38) himself accepted Marschak's suggestion to designate this figure as the Jevonian investment figure.
    ${ }^{49}$ Hayek $(1935,1941)$ called this difference the profit margin, or the price margin.

[^24]:    ${ }^{50}$ The critique of the Garrison's approach can be found in Hülsmann (2001), Fillieule (2005), the critique of Hayekian triangles in Barnett and Block (2006).

[^25]:    ${ }^{51}$ Modern literature and the New Keynesian model (Woodford 2003) assume that the natural rate of interest is consistent with the natural level of output, also known as the flexible-price output.
    ${ }^{52}$ Fisher (1930) himself attributed this explanation to Böhm-Bawerk, although he claimed that Böhm-Bawerk's inclusion of productivity was rather confused and misguided. On the other hand, Austrian economists that adhere to the pure time preference theory blame Böhm-Bawerk for inconsistencies and criticise the element of productivity in his theory. See, for example, Kirzner (2011).
    ${ }^{53}$ Chapter 3 of this dissertation is an attempt to expose the problems in the pure time preference theory.

[^26]:    ${ }^{54}$ This idea is best developed in Böhm-Bawerk (1891) and Hayek (1941).
    ${ }^{55}$ More thorough analysis of this process can be found in Böhm-Bawerk (1891) or Hayek (1931; 1935)

[^27]:    ${ }^{56}$ As can be seen in Figure No. 8, there is a point in the structure of production where the demand neither rises nor falls. It is exactly the point at which the so-called effect of the derived demand and the discount effect offset each other (Garrison 2001).

[^28]:    ${ }^{57}$ Almost one century ago Mises (1976) clarified, while criticising theories of the Banking School, that virtually any amount of newly created money can be placed on the market by a sufficient reduction in the interest rate.

[^29]:    ${ }^{58} \mathrm{~A}$ situation of the economy below its potential will be discussed later in the text.

[^30]:    ${ }^{59}$ In Figure No. 11, the transfer of factors of production from the market of consumption goods could be manifested as the inward shift of the supply curve.

[^31]:    ${ }^{60}$ The new longer process resulting in 15 units at the end creates different intermediate products than a relatively shorter process ending with 10 units. Thus, we should distinguish between $\mathrm{I}_{\mathrm{A}}, \mathrm{II}_{\mathrm{A}}, \mathrm{III}_{\mathrm{A}}, \mathrm{IV}_{\mathrm{A}}$, and $\mathrm{I}_{\mathrm{B}}, \mathrm{II}_{\mathrm{B}}, \mathrm{III}_{\mathrm{B}}, \mathrm{IV}_{\mathrm{B}}$, $\mathrm{V}_{\mathrm{B}}$. As before, capital is changing not only its size but also its structure.

[^32]:    ${ }^{61}$ This observation was made also by Wicksell (1936:155), who added that consumers might be partly compensated next period when the longer methods are completed.

[^33]:    ${ }^{62}$ The subsistence fund theory is criticised in Hayek (1935; 1941).

[^34]:    ${ }^{63}$ Similar statement can be found in Hayek (1935) and Garrison (2001).

[^35]:    ${ }^{64}$ Discussion about the correct calculation of the interest rate in the time-consuming process of production can be found in Rothbard (2004).

[^36]:    ${ }^{65}$ As can be seen in Figure No. 14, there is a time period (from t1 to t2) in which the nominal interest rate is actually rising, however, the real rate is still declining. This lag of the real interest rate behind the nominal rate was mentioned by Fisher (1930) who wrote that the nominal interest rate does not respond enough (both in time and height) to inflation, which leads to a decline in the real interest rate.
    ${ }^{66}$ The new approach can be found in Hayek (1939,1941,1942a). The critique that followed in Kaldor (1942) and the consequent response in Hayek (1942b).

[^37]:    ${ }^{67}$ We may say that this is exactly the point of time at which the phenomenon of the forced saving is about to end because workers have finally received the newly created money.
    ${ }^{68}(106 / 56)^{1 / 10}-1=6.6 \%$

[^38]:    ${ }^{69}$ Some traces of similar confusion can also be found in Mises (1996:553):
    Of course, in order to continue production on the enlarged scale brought about by the expansion of credit, all entrepreneurs, those who did expand their activities no less than those who produce only within the limits in which they produced previously, need additional funds as the costs of production are now higher. If the credit expansion consists merely in a single, not repeated injection of a definite amount of fiduciary media into the loan market and then ceases altogether, the boom must very soon stop. The entrepreneurs cannot procure the funds they need for the further conduct of their ventures. This gross market rate of interest rises because the increased demand for loans is not counterpoised by a corresponding increase in the quantity of money available for lending. [emphasis added]
    And in Garrison (2001:72):
    The bidding for increasingly scarce resources and the accompanying increased demands for credit put upward pressure on the interest rate. [emphasis added]

[^39]:    ${ }^{70}$ At this moment, it would be better to avoid the term - expansion of real savings. The injection of money into the system extends the supply of real funds firms have at their disposal. Using the term "increase in real savings" would suggest that monetary expansion brings about higher saving on the part of people, which is obviously not the case; rather the opposite is true.

[^40]:    ${ }^{71}$ If the real demand for money was not sensitive to the nominal interest rate, the LM curve would be vertical, and the economy might smoothly move to a higher interest rate with no booming period. In such a case, there will be no period of temporarily higher inflation rate - it will be perfectly equal to the rate of monetary expansion in all periods, and money would be super-neutral in the long run.

[^41]:    ${ }^{72} \mathrm{~A}$ situation of relative abundance of the factors of production will be analysed later in the text.
    ${ }^{73}$ Since the resources are moving to early stages of production, the term Pi (i.e. price of capital goods) falls down, which will further encourage the growth in the profit margins $(\mathrm{Pc} / \mathrm{Pi})$.
    ${ }^{74}$ This theory is at odds with the concept of the accelerator, thoroughly investigated in Hayek (1939).

[^42]:    ${ }^{75}$ This question was raised, for example, by Friedman (1968). The new Keynesian literature also suggests that the central bank does not have total freedom in setting the interest rate - it must obey the Taylor principle otherwise the inflation will get out of control (Woodford 2003).

[^43]:    ${ }^{76}$ The subsequent dynamics of the natural rate of interest is more complicated than the path suggested in this paper. New invention will bring about an increase in incomes and consequently savings that will push the interest rate downwards. However, the precise description of the path of the natural interest rate would require a more complicated model of the growth theory. Fisher (1930) himself suggested that the eventual level of the interest rate would be somewhat lower. On the other hand, modern growth models, such as the Solow or the Ramsey-Cass-Koopmans models, base their predictions on the distinction whether the initial shock improves only the level of technologies in the economy, or their growth rate.
    ${ }^{77}$ Rothbard (1963:32), as the adherent of the pure time preference theory, strictly rejects this type of analysis:
    In defense of the Mises "anti-bank" position, we must first point out that the natural interest rate or "profit rate" does not suddenly increase because of vague improvements in "investment opportunities." The natural rate increases because time preferences increase.

[^44]:    ${ }^{78}$ This change of the shape of the triangle can be deduced from Hayek (1941; 1942a).

[^45]:    ${ }^{79}$ For simplicity, we assume that the time structure of loans and time deposits is perfectly matched.

[^46]:    ${ }^{80}$ The presented example is highly stylized. One major inconsistency lies in the fact that investment and saving are flows, whereas monetary and credit aggregates are stocks. It would be more in line with the concept of flows if we assumed that the investment demand increased to 100 , and part of this demand (e.g. 50) was reduced due to higher interest. A similar reasoning would hold for saving. A more detailed analysis of the money creation process can be found in de Soto (2006).

[^47]:    ${ }^{81} \mathrm{~A}$ similar idea can be found in Cassel (1928:528).

[^48]:    ${ }^{82}$ More on this can be also found in Cochran and Call (1998), Cochran et al. (1999; 2003), and Cochran (2004).

[^49]:    ${ }^{83}$ It must be mentioned that the Austrian analysis is at odds with the approach of the real business cycle theory. King and Plosser (1984), for example, concluded that the expansion of deposits during the economic boom is just a neutral response of the banking sector to the positive technological shock.

[^50]:    ${ }^{84}$ Using a simple equation of exchange in the case just being studied, V increases rather than M in the expression MV. Further discussion can be found in Cochran and Call (1998; 2000).

[^51]:    ${ }^{85}$ In the new Keynesian literature, at least the first two may not be used interchangeably.

[^52]:    ${ }^{86}$ See, for example, the European Commission (2009), and Furceri and Mourougane (2009).
    ${ }^{87}$ In the Austrian literature, this phenomenon is known as the secondary deflation.

[^53]:    ${ }^{88}$ Some authors (Eggertsson and Woodford 2003) investigating optimal monetary policy when the zero bound on nominal interest rate has been hit explicitly assumed that the natural rate of interest is even negative. The idea that the natural rate is negative would require a thorough investigation, so it is postponed to Chapter 3 and 4.

[^54]:    ${ }^{89}$ See, for example, important articles of Koopmans (1960), Lancaster (1963), Olson and Bailey (1981), and Becker (1997) that referred to Böhm-Bawerk and that considered him a founder of the modern theory of interest.

[^55]:    ${ }^{90}$ In the Austrian tradition, the theory of diminishing marginal utility was developed by Menger (2007) and further refined by Böhm-Bawerk (1891) himself.
    ${ }^{91}$ Very similar ideas can be found in Jevons (1957) who talked about a lower intensity of future anticipated feelings (1957:34-35) and about the fact that future utilities must be weighed (i.e. scaled down) by the probability of their realisation (1957:72).

