University of Economics in Prague

Faculty of Finance and Accounting

Department of Banking and Insurance

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Analysis of Interest Rate Swaps and Selected Quantitative Methods

Author: Pavel Slabý

Supervisor: doc. Mgr. Jiří Málek, Ph.D.

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Declaration of Authorship

I hereby declare that I have written this master thesis "Analysis of Interest Rate Swaps and Selected Quantitative Methods" independently, using only literature and other resources that are properly listed in the bibliography.

In Prague, May 15th

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Pavel Slabý

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Abstract

The goal of this thesis is to present the concept of interest rate swaps and selected quantitative methods related to pricing of these derivatives and their risk management. The thesis starts with description of the plain vanilla interest rate swap and its most common variations. The author continues with explanation of selected quantitative methods, these include bootstrapping, linear interpolation, convexity adjustment, calculation of forward rates and constructing the zero-coupon curve. Duration, scenario analysis, DV01/PV01 indicator and several other risk management methods related to interest rate swaps are described as well. Principal component analysis is emphasized as one of the tools that is used to describe the development of interest rates. The analysis is applied on an example of EUR swap rates. The described methods are at the end of the thesis utilized to price a real-life interest rate swap.

Key words: interest rate swap, financial derivatives, interest rate risk, term structure of interest rates, interest rate risk management

JEL classification: G120, G150, G170, G130

Abstrakt

Cílem této diplomové na téma Analýza úrokových swapů a vybraných kvantitativních metod je prezentovat koncept úrokových swapů a vybrané kvantitativní metody, které lze využít k ocenění těchto swapů a řízení jejich rizik. V první části je popsán koncept úrokových swapů na jednoduchém příkladu “plain vanilla” úrokového swapu a dalších běžně používaných typů swapů. Vybrané kvantitativní metody jsou popsány v dalších částech, těmito metody jsou bootstrapping, lineární interpolace, očištění o konvexitu (convexity adjustment), odvození budoucích sazeb a konstrukce časové struktury úrokových sazeb. Některé metody využívané k řízení rizik úrokových swapů jsou popsány dále, mezi nimi například koncept durace, scenario analýza nebo indikátor DV01/PV01. Principal component analýza je podrobněji představena jako jeden z nástrojů, kterým lze popsat chování úrokových sazeb a tato metoda je také aplikována na eurové swapové sazby. Některé metody popsané v této práci jsou nakonec aplikovány na ocenění reálného úrokového swapu. Postup konstrukce struktury úrokových sazeb a ocenění tohoto konkrétního reálného swapu je popsán krok za krokem v poslední kapitole.

Klíčová slova: úrokový swap, finanční deriváty, úrokové riziko, struktura úrokových sazeb, řízení úrokových rizik

JEL klasifikace: G120, G150, G170, G130

CONTENTS

INTRODUCTION

The goal of this thesis is to describe selected quantitative methods related to interest rate swaps. These methods are described theoretically as well as applied on specific examples to show how they can be used in practice. There is already a lot of theoretical literature that explains what interest rate swaps are, what they can be used for and how they can be valued. Most of the literature however explains these topics only on simplified textbook-like examples. The aim of this thesis is to go beyond these simple examples and present more thorough approach applied on valuing a real-life interest rate swap using real life data.

There are numerous quantitative methods applicable to interest rate swaps that have been developed by the market participants as well as by scholars and are used in practice. These are methods used not only to value an interest rate swap or set a swap rate, but also methods related to risk management of interest rate swaps, modelling of interest rates and the construction of the term structure of interest rates. Every major institution uses its own models and methods and thus their results are virtually never the same. Nevertheless, all the methods have some common characteristics and are derived from the same basic principles. It is beyond the scope of this thesis to analyze all the methods; only selected ones are presented. Practitioners could thus easily argue that some of the described methods are simplified or inferior to other more sophisticated methods. The author is aware of this. Many of the quantitative methods described in this thesis are applied to value a real-life interest rate swap in the last chapter. Most of the concepts presented in this thesis are applicable not only to the interest rate swaps but also to many other financial instruments, especially the fixed income. The author assumes that the reader is already familiar with some basic financial concepts and terminology.

The first chapter describes the plain vanilla interest rate swap contract in general. It explains on a very simplified example the mechanics of an interest rate swap contract and illustrates why these contracts are used in modern finance. The chapter is concluded by a brief introduction to the swap valuation principle. The second chapter briefly describes other types of interest rate swaps. Third chapter is dedicated to interest rates. It frameworks the concept of interest rates and most importantly presents some basic concepts on how to calculate interest rates using bootstrapping, interpolation techniques and convexity adjustments. It also describes the relationships between spot, forward and par rates. Fourth chapter presents a simplified process of valuing an interest rate swap and determining the swap rate. Fifth chapter explains some approaches to interest rate swap risk management, it defines some basic fixed income concepts such as duration and convexity and further describes specific approaches applicable to interest rate swaps such as scenario analysis or DV01/PV01 approach. Analysis of the term structure of interest rates is certainly important when talking about interest rate risk management. This approach is described in chapter six, which focuses on principal component analysis of interest rates. Chapter seven uses the concepts described in previous chapters and uses them to value a real-life interest rate swap contract during its life.

Reader interested in studying derivatives literature can be advised to see J. Hull (2012), where he can find very thorough and wide description of financial derivatives. Fabozzi (2005) and Martellini (2003) describe the fixed income instruments. Many financial institutions also issue documents related to financial derivatives for example Bank of Canada (2000) provided a thorough manual on constructing the zero-coupon yield curve, which is necessary to value not only interest rate swaps but also many other fixed income instruments.

1. INTEREST RATE SWAPS

Interest rate swaps are financial derivatives that are frequently used to hedge against the changes in interest rates. They are OTC contracts between two counterparties. Each interest rate swap contract is different and can be structured in such a way that it suits the specific needs of the two counterparties entering the contract, interest rate swap contracts are thus largely not standardized. Nevertheless, there are some conventions that are adopted and followed by the market participants and that are specified for example by ISDA[1], which seeks to set at least some standards for (not only) the interest rate swap market, with a goal of making the trading of derivatives more efficient and safer.

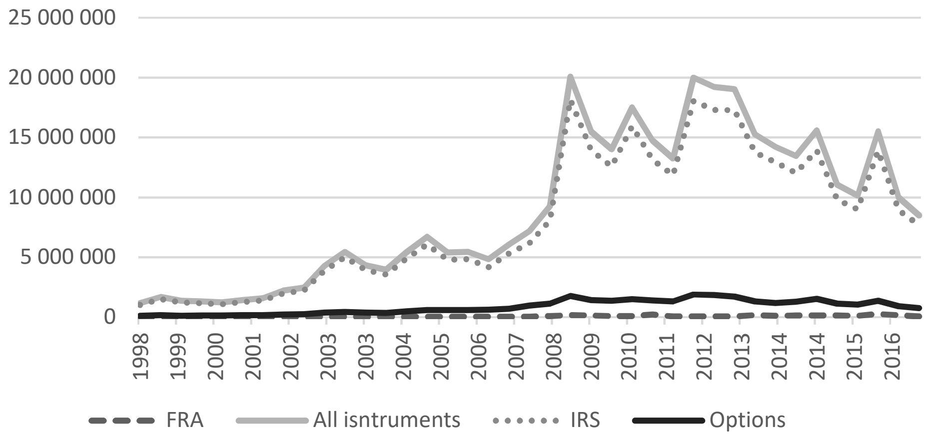

Figure 1.1 Gross market value of OTC interest rate derivatives market (in millions of USD)

Source: Bank for International Settlements

Interest rate derivatives became more popular in the 1980s after many market instruments became more volatile and the markets became more globalized[2]. Since then, the interest rate swap market has grown into huge proportions. The composition and size of the derivatives markets are visible in the following graphs, data are published by the BIS[3]. Gross market value is defined by the BIS as "the sum of the absolute values of all outstanding derivatives contracts with either positive or negative replacement values evaluated at market prices prevailing on the reporting date."[4]

[1] International Swaps and Derivatives Association. ISDA prepares the ISDA Master Agreement and a lot of other supporting documentation, which reduces legal and credit risk when trading financial derivatives. The Master Agreement is a document that sets terms between two parties that enter a derivative transaction. Once the Master Agreement is signed, it is not necessary to negotiate the terms again when the parties want to enter a new transaction in the future.

[2] The end of Bretton Woods agreement, Oil crises of the 70s, Reagonomics, but also gradually growing interconnectedness of the global financial system can be listed among many factors that contributed to the rise in popularity of interest rate derivatives.

[3] Bank for International Settlements performs regularly surveys of foreign exchange and OTC derivatives markets and publishes the collected data.