[^56]:    ${ }^{92}$ Böhm-Bawerk admitted that not only future goods might possess higher valuation than present goods (1891:252), but he briefly mentioned exceptions also for the second cause of interest, i.e. future might not be always undervalued compared with the present (1891:257). Frank Fetter offered an example for the nonexistence of time preference: If the needs of men were supplied from day to day by some outside agency, if the things we need fell like manna from the skies,..., there would be no such thing as time-preference or time-value. Fetter (1928:236)

[^57]:    ${ }^{93}$ The ideas of these two Austrian authors are almost indistinguishable. Rothbard's reasoning is sometimes clearer and more understandable. Nevertheless, any critique raised against Mises is readily applicable to Rothbard as well, unless we explicitly specify that the given statement is to be attributed only to Rothbard.
    ${ }^{94}$ Topan and Păun (2013) remarked that Mises was inconsistent from the pure logic point of view - a general statement may not be negated by another general statement. Hence, the correct negation should be - at least once, man would prefer consumption in the future. Furthermore, they also noted that the preference for future would lead to "conscious non-action", which is a logical contradiction. In this respect, they were inspired by Block (1990:199) who claimed that the lack of time preference would lead to eternal non-action. Thus, they generalized the idea of Mises of the eternal postponement of consumption. A comment on their approach can be found in Herbener (2013).
    ${ }^{95}$ This observation is also mentioned in Fetter (1928:241).

[^58]:    ${ }^{96}$ The same argument, along with other comments, can be found in Block (1978).

[^59]:    ${ }^{97}$ Obviously, future ice is a different good from present ice due to their different time position. However, apart from that, Mises and Rothbard added the reason of a different technique of production or a different wants the given goods satisfy. Thus, we question these two approaches, not the obvious fact that present and future ice might be considered different goods solely due to the different position in time.
    ${ }^{98}$ Pellengahr (1996:26-27) pointed out (in considering the phenomenon which we call here the time preference in the second sense) that it must be distinguished whether, on the one hand, the analysis is carried out as the comparison between the same present and the same future satisfaction, where the former is preferred to the later. Or whether, on the other hand, it is stated that the given satisfaction is preferred earlier rather than later. According to him, the first approach is misleading because the acting man cannot declare that the given satisfaction being compared is the same both in the present and in the future, and at the same time claim that the former is preferred to the later. Thus, it is the second approach, comparing the moment at which the given satisfaction might be gratified, which should be used as the genuine representative of the time preference (in the second sense). We perfectly agree with this objection, however, as a shortcut, the first approach will be sometimes used. This may be defended also by the fact that it is so widespread in the literature. See, for example, Fisher (1907:246), Fisher (1930:24), Rothbard (2004:51), Murphy (2003:61), Herbener (2011:14).
    ${ }^{99}$ Following Menger (2007:122ff) and Böhm-Bawerk (1891:146ff), want "I" could be to "eat", want "II" to "feed the dog", want "III" to "feed the cat", want "IV" to "feed the fish". Good A can be represented by a baker's roll.

[^60]:    ${ }^{100}$ In our example, "eat now" is preferred to "eat tomorrow". However, "eat tomorrow" and "feed the dog tomorrow" might be preferable to "eat now".
    ${ }^{101}$ It might be preferable for the consumer (from the present perspective) to "feed the dog tomorrow" to "feed the fish today".

[^61]:    ${ }^{102}$ More on this can be found in Chapter 1 of this dissertation.
    ${ }^{103}$ Fisher (1930) would prefer the term "impatience" rather than "time preference."

[^62]:    ${ }^{104}$ See, for example, the mathematical and graphical apparatus used in important articles of Becker and Mulligan (1997), Ghez and Becker (1975), Olson and Bailey (1981), Broome (1994), and (text)books of Stigler (1987) and Becker (2007).
    ${ }^{105}$ This assumption is briefly discussed in Hayek (1941). In section 3, we will explore its validity in a more detail.
    ${ }^{106}$ Fisher listed the following set of assumptions in his model: It condenses a year's income into an infinitesimal time; it confines our variations to two years only; it disregards the element of risk; it pictures next year's income as a certainty; it disregards the lack of security that limits the ease with which an individual can slide his series of transactions along the m line; it assumes that the market is perfect. (Fisher 1930:259)
    ${ }^{107}$ Obviously, we require not only decreasing indifference curves (i.e. $\mathrm{dC}_{1} /\left.\mathrm{dC}_{0}\right|_{\text {Uconstant }}<0$ ), but also convex indifference curves (i.e. decreasing MRS). This means that the better set must be a convex set. In other words, the contour of the utility function must be concave (sometimes expressed as convex to the origin), which implies quasi-concave utility function $\mathrm{U}\left(\mathrm{C}_{0}, \mathrm{C}_{1}\right)$. This is satisfied if $\mathrm{d}^{2} \mathrm{C}_{1} /\left.\mathrm{dC}_{0}{ }^{2}\right|_{\text {Uconstant }}>0$. It can be shown (see Appendix 1) that this requires: $1 /\left(\mathrm{U}_{1}\right)^{3}\left[\left(\mathrm{U}_{0}\right)^{2} \mathrm{U}_{11}-2 \mathrm{U}_{0} \mathrm{U}_{1} \mathrm{U}_{01}+\left(\mathrm{U}_{1}\right)^{2} \mathrm{U}_{00}\right]<0$. $\mathrm{U}_{0}$ is the marginal utility of the present consumption good, and $U_{1}$ is the marginal utility of the future consumption good. $\mathrm{U}_{00}, \mathrm{U}_{11}$ represent a change in the marginal utility of the present good due to an additional unit of the present good, and a change in the marginal utility of the future good due to an additional unit of the future good respectively. $\mathrm{U}_{10}$ (where $\mathrm{U}_{10}=\mathrm{U}_{01}$ owing to Young's theorem that applies if $\mathrm{U}\left(\mathrm{C}_{0}, \mathrm{C}_{1}\right)$ is twice continuously differentiable) stands for a change in the marginal utility of the present good resulting from an additional unit of the future good, and $U_{01}$ represents a change in the marginal utility of the future good resulting from an additional unit of the present good. Since $U_{i}$ is positive, and $\mathrm{U}_{\mathrm{ii}}$ is negative by assumption, the above-mentioned condition requires that $\left(\mathrm{U}_{0}\right)^{2} \mathrm{U}_{11}+\left(\mathrm{U}_{1}\right)^{2} \mathrm{U}_{00}<$ $2 \mathrm{U}_{0} \mathrm{U}_{1} \mathrm{U}_{01}$. But this implies that $\mathrm{U}_{01}$ is not (too) negative. Thus, we require that the marginal utility of present consumption good does not decrease (too abruptly) with more units of future consumption goods and vice versa. However, in section 2.4 we will assume additively separable utility function, $\mathrm{U}=\mathrm{u}(\mathrm{C} 0)+\beta \mathrm{u}(\mathrm{C} 1)$, that leads to $\mathrm{U}_{01}$ $=\mathrm{U}_{10}=0$. Hence, marginal utility of consumption in one period is not affected by a change in the amount of consumption goods in some other period. Thus, decreasing and convex indifference curves might be considered, in the first place, as an appropriate representation of the intertemporal preferences of our consumer and, in the second place, as a suitable tool representing the Mengerian theory - satisfaction increases with more consumption goods, but the marginal utility diminishes (both in the present and in the future).

[^63]:    ${ }^{108}$ The fact that the slope of the indifference curve at the diagonal line reflects the second Böhm-Bawerkian cause and/or that it reflects the pure time preference (and thus the slope should exceed one) was stressed by Stigler (1987), Becker (2007), Olson and Bailey (1981), Loewenstein (1992), Becker and Mulligan (1997), and by many other authors. It is also implicitly included in the analysis of Fisher (1930). Hayek in this connection explicitly wrote: The rate of time preference is here the rate at which a person would just be indifferent towards giving up a marginal quantity of his present income in return for a corresponding addition to his otherwise equal future income. (Hayek 1941:235,n.1)

[^64]:    109 As was mentioned before, it would be more appropriate to talk about the given want with different time moment of satisfaction rather than talking about the same want in the present and in the future. Thus, we may denote $I^{0}$ as want $I$ that is satisfied in the present, and $I^{1}$ as the same want that is satisfied later. According to the fundamental law of time preference, $\mathrm{I}^{0}$ is always preferred to $\mathrm{I}^{1}$. However, even though $\mathrm{III}^{0}$ is preferred to $\mathrm{III}^{1}$, satisfaction of want III in the present may persuade the individual to save the marginal present good to the future if he expects a very poor future endowment. Thus, he may prefer the given marginal good to be delivered later rather than sooner. The economic reason is that $\mathrm{I}^{1}$ may be preferred to $\mathrm{IV}^{0}$, which means that the satisfaction of want I later is preferred to the satisfaction of want IV in the present. Even though the difference might seem imperceptible, $I^{1}$ is not the same want as $I^{0} . I^{1}$ is not the future want and $I^{0}$ the present want. As was said earlier, these two stand for a different time moment of the satisfaction of the identical want I. I is the want that is to be satisfied, and it is always felt in the present. However, it might be preferable for an acting man to satisfy want I later to gratify want IV in the present. As a result, the given good may be preferred later rather than sooner, future goods may be preferred to present goods.

[^65]:    ${ }^{110}$ Fisher (1930) had clearly this meaning of time preference in mind when he was talking about the shape of the income stream to be the crucial determinant of the time preference.

[^66]:    ${ }^{111}$ As will be seen in section 3, the resulting equalization of (properly defined) marginal utilities suggested by Rothbard is in perfect conformity with the Euler equation. In this section, we will demonstrate that time preference in sense one (i.e. the subjective exchange ratio between present goods and future goods) will adjust to the objective constant marginal productivity that will ultimately determine the interest rate, and the equalization of marginal utilities will play a crucial role in this process.

[^67]:    ${ }^{112}$ His distaste to the indifference analysis can be found in Rothbard (1956).

[^68]:    ${ }^{113}$ In the terms of microeconomic analysis, we have to distinguish between the feasible set of the consumer and his optimum.

[^69]:    ${ }^{114}$ Especially, if the marginal utility rises beyond all limits, as consumption approaches zero.
    ${ }^{115}$ The time preference in the second sense is obviously effective in our representation, as can be seen in Figure No. 6 and Figure No. 7. Along the $45^{\circ}$ line, we may consider that it is the same want that is at the centre of the analysis. And this want might be satisfied either in the present or in the future by the dose of 10 goods (present or future). Since the given want is preferred to be satisfied earlier rather than later, the loss of 10 present goods is not accepted even for the compensation of 11 future goods (see Figure No. 6). Point B lies on a lower indifference curve than point A. Alternatively, we can see that the consumer is willing to accept ten present units in exchange for 10 future units (compare point $\mathrm{N}^{\prime}$ and A in Figure No. 7). 10 present goods are then preferred to 10 future goods.

[^70]:    ${ }_{116}$ Another famous reason to include $\rho$ to the utility function was provided by Parfit (1971). He assumed that every person has many selves, hence the model of multiple selves. In present, the future self has a lower weight than the present self in the valuations of the present self. A critique of this approach can be found in Frederick (2003).
    ${ }^{117}$ The fact that the subjective discount rate does not depend on time, i.e. $d \rho(t) / d t=0$ for all $t$, hence $\rho$ is time invariant, should not be confused with the increasing discount of more remote time periods $1 /(1+\rho)^{t}$.
    ${ }^{118}$ This is not the case for other forms of discounting. See, for example, Laibson (1997) with (quasi-)hyperbolic form of discounting.
    ${ }^{119}$ More about the assumption of constant tastes in the intertemporal analysis can be found in Hayek (1936; 1941). This assumption is relaxed, for example, in Trostel and Taylor (2001). They built a model in which tastes deteriorate with the age of an individual.

[^71]:    ${ }^{120}$ As a result, the functional form of $u()$ is identical in every period. Furthermore, the time variable is not present as an argument in this function. Exactly the opposite might hold if a cake on one's birthday or champagne on New Year's Eve were particularly enjoyed on these specific days (Strotz 1956:168). In such a case, the instantaneous utility function would be expressed as $u(C, t)$.
    ${ }^{121}$ Technical details for a general form of the utility function are provided in Appendix 1.
    ${ }^{122}$ Since the discounted utility model is cardinal, we may write that $u^{\prime}\left(C_{0}{ }^{A}\right)>u^{\prime}\left(C_{1}{ }^{A}\right)$.

[^72]:    ${ }^{123}$ However, Becker and Mulligan (1997:731) would call $\varepsilon$ the time preference and $\rho$ the rate of time preference.
    ${ }^{124}$ Subscripts 0 and 1 in $u_{0}$ and $u_{1}$ are added to stress the time position of each instantaneous utility function even though we know that both functional forms are the same.

[^73]:    ${ }^{125}$ Of course, the cardinal reasoning is very unfortunate at this place. We can say that $\rho$ measures the required increase in the future satisfaction if losing some given amount of present satisfaction. According to Mises, man will never accept both to be the same (forgone present satisfaction and acquired future satisfaction), $\rho$ must be therefore positive.

[^74]:    ${ }^{126}$ Technical details of the concept of the intertemporal elasticity of substitution in consumption are provided in Appendix 2.

[^75]:    ${ }^{127}$ It can be shown that in the case of very high elasticity of substitution, the response of the optimum consumption path to a change in the interest rate is very strong.
    ${ }^{128}$ Clearly, consumption at time 0 is a perfect substitute to consumption at time 1 in this case. However, greater weight is put on present consumption because the slope of the indifference curve is higher than one (in absolute value). The lifetime utility function (for two periods) has the form: $U=C_{0}+C_{1} /(1+\rho)$. Obviously, the assumption of diminishing marginal utility is violated in this case.
    ${ }^{129}$ Only in this case, the MRS between consumption levels is the same as the MRS between utility levels $(1+\rho)$.

[^76]:    ${ }^{130} \mathrm{u}^{\prime}(\mathrm{C})$ is simply $\mathrm{C}^{-\theta}$, so $\mathrm{u}^{\prime}\left(\mathrm{C}_{0}\right) / \mathrm{u}^{\prime}\left(\mathrm{C}_{1}\right)=\mathrm{C}_{0}{ }^{-\theta} / \mathrm{C}_{1}{ }^{-\theta}=\left(\mathrm{C}_{1} / \mathrm{C}_{0}\right)^{\theta}$
    ${ }^{131}$ Barro (2004:91) calls the form (10) the constant intertemporal elasticity of substitution (CIES) utility function.
    ${ }^{132}$ Formal proof uses the L'Hospital rule (see Appendix 3).

[^77]:    ${ }^{133}$ In the real world, we do not observe such concentration of wealth in the hands of several of the most patient families. The reason could be found in the fact that each generation may have a different subjective discount rate, so wealth accumulated by one generation may be easily exhausted by the next generation. Other explanations can be found in Epstein and Hynes (1983).
    ${ }^{134}$ Other deviations from the standard model can be found in the behavioral or experimental literature. A very readable overview was provided by Frederick et al. (2002).

[^78]:    ${ }^{135}$ A continuous version of this model with lifetime T will be presented in section 5 .

[^79]:    ${ }^{136}$ Mises (1996:532) claimed that if the interest rate was artificially depressed to zero, the consumption of capital would ensue.

[^80]:    ${ }^{137}$ Using the loanable funds market interpretation, the horizontal investment curve, which is posited at the horizontal axis due to zero and constant marginal productivity of capital in this hard-tack economy, is intersected by the saving curve such that the saving of $\left(\mathrm{A}-\mathrm{C}_{0}{ }^{*}\right)$ is generated by every sailor. The loanable funds representation will be discussed in a more detail in section 4 .
    ${ }^{138}$ Notice that the hypothetical future MU schedule must be scaled down by the discount factor $1 /(1+\rho)$.

[^81]:    ${ }^{139}$ This tendency is well known from the basic microeconomic intra-temporal analysis. Income is allocated so as to equalize MU of various goods (weighted by the inverse of their prices). Austrian theorists use the second Gossen law very often, so it is quite surprising that they neglect this approach in the theory of interest.

[^82]:    ${ }^{140}$ Negative time preference can also emerge if the framing effect is present (Loewenstein \& Prelec 1991).
    ${ }^{141}$ Restriction $\rho>-1$ must be imposed on the model; otherwise, future consumption would be considered by the representative consumer as an economic bad.
    ${ }^{142}$ On the other hand, it might be costly (dangerous) to protect the given stock of hard-tacks. Hence, the act of lending for zero interest might be a good way to pass the costs of storing hard-tacks to someone else. However, we implicitly assume no costs of this kind. The impact of storing costs on the intertemporal exchange ratio will be discussed later.

[^83]:    ${ }^{143}$ This situation would be similar to the famous Böhm-Bawerkian horse market if all individuals possessed the same subjective exchange ratio. The only market equilibrium would be that particular ratio, even though no horse would be traded in this market. There would be no horse market at all.
    ${ }^{144}$ The Austrian capital theory usually deals with a much more complex definition of capital following the seminal work of Böhm-Bawerk (1891) or Hayek (1941). More on this topic can be found in chapter one of this dissertation.

[^84]:    ${ }^{145}$ The endogenous concept of time preference (in sense one) may be interchangeably analyzed either in terms of the marginal rate of substitution MRS, or in terms of the marginal rate of time preference: $\varepsilon=$ MRS -1 . It should be stressed that $\varepsilon$ must be conceptually distinguished from $\rho$. As we move along the indifference curve, $\varepsilon$ (time preference in sense one) gradually falls, whereas $\rho$ (time preference in sense two) remains constant. They coincide only at the 45 -degree line. So far, we have introduced two parameters with a different role - $\rho$ and $\theta$. It will not be much helpful to introduce the third one - $\varepsilon$. As a result, we will keep using MRS as a representative of the first sense of time preference. This strategy might be more convenient to the reader because the MRS can be immediately identified with the slope of the indifference curve. The remaining two parameters, $\rho$ - the subjective discount rate, and $\theta$ - the coefficient of the relative risk aversion directly affecting the elasticity of substitution, have a perfect and clear economic interpretation. Moreover, both are present in the lifetime utility function, both are independent of each other in our model and both, finally, affect the MRS (or $\varepsilon$ ). This strategy, the author of this paper believes, should preserve clarity of the exposition at every step rather than the introduction of the third Greek letter $\varepsilon$.

[^85]:    ${ }^{146}$ In this example as well as in the previous example with hard-tacks, there will be no intertemporal market, even though there must exist a positive interest rate of $10 \%$. If the market interest rate was lower than $10 \%$, there

[^86]:    would be an excess of borrowing over lending. If the interest rate exceeded $10 \%$, the opposite situation would emerge.
    ${ }^{147}$ Similar reasoning would also hold for a negative time preference in sense two ( $\rho<0$ ), i.e. if the future satisfaction was preferred to present satisfaction.
    ${ }^{148}$ Surprisingly, this conclusion was made by young Hayek (1927) in his very early paper. In section 5, we will extend the time horizon to infinity, where the Misesian argument must be reconsidered anew. We will see that our conclusions hold also in the infinite horizon. Nevertheless, the problem with the two period model is that implicitly, there is an infinite time preference (in sense two, i.e. $\rho \rightarrow \infty$ ) in all periods following the death of the representative consumer (i.e. in our timing at $t=2, t=3, \ldots$ ). Thus, we are inconsistent in saying that the consumer has no time preference in sense two (i.e. $\rho=0$ ). This statement holds only as regards one future period. All the other periods are infinitely discounted. Hence, the attack against the Mises's theory that claims that all consumption will be postponed to the future loses its power because there is an infinite, not zero, time preference in the other periods. However, even considering an infinitely lived individual with zero time preference (i.e. $\rho=$ 0 ) over his eternal life will confirm our critique provided that his elasticity of substitution is low enough. Thus, it can be shown that the simultaneous existence of positive interest rate and zero (or even negative) time preference

[^87]:    (in sense two) will not result in a complete postponement of the act of consumption to the indefinite future. A thorough analysis will be provided in section 5 .
    ${ }^{149}$ See also Brown (1913) who discussed the same point in his defense of Böhm-Bawerk against I. Fisher. Furthermore, Seager (1912) paradoxically accused Fisher of not putting enough emphasis on the productivity element. Fisher's own defense can be found in Fisher (1913).

[^88]:    ${ }^{150}$ This is the case in our model because $u^{\prime}(C)>0$. Böhm-Bawerk (1891) imposed the same assumption in his theory.
    ${ }^{151}$ For the equilibrium of the economy to be well defined, $\mathrm{u}^{\prime}(\mathrm{C})>0, \mathrm{u}^{\prime \prime}(\mathrm{C})<0$, and convex indifference curves are required. Thus, more must be preferred to less, MU must be diminishing, and the interaction between MU of present goods with respect to future goods and vice versa may not be too "perverse" (see Appendix 1).

[^89]:    ${ }^{152}$ The graphical proof would be the same as in the case of hard-tacks in panel (b) of Figure No. 20. The only difference is that the slope of the IBC would be 1 rather than 1.1 (in absolute value), and the slope of the PPF would be 0.9 rather than 1 .
    ${ }^{153}$ Nowadays, the theoretical existence of a negative natural rate of interest is a big topic. If the central bank is to follow the natural rate (as, for example, Wicksell (1936) suggested), monetary policy might be seriously limited due to the zero lower bound of the nominal interest rate. The modern approach can be found, for example, in McCallum (2000), Eggertsson and Woodford (2003), Orphanides (2004), Wolman (2005), Jung et al (2005).

[^90]:    ${ }^{154}$ See, for example, Block (1978), Garrison (1979), Kirzner (1993), Lewin (1997), Hülsmann (2002), Cwik (2004), Herbener (2011), Topan and Păun (2013), Herbener (2013).
    ${ }^{155}$ The problem in this particular way was first raised by Böhm-Bawerk (1891). It was then followed by Fetter (1928), Mises (1996), Rothbard (2004) and all other proponents of the PTPT approach.
    ${ }^{156}$ However, we will see in this section that the questions asked in this paragraph are by no means interchangeable, and they must be analyzed separately. Their separation is of fundamental importance to find a satisfactory answer to the most pressing problems in the theory of interest.

[^91]:    ${ }^{157}$ See Cwik (2004:3). However, notice that he made a mistake in his formula - there must be "interest rate" not "interest" on the left hand side of his equation. Furthermore, the formula should be modified as follows: "future value of future goods" and "present value of present goods".

[^92]:    ${ }^{158} \$ 2,200$ invested in rice this year can purchase 100 units $(\$ 2,200 / \$ 22)$. The eventual return of $\$ 2,200$ will purchase 110 units of rice $(\$ 2,200 / \$ 20)$ next year. The real interest is 10 units. Hence, the real interest rate is $10 \%$ ( $=10 / 100$ ).
    ${ }^{159}$ Would it be useful to know that the investment of $\$ 2,000$ in rice will earn $\$ 2,200$ or $\$ 2,000$ or $\$ 2,000,000$ next year? Without the knowledge of the evolution of prices in each particular case, this information is almost useless because, in the end, it is the problem of allocation of real goods that must be ultimately solved by every agent.

[^93]:    ${ }^{160}$ Do not forget that we still assume that the amount of money is constant over time. Consequently, the nominal demand is constant as well. If we let the amount of money increase, for example at the same rate as output, which is $6 \%$ in panel (c), the future nominal demand will raise future prices. As a result, the price level will be constant $\left(\mathrm{P}_{1}=\mathrm{P}_{0}\right)$, but the nominal interest rate will increase from $4 \%$ to $10 \%$ (it will go up due to the positive growth rate in the money supply). It will then perfectly coincide with the real interest rate, but the equality between the subjective discount rate and the nominal interest rate will collapse.

[^94]:    ${ }^{161}$ There is one more case of equality between $\rho$ and i. If the real rate of interest $r$ is equal to the subjective discount rate $\rho$ (and this possibility, as we will see in section 4, is very plausible in an economy with stationary technology and diminishing marginal productivity of capital), parameter $\theta$ plays no role. Consumption will be perfectly smoothed, and the inflation rate will be zero. As a result, the nominal rate of interest will then perfectly coincide with the subjective discount rate (and the real rate of interest).
    ${ }^{162}$ This conclusion holds only for $\mathrm{r}>\rho$. If $\mathrm{r}<\rho$, the opposite statement is correct.
    ${ }^{163}$ This phenomenon will be discussed in a more detail in chapter 4 of this dissertation, where we explore the validity of "neutral money" in a growing economy.

[^95]:    ${ }^{164}$ Negative real interest rate is sometimes observed in modern economies when the rate of inflation exceeds the nominal interest rate.
    ${ }^{165}$ The Euler equation with $\rho=0, \theta=1$, and $r=-10 \%$ suggests that consumption should fall by $10 \%$ over time. With constant money, prices must go up by $10 \%$. See Figure No. 25 for a possible graphical representation of the optimum path of consumption in this case.
    ${ }^{166}$ Standard growth theory would interpret this economy as dynamically efficient because the real interest rate exceeds the output growth. At the same time, the nominal interest rate exceeds zero. As we will see in section 3.1.3, this is not a coincidence.

[^96]:    ${ }^{167}$ See, for example, the pure productivity approach in Knight (1934, 1935a, 1935b, 1936a, 1936b).
    ${ }^{168}$ One might argue that nobody would invest money in production process No. 3. If the representative good was perishable and money existed, everybody would be willing to keep saving in the form of cash as it would give the same real return as that particular production process. Surprisingly, such a world would represent Friedman's optimum quantity of money (Friedman 1969). However, if it was costly or dangerous to keep (too much) money, production process No. 3 might dominate pure hoarding.
    ${ }^{169}$ This does not mean that the production process and its structure are not important in the capital and interest theory. The exact opposite is true as is demonstrated in chapter 1 . However, the reasoning in this section is

[^97]:    focused just on the fact that one must distinguish between the nominal and real approach in the theory of interest. And this might dispense with the explicit considerations of the factors of production.
    ${ }^{170}$ Obviously, output in the hard-tack economy is given in the beginning. Thus, we should rather talk about the output consumed that is gradually falling at the rate of $10 \%$. A more genuine example might be represented by a contracting economy in which the rate of decline in output is $10 \%$. More on this will be said in the next section.

[^98]:    ${ }^{171}$ Hayek (1941:419) wrote in this connection: The answer to Böhm-Bawerk's question as to why there is a difference between the value of the present factors and the value of their present product is that there is no such difference.

[^99]:    ${ }^{172}$ In the example above, the own rate of interest on apples is $3 / 2-1=50 \%$ and on oranges $4 / 3-1=25 \%$. More on this problem can be found in Sraffa (1932), Hayek (1932; 1941), or more recently in Murphy (2003).
    ${ }^{173}$ Fisher (1930) stressed that one of the key determinants of the rate of impatience is the composition of (real) income.

[^100]:    ${ }^{174}$ More on this will be said in Chapter 4 of this dissertation.

[^101]:    ${ }^{175}$ Böhm-Bawerk associated the productivity of roundabout methods with his third ground for interest. Mises (1996) preferred the term "longer methods" rather than "roundabout methods". The Austrian theory of capital is discussed in greater detail in Chapter 1 of this dissertation.
    ${ }^{176}$ Here we should add time preference in the first sense because a huge under-provision in the present, thus almost infinite marginal utility from present consumption and the resulting enormous size of the MRS (i.e. time preference in the first sense) will not allow never-ending postponement of present consumption and indefinite lengthening of the production process.

[^102]:    ${ }^{177}$ To be more precise, we have to add that there is no endowment of future factors of production. In other words, future output is zero unless present factors of production are engaged in longer processes. A picture of an economy with future endowment of factors of production is presented in Figure No. 43.

[^103]:    ${ }^{178}$ Hayek (1945) in his later article on capital theory predicted that a sudden (unexpected) decrease in saving may result in a very unfortunate interruption of the creation of capital structures. This is then reflected in a highly curved opportunity line, where the restructuring of the process of production from longer methods to shorter methods requires a very high sacrifice of future output in order to obtain one unit of present output. Such an abrupt change in the time preference has similar consequences as a halt of the monetary expansion at the very peak of the boom. More on this is said in Chapter 2 of this dissertation. As a result, if it is difficult to reallocate resources from longer methods to shorter methods, an unexpected fall in saving leads to the fact that the natural rate of interest is determined mainly by the time preference.

[^104]:    ${ }^{179}$ Obviously, the investment and the saving curves should not be linear. Yet, the linear shape is constructed just for simplicity and as a reasonable approximation around the given equilibrium.

[^105]:    ${ }^{180}$ In the next section, we will relax the assumption of zero future (income) endowment, hence we will allow any shape of the income stream, not only $(\mathrm{A}, 0)$.

[^106]:    ${ }^{181}$ However, similar reasoning might be found also in Fisher (1907; 1913).
    ${ }^{182}$ This determination of the real rate of interest was especially emphasized by F. Knight (1934, 1935a, 1935b, 1936a, 1936b, 1941).

[^107]:    ${ }^{183}$ Recall that we assume $\rho=0 \%$ and $\theta=2$ in panel (b), and $\rho=4 \%$ and $\theta=2$ in panel (d).

[^108]:    ${ }^{184}$ The solution that $\mathrm{r}=\rho$ for a constant income flow is valid regardless of the fact whether people have identical size of income or earn different incomes. See Appendix 2 B, section A and B. Furthermore, since $r=\rho$,

[^109]:    consumption will be smoothed for all levels of constant income. As a result, there will be no individual saving either on the part of the rich or the poor. Income in every period will be consumed in full regardless of its size.
    ${ }^{185}$ We can add that if $\rho$ was zero, the natural rate would be zero. If $\rho$ was negative, the natural rate would be negative.

[^110]:    ${ }^{186}$ According to empirical studies (see references in Laibson 1997:454), growth rate of consumption is tightly connected with the growth rate in income. Several theoretical models were developed to address this issue that is at variance with the standard neoclassical model presented here. See, for example, Laibson (1997) or Carrol (1997).
    ${ }^{187}$ One should not be confused by the fact that a low growth rate of consumption leads to a borrowing position of the individual. We have to realize that this does not say anything about the absolute size of consumption in

[^111]:    each period. If present consumption is close to future consumption and if the income stream is increasing, present consumption must exceed present income, which results in negative saving on the part of this individual.

[^112]:    ${ }^{188}$ In simulation 4 in Appendix 2 B, an individual with a lower subjective discount rate (i.e. individual A) is a borrower due to his sharply increasing flow of income. Thus, a relatively low time preference (in sense two) does not guarantee a net lending position if the flow of income of the individual is growing at a sufficient rate (or if it is falling at a lower rate) compared with others.

[^113]:    ${ }^{189}$ One may wonder why the real interest rate in normal conditions is not between $\rho_{\mathrm{A}}$ and $\rho_{\mathrm{B}}$. More on this will be said in the final section. The fundamental reason is, however, the increasing time shape of the (aggregate) income stream (i.e. the first Böhm-Bawerkian cause for interest) that raises the interest rate above the subjective discount rate of even the most impatient individual (in our case $B$ with $\rho_{\mathrm{B}}$ ). Furthermore, the least patient individual (as measured by $\rho$ ) might not be a borrower if the growth rate of his income stream is lower than that of the others.

[^114]:    ${ }^{190}$ This model is outlined in section C of Appendix 3 B.

[^115]:    ${ }^{191}$ The derivation of this particular intertemporal budget constraint is presented in Appendix 5 B.

[^116]:    ${ }^{192}$ If we used equation (35) instead of (34), the Euler equation would be $u^{\prime}\left(\mathrm{C}_{\mathrm{t}+1}\right) / \mathrm{u}^{\prime}\left(\mathrm{C}_{\mathrm{t}}\right)=(1+\rho) /\left(1+\mathrm{r}_{\mathrm{t}+1}\right)$

[^117]:    193 In Appendix 4 B, we will demonstrate that present consumption need not be curtailed to negligible levels even if the subjective discount rate is zero and the interest rate is positive. The reason lies in the fact that infinite consumption in infinite horizon might be easily achieved just with moderate present savings. As a result, equation (38) is satisfied due to zero numerator rather than infinite denominator. This appendix will also reveal further problems of the approach presented in Olson and Bailey (1981). The key point is that they underestimated interesting and unexpected implications when the time horizon is extended to infinity. However, our correction of their theory will not support the Misesian PTPT. Rather the opposite is true. It will be shown that a positive interest rate accompanied by zero time preference might lead to non-zero present consumption even under weaker assumptions than Olson and Bailey believed. Yet, these corrections are not presented in the main text, because we are mainly interested in problems of the Misesian PTPT. The inclusion of long proofs will make our exposition less lucid. However, even the first approximation of Bailey and Olson suffices to reveal fundamental problems in the PTPT, although their theory is not perfectly accurate.
    ${ }^{194}$ Trostel and Taylor (2001) offered a different explanation for the Euler equation (38) to hold. Consider again the left hand part of equation (38). If tastes of people deteriorate over time, marginal utility in very remote future may converge to zero. Thus, the Euler equation (38) might be satisfied even with finite consumption in the future.

[^118]:    ${ }^{195}$ According to Bailey and Olson (1981:19), this seems to be an empirical fact.
    ${ }^{196}$ A thorough analysis of the growth in income and of the requirement on $\theta$ is presented in Appendix 4 B .
    ${ }^{197}$ In Appendix 4 B, we will show that infinite consumption in infinity, positive interest rate, zero time preference and positive (i.e. non-zero!) present consumption are attainable even if the flow of (labor) income is constant.

[^119]:    ${ }^{198}$ Hayek (1932) dealt with consumption of capital in greater detail.
    ${ }^{199}$ Obviously, equation (43) is a final step in deriving the sum of an infinite series provided that the real rate of interest is the same in all periods. However, if the interest date differs over time, the sum need not be infinite if the interest is zero in a finite number of periods, but positive in the rest.

[^120]:    ${ }^{200}$ Hence, a finite price of land requires that the economy is dynamically efficient. If the growth rate of income from land exceeded the rate of interest, the price of land would grow beyond all limits. Furthermore, zero interest rate is possible, but it must exceed the growth rate of income (in this case $g$ must be negative).
    ${ }^{201}$ Consumption will be then decreasing at the same rate as real income, i.e. $4 \%$ per year.

[^121]:    ${ }^{202}$ As can be seen, $\rho$ and $\theta$ have opposing effects on the optimum consumption path. Higher $\theta$ leads to a smoother profile of consumption, which, however, implies that present consumption is lower. Thus, for a fixed present endowment, higher $\theta$ has similar effects as lower time preference. This could be compared with the discussion in Appendix 4 B . It is derived that if the income stream is increasing over time, higher $\theta$ results in

[^122]:    higher present consumption because the individual is trying to move higher future income closer to the present. Thus, higher $\theta$ had similar effects as higher time preference. As can been seen, for a fixed present endowment, high preference for consumption smoothing (high $\theta$ ) motivates the consumer to spread his consumption over the entire planning horizon. Yet, this leads to a lower present consumption.
    ${ }^{203}$ A similar statement can be found in Woodford (2003:5) about Wicksell and Hayek when considering their contribution to (modern) monetary theory.

[^123]:    ${ }^{204}$ Similar objection along with the presentation of his own position can be found in Knight (1941).

[^124]:    ${ }^{205}$ See the seminal contributions of Ramsey (1928), Cass (1965), and Koopmans (1963). It is quite fascinating that the article of Ramsey was published not only before Solow (1956) but even before Fisher (1930). Yet, it must be admitted that Fisher presented his key ideas much earlier in Fisher (1907).
    ${ }^{206}$ We assume the CRRA instantaneous utility function. Convergence of life-time utility then requires that $\rho-\mathrm{n}$ $-(1-\theta) \mathrm{g}>0$. More on this can be found in Appendix 7.
    ${ }^{207} \delta$ is the depreciation rate.

[^125]:    ${ }^{208}$ Even a negative natural rate of interest can be achieved with positive time preference (in sense two). Set, for example, $\rho=4 \%, \theta=1$, and $\mathrm{g}=-5 \%$.
    ${ }^{209}$ A path of the economy to the zero natural rate of interest is displayed in Appendix 7, section G, Figure No. 28_A7. Furthermore, Section G in Appendix 7 discusses other notable phenomena in the contracting economy.
    ${ }^{210}$ At first sight, it seems that the integral in (45) should diverge for $\rho=0$. However, for the CRRA utility function with $\theta>1$ and an increasing consumption path this will not happen, as is shown further in the text and in Appendix 5B, Figure No. 1_A5 and 2_A5. Conversely, it can be shown that the lifetime utility will not diverge if $\theta<1$ and the optimum consumption path is decreasing (due to negative $g$ and hence negative $r^{*}$ ).
    ${ }^{211}$ This fact is stressed by Broome (1994) in justifying the discounting of wants of future generations.

[^126]:    ${ }^{212}$ Hayek used horizontal axis for period $t+1$ and vertical axis for period $t$. We swapped the axes to make his model more comparable to our approach in this paper.

[^127]:    ${ }^{213}$ In this particular respect, Hayek accepted the theory of Frank Knight. See, for example, Knight (1936a, 1936b, 1941).
    ${ }^{214}$ Hayek (1941:233) explicitly wrote that in the process of the accumulation of capital, the time preference (in the sense of the slope of the indifference curve at the $45^{\circ}$ line) might be zero or even negative; yet, the interest rate is still positive as it is determined by the productivity of capital.
    ${ }^{215}$ This model is derived in Appendix 7. Discussion about the fall in $\rho$ is provided in section D of this appendix.

[^128]:    ${ }^{216}$ However, there is one exception. MPK at the given time depends on the level of capital, which in turn depends on the speed of convergence $\lambda$. The size of $\lambda$ is also influenced by $\rho$. Thus, it might be said that the natural rate of interest along the convergence path is affected not only by the marginal productivity of capital but, to some extent, by the subjective discount rate as well.

[^129]:    ${ }^{217}$ In Appendix 7, this model is derived and the whole discussion about the increase in $g$ is provided (section D).

[^130]:    ${ }^{218}$ For simplicity, we assume $g=0 \%$. The general discussion about the outcomes of this model is provided in Appendix 7. In this appendix, we also discuss the most important differences between the increase in $g$ and in $A$ (section D). Some of them (e.g. the impact on the saving rate and the role of $\theta$ ) are very important. However, to keep a continuous flow of our discussion about the natural rate of interest undisturbed, they are all postponed to Appendix 7.
    ${ }^{219}$ In the transition process, the representative consumer is maximizing his utility because the Euler equation is still effective. This equation guarantees stability in this model and a smooth path to the new steady state. In other words, the stabilizing effect is played by the effort to equalize (discounted) marginal utilities over time, or alternatively to equalize the marginal rate of time preference ( $\varepsilon=\mathrm{MRS}-1$ ) with the ongoing real rate of interest.

[^131]:    ${ }^{220}$ It should be stressed that even along the transition path, the natural rate of interest is in equilibrium. In the growth theory, we have to distinguish between "non-steady state equilibria" and the steady state equilibrium. From the static point of view, the first type of equilibrium is qualitatively the same as the second one. The key difference lies in dynamic considerations.
    ${ }^{221}$ This assumption obviously means that the (real/nominal) demand for money is homogenous of degree one in (real/nominal) income and it is independent of the (nominal) interest rate.

[^132]:    ${ }^{222}$ Recall that for constant money (and velocity) the inflation rate $\pi$ should be equal to the negative of the growth rate of output (which is $n+g$ on the BGP).

[^133]:    ${ }^{223}$ Note that we assumed ZPG, i.e. $\mathrm{n}=0 \%$, in this discussion.
    ${ }^{224}$ It can be perfectly seen, that the steady state optimum saving rate is negatively related to parameter $\theta$ provided that the technological progress is positive.

[^134]:    ${ }^{225}$ Condition A7_14, $\rho-\mathrm{n}-(1-\theta) \mathrm{g}>0$, requires $\theta<1$ if $\rho=\mathrm{n}=0$ and $\mathrm{g}<0$. This means that if there is an economic decay, elasticity of substitution must be high enough. The economic reason for such a conclusion is as follows. Technological decline leads to a lower interest rate. For low enough $\theta$, the saving curve is upward sloping, hence people do not save much for low interest rate. As a result, the economy will not over-accumulate capital, it is dynamically efficient, and the nominal interest rate is positive (see section G in Appendix 7). On the other hand, if the preference for consumption smoothing was relatively high $(\theta>1)$, technological decline and the resulting expected fall in the income endowment would lead to excessive saving in the present. Thus, oversaving and dynamic inefficient character of the economy might emerge along with the negative nominal interest rate. Hence, condition A7_14 prevents this possibility.

[^135]:    ${ }^{226}$ As is well known, according to the ordinal utility theory the numerical value of $U$ is of no importance.

[^136]:    ${ }^{227}$ Since any indifference curve is drawn in the $\mathrm{C}_{1}-\mathrm{C}_{0}$ space, along the given indifference curve $\mathrm{C}_{1}$ can be considered as a function of $\mathrm{C}_{0}$.

[^137]:    ${ }^{228}$ Both individuals have identical subjective discount rate since the slope of the indifference curve at the diagonal line is the same for both of them. Nonetheless, $\rho$ plays no role in determining $\sigma$ in the case of CRRA (CIES).

[^138]:    ${ }^{229}$ Term $1 /(1-\theta)$ disappears in the expression of the marginal utility: $\mathrm{MU}\left(\mathrm{C}_{\mathrm{t}}\right)=\mathrm{C}_{\mathrm{t}}{ }^{-\theta}$, which is the same as the MU for the CRRA in (10) in the main text. Hence, the MRS must be also the same.

[^139]:    ${ }^{230}$ We assume $u^{\prime}(\mathrm{C})>0$ and $u^{\prime \prime}(\mathrm{C})<0$, hence the budget constraint is binding. At the same time, if the marginal utility rises sky high as consumption goes to zero, we do not have to bother with non-negativity constraints.

[^140]:    ${ }^{231}$ Differences between income and consumption in absolute value in period 0 are identical for both individuals, even though it might not be visible from the figures as they do not preserve identical scales.

[^141]:    ${ }^{232}$ The model that follows is an extension of a textbook model from Romer (2006).

[^142]:    ${ }^{233}$ In supplement 2 to this Appendix, we show that the endowment point $\mathrm{A}^{2}$ is below the old budget line, if (for a constant stream of wages) the interest rate is lower than the subjective discount rate (i.e. $r<\rho$ ) and vice versa.

[^143]:    ${ }^{234}$ However, one possible (and most probably correct) interpretation is as follows: Reduction in the interest rate leads to a shift of the optimal point which represents the ideal intertemporal allocation of labour and the resulting income endowment (along a hypothetical PPF) to the left (i.e. the growth rate in income rises). At the same time, a decline in the interest rate results in a decrease in the optimal growth rate of consumption. Thus, the equilibrium interest rate can be found where these two tendencies offset each other. On the PPF, an increasing income stream is consistent with a lower interest rate. In case of the (consumption) indifference curve, increasing consumption stream is associated with a higher interest rate. Thus, it can be said that the lower interest rate decreases the supply of present goods (due to the reduction in the supply of present labour) and raises the demand for present goods (due to higher consumption demand). An increase in the interest rate has the opposite diverging effects. Thus, the interest rate must adjust to equilibrate these two tendencies.

[^144]:    ${ }^{235}$ Even though this indifference curve represents utility only from consumption and not from leisure, our conclusion seems to correct, because present leisure increases, even though the future leisure falls. Furthermore, the increase in consumption in both periods due to the reduction in the interest rate will surely benefit the debtor.

[^145]:    ${ }^{236}$ Linear saving curve is constructed just for simplicity. As can be seen from (19) and (28), the relationship between optimum saving and real interest rate must be clearly non-linear. Furthermore, logarithmic utility function and the presence of future labour income lead to an upward sloping saving curve. The response of present consumption to the change in the interest rate is negative (see equation 28 ): $\partial \mathrm{C}_{0} * / \partial \mathrm{r}=-\mathrm{K} . \mathrm{W}_{1} /(1+\mathrm{r})^{2}$, where $K=(1+\rho) /[(2+\rho)(1+b)]$. Thus, with a lower interest rate, optimum saving declines. If there was no future wage ( $\mathrm{W}_{1}=0$ ), the saving curve would be vertical (neither the optimum present consumption nor the optimum present leisure would depend on the interest rate).

[^146]:    ${ }^{237}$ In supplement 1 at the end of this section it is proved that the new endowment point must lie on the original budget line because a change in the subjective discount rate leaves the present value of the income stream unaffected. Alternatively, it is demonstrated that the relative shift of the endowment point is in the direction of ( $1+\mathrm{r}$ ), which perfectly coincides with the slope (in absolute value) of the intertemporal budget constraint.

[^147]:    ${ }^{238}$ It seems that the impact on the equilibrium interest rate is not stronger for logarithmic utility function (i.e. for $\theta=1$ ), but it might be for $\theta$ different from 1 .

[^148]:    ${ }^{239}$ Condition (70) implies that the response of the endowment point to a change in the real interest rate depends on the position of the original endowment point. If the initial endowment point is above the $45^{\circ}$ line, the new endowment point will be below the original budget line and vice versa. The reason lies in the fact that point A is above the $45^{\circ}$ line (i.e. $\left.\mathrm{Y}_{1}=\mathrm{W}_{1} \mathrm{~L}_{1}{ }^{*}>\mathrm{Y}_{0}=\mathrm{W}_{0} \mathrm{~L}_{0}{ }^{*}\right)$ if $\mathrm{W}_{1}>\mathrm{W}_{0}$ and $\mathrm{r}\left\langle\rho\left(\right.\right.$ and hence $\mathrm{L}_{1}{ }^{*}>\mathrm{L}_{0}{ }^{*}$ ) or if $\mathrm{W}_{1}>\mathrm{W}_{0}$, r$\rangle$ $\rho(!)$ and the growth rate of wages $\left(\mathrm{W}_{1} / \mathrm{W}_{0}-1\right)$ exceeds the difference between r and $\rho$ (see equation 12).
    ${ }^{240}$ This IBC for constant income is assumed in Olson and Bailey (1981:9). Its derivation is presented in Appendix 5 B. For a better understanding, the reader is advised to read Appendix 5 B first.

[^149]:    ${ }^{241}$ As will be seen below, this outcome is due to the approximation made by Olson and Bailey.

[^150]:    ${ }^{242}$ E.g. for $\theta=0.7$ and $\mathrm{T}=50$ years, optimum present consumption is about 25 . For $\mathrm{T}=200$ years it is depressed to 0.86 .

[^151]:    ${ }^{243}$ In the 500 -year planning horizon, the development of crucial variables was separated in more graphs for greater clarity. As can be seen in Figure no. 11_A4, the eventual future consumption is more than 100 times higher than the labour income. Furthermore, the first 100 years of this planning horizon were also reported separately to track the initial accumulation of assets.
    ${ }^{244}$ It can be perfectly seen (especially in Figures 12,13, and 14) that relatively modest savings in every period (a flow concept) might generate huge assets (stock concept) over a very long period of time.
    ${ }^{245}$ In the balance of payments jargon, the difference between Y and C might represent the trade balance, the difference between the disposable income and consumption then the entire current account reporting the sum of the trade balance and the income balance rB. Thus, the current account might be positive (and the financial account therefore negative indicating a positive net outflow of capital, i.e. positive net foreign investment) even when the trade balance is negative provided that the income balance reports a sufficient surplus.

[^152]:    ${ }^{246}$ Notice that the same result is obtained for the condition of the convergence of life-time utility. See section B in Appendix 5 B.

[^153]:    ${ }^{247}$ It should be stressed that we do not blame these authors for this approximation, because this set of parameters will perfectly hold in a continuous-time model (see Appendix 7 equation A7_58). However, the infinite horizon model requires perfect accuracy since any rounding error will be magnified sky high. On the other hand, this rounding error allowed us to report a specific behaviour of consumption and debt in the finite-horizon model, which was useful as well.

[^154]:    ${ }^{248}$ From now on, we may indicate utility maximizing values by asterisk.
    ${ }^{249}$ Notice that for parameters $\mathrm{r}=5 \%, \mathrm{~g}=1 \%$ and $\mathrm{Y}_{0}=100$, the PV is 2625 , which is the same figure as the PV for a 500-year planning horizon presented above. Thus, it seems at first glance that very long horizons might be approximated by an infinite-horizon model.

[^155]:    ${ }^{250}$ Notice that even though the PV is (almost) the same as in the 500 -year finite horizon model and the present optimum consumption is also very close, debt is not repaid in period 500 . The reason is that in the infinite horizon, the present consumption is a little bit higher so is the consumption in every subsequent period.

[^156]:    ${ }^{251}$ It might be said that saving is a flow concept and debt is a stock concept. Thus, saving must be a difference between two flow variables (disposable income and consumption), not between a stock variable (wealth) and a flow variable (consumption). As a result, equation (A5_2) indicates a link between stocks $\mathrm{B}_{0}$ and $\mathrm{B}_{1}$ that are adjusted by two flows - $\mathrm{Y}_{1}$ and $\mathrm{C}_{1}$. On the other hand, equation (A5_3) includes only flow variables - savings and a change in debt (i.e. a change in the stock variable).

[^157]:    ${ }^{252}$ I.e. for utility function having no satiation point. This might be concisely expressed as $u^{\prime}(\mathrm{C})>0$ in the entire domain of non-negative real numbers $\left(\mathrm{R}_{0}{ }^{+}\right)$.

[^158]:    ${ }^{253}$ Economists usually say that NPG implies that debts cannot be rolled over forever (Romer 2006).

[^159]:    ${ }^{254}$ The exponent in the numerator must be $(\mathrm{T}+1)$ rather than $T$ because we start the sequence with $\mathrm{T}=0$.

[^160]:    ${ }^{255}$ Nevertheless, a simple monotonic transformation may easily shift the utility function to a positive quadrant leaving the behavior of the consumer unaffected. Furthermore, marginal utility is positive and diminishing even for $\theta>1$. The marginal rate of substitution between consumption in any time has typical properties as well. Thus, utility function posited in a negative quadrant poses no problem to our analysis.

[^161]:    ${ }^{256}$ This form was first presented in Samuelson (1937).
    ${ }^{257}$ See e.g. Strotz (1956:169).
    ${ }^{258}$ All the dynamic optimization methods presented here and in Appendix 7 can be found in Kamien and Schwarz (1991).

[^162]:    ${ }^{259}$ Assumptions (A7_2) - (A7_5) hold also for labour. See Acemoglu (2011).

[^163]:    ${ }^{260}$ See Hayek (1935) for a discussion about a usual approach to depreciation and the poor legitimacy of the theoretical separation of the pure replacement of capital and net investment.

[^164]:    ${ }^{261}$ See Appendix 4 B for a thorough discussion about a similar condition. At the steady state, $\mathrm{c}(\mathrm{t})$ is constant, hence the first term inside the integral must approach zero in infinite time.

[^165]:    ${ }^{262}$ Surprisingly, Hayek developed part of his theory of capital on the assumption of a benevolent central planner. We will follow this assumption in this section. Nonetheless, the decentralized economy solution of the model is exactly the same as for the central planner. See e.g. Romer (2006) or Barro, Sala-i-Martin (2004)
    ${ }^{263}$ This procedure might be found e.g. in Blanchard and Fischer (1989:39-41)

[^166]:    ${ }^{264}$ For the labour income of one individual in the general equilibrium model, we will use expression $\mathrm{W}(\mathrm{t}) \mathrm{l}(\mathrm{t})$ rather than $\mathrm{Y}(\mathrm{t})$. However, each member of household offers just 1 unit of labour in every period, i.e. $\mathrm{l}(\mathrm{t})=1$. Thus, we get $\mathrm{W}(\mathrm{t})$ as the labour income of each individual at time $t$. However, all variables in (A7_24) are of instantaneous nature. See section C of this appendix to derive (A7_24) from (A7_23).

[^167]:    ${ }^{265}$ It can be shown that if our modeling technique was slightly different, namely if we modeled a representative individual, who does not care about population growth, rather than a representative household which is concerned about the rate of its expansion, the natural real rate of interest at the steady state (i.e. in the dynamic general equilibrium) would also positively depend on the rate of population growth (see Section E).

[^168]:    ${ }^{266}$ This stems from the fact that $R(t)=\int_{0}^{t} r(\tau) d \tau$, thus $R(0)=\int_{0}^{0} r(\tau) d \tau=0$. Similar idea holds for the

[^169]:    ${ }^{267}$ Production function in the extensive form assumed in (A7_2D) is $Y=A F(K, L)$, in the intensive form $y=A f(k)$. Furthermore, there is no need to distinguish between consumption per worker C and consumption per effective worker $\mathrm{c}=\mathrm{C} / \mathrm{A}$ since (labour-augmenting) technology is constant.

[^170]:    ${ }^{268} \mathrm{As}$ can be seen, the golden rule for $\mathrm{n}=\mathrm{g}=0$ is $\mathrm{MPK}=\delta$.

[^171]:    ${ }^{269}$ As can be seen in Figure No. 0a_A7, there is no peak point on the capital locus $\mathrm{dk} / \mathrm{dt}=0$ since its equation (see equation A7_11 for $\mathrm{n}=\mathrm{g}=\delta=0$ ) implies $\mathrm{c}(\mathrm{t})=\mathrm{Af}(\mathrm{k}(\mathrm{t}))$ which in this case expands beyond all bounds with higher k .
    ${ }^{270} \mathrm{c}_{\text {sub }}$ is not optimal, because the interest rate for such a state would be lower than the subjective discount rate. This would lead the consumer to choose much larger consumption (larger than income, hence consumers would directly consume capital) and a decreasing time shape of consumption. The economic reason is that consumers are too impatient to keep positive saving for such a low interest rate.

[^172]:    ${ }^{271}$ since $y=\operatorname{Af}(k)$, then for $\operatorname{CDPF} y=A k^{\alpha}$, and $f(k)=k^{\alpha}$

[^173]:    ${ }^{272}$ The consumption locus in Figure No. 1_A7 shifts to the right because the term $A$ was increased in equation (A7_34D). Furthermore, according to (A7_37D) the golden rule level of capital is higher than before along with the golden rule level of consumption (see A7_38 D). Hence, the capital locus expands, as can be seen in this figure as well.
    ${ }^{273}$ It can be shown that if $\theta$ exceeds 9.2, the initial saving rate decreases. However, this does not mean that total saving falls. There are two simultaneously operating phenomena. First, the increase in $A$ raises income and hence saving curve shifts to the right. Second, saving also depends on the shape of the saving curve, which is determined by $\theta$. See section F that analyses the role of $\theta$ in determining the impact of the real interest rate on present consumption. Furthermore, it can be shown that present consumption is not affected after the shock if $\theta=$ 0.38. Again, this does not mean that the saving curve is vertical, as one would suggest for invariant income, or horizontal as one may think if income moves with higher technology. The saving curve rather shifts to the right due to higher income. At the same time, it is very flat. Hence, an increase in income accompanied by a considerable increase in saving leads to zero change in present consumption for this very low value of $\theta$.

[^174]:    ${ }^{274}$ Before any accumulation of capital starts, i.e. in the period of the shock, output rises by $10 \%$ due to an increase in the level of technologies by the same magnitude.
    ${ }^{275}$ It should be stressed that the time unit in the figures is one year. However, since the model is continuous, the growth rate of $10 \%$ is valid not in the entire year, but rather in the given instantaneous moment of the shock. Immediately after this shock, the growth rate falls below the real interest rate. See Figure No. 7_A7b for the dynamics of the growth rate per worker within the first year, in which the time interval between subsequent periods is much shorter.

[^175]:    ${ }^{276}$ In this model, we assume perfect foresight, hence expected inflation is always equal to actual inflation. However, the shock itself is unexpected. Thus, the actual inflation rate cannot be reflected in the expected inflation rate at the moment of the shock and, as a result, in the nominal interest rate. It seems to be more appropriate to assume that the nominal interest rate at the time of shock is equal to the previous steady state level, so we can focus on the behaviour of the nominal interest rate in periods that follow the shock. Furthermore, this disturbing fact (which also creates a difference between the natural real rate of interest depicted in Figure No. 8_A7 and the ex post real interest rate) is valid only at one instantaneous moment of the shock, not in the whole year.

[^176]:    ${ }^{277}$ However, if the depreciation of capital is zero $(\delta=0)$, the eventual saving rate is the same as before the shock, namely zero (See Figure No. 0d_A7)
    ${ }^{278}$ This $\theta$ is such that $1 / \theta<s^{*}$, which, according to equation (A7_42 D), means $\theta=(\rho+\delta) /(\alpha \delta)$. By inserting parameters from our simulation, we get the critical value: $\theta=7$.

[^177]:    ${ }^{279}$ On the other hand, very low $\theta$ (namely 0.3 for our set of parameters) leads to an increase in the saving rate immediately after the shock. In this case, the consumer easily shifts consumption over time and responds to a gradually increasing real interest rate by reducing the present consumption.

[^178]:    ${ }^{280}$ Recall (see equation 51 in the main text) that the steady state nominal rate of interest is $\mathrm{i}^{*}=\rho-\mathrm{n}-(1-\theta) \mathrm{g}$. Hence its level is unaffected by changes in $g$ if the utility function is logarithmic $(\theta=1)$.

[^179]:    ${ }^{281}$ Notice that the only difference is the absence of $\mathrm{L}(\mathrm{t}) / \mathrm{H}$ in the life-time utility function (A7_1 E ). Thus, the "exp" term in (A7_35) is without $n$. As a result, step (A7_41), for example, takes the form:
    $[r(t)-n]=-\left[(-\rho)+(-\theta) \frac{\dot{C}(t)}{C(t)}\right]$

[^180]:    ${ }^{282}$ Böhm-Bawerk (1891) claimed that the interest rate is positively affected by the undervaluation of future wants, productivity of capital, and the population growth. The first phenomenon is reflected by $\rho$ in our model, the second by $g$, and the third by $n$.
    ${ }^{283}$ The impact on the optimum initial consumption critically depends on the shape of the saddle path, which in turn is affected by parameter $\theta$. The precise analysis of this path would require similar approximation around the steady state as was performed before. However, the idea is similar to a change in $A$ described in the previous section. Thus, a simple graphical representation seems to be adequate for our purposes. In other words, there is no need to derive the solution of the entire system of differential equations for this minor modification of our model. Since the steady state consumption in this case decreases, higher preference for consumption smoothing (higher $\theta$ ) is consistent with a drop in present optimum consumption and thus with the saddle path that is closer to the new capital locus. On the other, lower $\theta$ might lead to an increase in present optimum consumption.

[^181]:    ${ }^{284}$ Since wages are constant by assumption, real interest rate must be positive to get converging integral. Nonetheless, as will be seen in section G, negative real interest rate can be obtained only with generally diminishing labour income endowment. Thus, assuming positive subjective discount rate and constant wages over time, the general equilibrium requires that the real interest rate is surely positive as well.

[^182]:    ${ }^{285}$ Consumption of capital is thoroughly discussed in Hayek (1984b; 1941; 1935).

[^183]:    ${ }^{286}$ In this case, the break-even investment line in the Solow model will be downward sloping. Then, the resulting negativity of the optimum saving rate is gigantic.

[^184]:    ${ }^{287}$ We are obviously neglecting the crucial time dimension in this process. At one moment the credit contract is negotiated, at some future moment it is settled. The eventual level of the natural rate of interest could be different. Obviously, the entire optimization problem should be redefined if we allow for uncertainty in the model. Euler equations will be modified due to the presence of the stochastic element. In our approach, we only analyze optimum response of present consumption to an unpredictable change in the real interest rate. This will in turn affect future accumulation of capital, marginal product of capital, and hence the natural rate of interest.
    ${ }^{288}$ Obviously, one of the main building blocks of the RBC theory, intertemporal substitution of labour, is missing in our model. Yet, we are mainly interested in the optimum intertemporal consumption behaviour and the resulting natural rate of interest.

[^185]:    ${ }^{289}$ The subjective discount rate, reflecting the desire to achieve the given want sooner rather than later, affects rather the next period interest rate in our model.

[^186]:    ${ }^{290}$ The relevant literature and comparison of various authors will be presented in section 2.3.

[^187]:    ${ }^{291}$ Haberler (1946:35) pointed out that this conclusion was first made by David Davidson. This conclusion will be reconsidered in section 5 in which the expected inflation will be discussed in more detail.

[^188]:    ${ }^{292}$ Discussions of the decline in potential output during economic crises can be found in Furceri and Mourougane (2009)

[^189]:    ${ }^{293}$ Terms "natural output" and "potential output" are used interchangeably. More on this will be said in the following section.
    ${ }^{294}$ In Chapter 3, it was derived that the growth rate in potential output on the balanced growth path is equal to the sum of the rate of technological progress and the population growth, i.e. $\mathrm{g}_{\mathrm{Y}^{*}}=\mathrm{g}+\mathrm{n}$.
    ${ }^{295}$ The money market equilibrium is to be derived for the nominal interest rate. However, assuming zero expected inflation, nominal interest and real interest coincide. This assumption will be relaxed in the next section.

[^190]:    ${ }^{296}$ In the fixed price level framework, higher interest rate will increase velocity and this will in turn raise output - we move along an upward sloping LM curve (Mankiw 2003:276).
    ${ }^{297}$ Compare this interpretation with the neo-Wicksellian model presented in Woodford (2003).

[^191]:    ${ }^{298}$ Temporary fiscal expansion means higher taxes either in the present or in the future. People may feel poorer, and the intertemporal model predicts that they will work more. This would be reflected by a rightward shift in the $\mathrm{Y}^{*}$ curve. However, we neglect this effect.
    ${ }^{299}$ Compare the flexible price IS-LM-Y* model with models in Barro (1997). In our case, we integrated his Ys Yd model with his neoclassical interpretation of the money market ( $\mathrm{Ms}-\mathrm{Md}$ ). Ys curve is represented by $\mathrm{Y}^{*}$ curve, whereas Yd is IS in our model. The money market is directly integrated as the LM curve since Barro also assumed that the real demand for money is positively related to the interest rate.

[^192]:    ${ }^{300}$ The discussion about the expected deflation that will surely emerge in this situation and the impact of this phenomenon on the real demand for money will be postponed to the next section.

[^193]:    ${ }^{301}$ The discussion about self-perpetuating deflation caused by deflationary expectations will be postponed to the next section.

[^194]:    ${ }^{302}$ It is obvious that in the past, the supply of goods must have taken place. However, we assume a steady state in which the flows of incomes and consumption are constant and in which the demand for nominal balances is held constant as well. A sudden decrease in " $k$ " represents a disruption of this equilibrium leading to an increase in the flow of nominal demand and subsequently in nominal incomes.

[^195]:    ${ }^{303}$ IS-shocks are, in the end, included in V-shocks as well because they affect interest rate and consequently $\mathrm{k}(\mathrm{i}, \ldots)$. Yet, to be more accurate, we may say that the MV-rule will automatically offset LM-shocks coming either from changes in money supply or external changes in money demand, and the IS-shocks.

[^196]:    ${ }^{304}$ Panel (b) is constructed for logarithms to stress the slope of the MV-rule curve. Quantity equation then takes the form $\ln \mathrm{M}+\ln \mathrm{V}=\ln \mathrm{P}+\ln \mathrm{Y}$. The MV-rule curve then coincides with the XX curve in Bean (1983).
    ${ }^{305}$ See important discussions in Bradley and Jansen (1989), Hall and Mankiw (1994), Taylor (1985), West (1986), or Clark (1994).

[^197]:    ${ }^{306}$ A careful reading of Friedman (1969) may uncover that he also did not distinguish between the two meanings of time preference, i.e. $\rho$ and MRS. More on this can be found in Chapter 3 of this dissertation.

[^198]:    ${ }^{307}$ The growth rate of the money supply under the Friedman rule is: $g_{M}=-r+(n+g)-g_{V}$ because for $i=0 \%$ : $r$ $=-\pi \mathrm{e}$. Furthermore $\pi \mathrm{e}=\pi=\mathrm{gM}+\mathrm{gV}-\mathrm{gY}$ and $\mathrm{g} \mathrm{Y}^{*}=\mathrm{n}+\mathrm{g}$

[^199]:    ${ }^{308}$ The difference between neutrality and super-neutrality is as follows: money is neutral if it is not important for real economy whether the money supply is one trillion or ten trillion. Money is super-neutral if the real variables are not affected by a change in the growth rate in the money supply (e.g. $3 \% \mathrm{vs} .5 \%$ ). In the exposition presented above, money is neutral, but not super-neutral.
    ${ }^{309}$ At this point, we encounter a specific problem - forces of the LM curve may shift the entire IS curve due to this specific form of the "Piguovian" effect of real money balances. The forces of the LM curve, i.e. changes in money supply or money demand (growth), may then affect the natural rate of interest. In modern models, this property may be derived if the utility function is not additively separable in real consumption and real money balances (Woodford 2003:115; McCallum 2000:882).

[^200]:    ${ }^{310}$ On the BGP in the economy with neutral money, the growth rate in nominal wages is equal to the growth rate in real wages " g " plus the inflation rate $\pi$. Under the MV-rule, we get that the growth rate in nominal wage $=\mathrm{g}+$ $\pi=\mathrm{g}-(\mathrm{n}+\mathrm{g})=-\mathrm{n}$.
    ${ }^{311}$ Compare this recommendation with the discussion of Milton Friedman (1969:92) about proposals of Henry Simons. One of these was to freeze the money supply.
    ${ }^{312}$ We can check that this conclusion holds. The rate of inflation is equal to the monetary growth minus the growth in output, i.e. $(\mathrm{n}+\alpha \mathrm{g})-(\mathrm{n}+\mathrm{g})=-(1-\alpha) \mathrm{g}$.

[^201]:    ${ }^{313}$ The change in velocity was due to the reduction in the interest rate.

[^202]:    ${ }^{314}$ It is the OLG structure of the model with finite lifetime horizon of individual consumers that may lead to dynamic inefficiency. The economy may over-save and hence over-accumulate capital, if the (labour) income stream of individuals is sharply decreasing - in other words, if they expect that they will earn much less in the future. Notice that it is exactly this assumption we used to explain the sudden downward shift in the IS curve.
    ${ }^{315}$ As we know from Chapter 3, the real natural rate of interest may fall below zero (even when the subjective discount rate is positive ( $\rho>0$ ), i.e. even when the given satisfaction is preferred sooner rather than later) if the income stream of the majority of population is decreasing. In Figure No. 27, we assume that the real natural rate of interest after the IS-shock, $\mathrm{r}_{\text {nat,IS-shock }}$, is positive; yet, we could depress its value below zero.

[^203]:    ${ }^{316}$ Our analysis is in perfect accordance with the model in Eggertsson and Woodford (2003:168), in which the natural rate of interest drastically falls with lower future natural income since they assume low elasticity of substitution in consumption (high $\theta$ in our model).

[^204]:    ${ }^{317}$ Articles analyzing ZLB usually assume negative natural rate of interest. But they also assume that the natural rate is solely determined by $\rho$. Hence, they imply temporary negative time preference, which is rather disturbing especially for the Austrian economists. It would be quite difficult to find economic reasons for the preference of future utility. It is therefore more consistent to assume a priori positive subjective discount rate and the decline in the natural rate of interest caused by a sharply decreasing income stream.
    ${ }^{318}$ There is one more inconsistency in the model presented above. People do expect a drop in their future income; yet, their inflation (deflation) expectations are still $\pi^{\mathrm{e}}=-(\mathrm{n}+\mathrm{g})$. If they realize that lower future output implies higher future prices, their inflation expectations may increase. This will shift the IS curve in panel (a) of Figure No. 27 outwards, and the nominal rate of interest will rise above zero. The "Fisher equation" curve in panel (b) will shift upwards. The nominal rate of interest will not hit the zero lower bound even though the real natural interest rate can be negative. This mechanism stayed behind the fact that in the RCK model, the nominal rate of interest can never be negative.

[^205]:    ${ }^{319}$ The steady state real interest should rise with higher $\mathrm{g}_{3}$. However, let us assume that the economy suddenly over-saves, so that $\mathrm{r}_{\mathrm{SS}}<\mathrm{n}+\mathrm{g}_{3}$.

[^206]:    ${ }^{320}$ The entire textbook of Woodford (2003) was designed to present this model.
    321 " $y$ " stands for the logarithm of Y. "p" stands for the logarithm of P.