[4] https://www.bis.org/statistics/bulletin_glossary.pdf

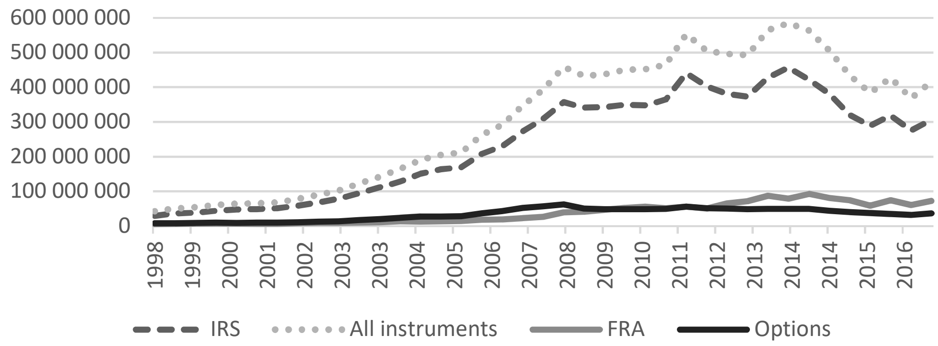

Figure 1.2 Notional amounts of OTC interest rate derivatives (in millions of USD)

Source: Bank for International Settlements

It can be seen from the graphs that the value of interest rate swaps is much higher than of any other interest rate derivative instrument. The OTC interest rate derivatives market is dominated by only few currencies. USD, EUR, JPY and GBP account together for 85% of the total market, while only USD derivatives represent 38%.

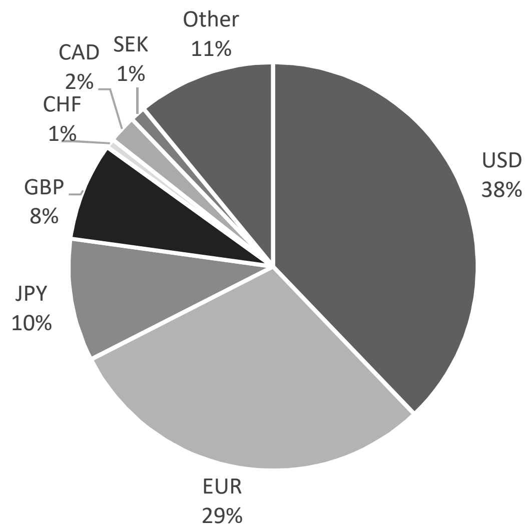

Figure 1.3 Currency composition of the OTC interest rate derivatives market

Source: Bank for International Settlements

1.1. Interest Rate Swap Mechanics

When two parties enter an interest rate swap contract, they agree to exchange (to swap) different cashflows. The mechanics of a general interest rate swap contract is further explained on an example of the “plain vanilla” interest rate swap, which is suitable for explanatory purposes and can be thought to be a cornerstone of more sophisticated contracts. In plain vanilla interest rate swap contract, two entities specify a notional principal amount and interest rates (or reference interest rates) that are used to calculate interest payments. Notional principal amounts are however not exchanged, only the interest payments are, the notional principal amount is used only for calculation of the interest payments and is constant during the life of the contract. In plain vanilla interest rate swap, one entity pays periodically a fixed interest and receives floating interest payments. The counterparty then receives fixed rate payments and pays floating rate payments. The floating interest payments are calculated from a market rate specified in the contract e.g. LIBOR[5], EURIBOR, PRIBOR etc. Fixed rate (also called the swap rate) is specified in the contract. Sums of floating rate and fixed rate payments are also known as “floating leg” and “fixed leg”. The duration and the periodicity of the payments are also specified in the contract, one entity might e.g. pay annually 3.15% from the notional of 100mn EUR. That is 3.15mn EUR and the other might pay semiannually a 6-Month EUR LIBOR + 100bps. The first floating rate payment is usually known at the beginning of the contract. Cashflows for such a hypothetical interest rate swap agreement are illustrated in the table below.[6] The hypothetical swap has a duration of four years; the payments happen always on 1.1 and 1.7. The floating rate payments are often determined one period before the payment, i.e. the LIBOR rate used for calculating the payment on 1.7. is always observed on 1.1.

Year	Date	6-Month Libor (%)	Floating rate cash flow	Fixed rate cash flow	Net cash flow
0.5	1.1.2018	2.00%			
1	1.7.2018	2.06%	3.00	-3.15	-0.15
1.5	1.1.2019	2.12%	3.06	-3.15	-0.09
2	1.7.2019	2.18%	3.12	-3.15	-0.03
2.5	1.1.2020	2.24%	3.18	-3.15	0.03
3	1.7.2020	2.30%	3.24	-3.15	0.09
3.5	1.1.2021	2.36%	3.30	-3.15	0.15
4	1.7.2021		3.36	-3.15	0.21

Table 1.1 Cash flows from a hypothetical interest rate swap

Cash flows are calculated using the following formula:

$$Float\ rate\ cash\ flow = 100mil\ EUR * (6 - Month\ Libor + 100bps) \qquad (1.1)$$

$$3.15\ mil\ EUR = 100mil\ EUR * 3.15\,\% = 3.15\ mil\ EUR$$

The table shows the net cashflows from the perspective of the fixed rate payer, who is by convention known as the buyer of the swap. The net cashflows for the floating rate payer (the seller of the swap) are opposite. Because the interest rate is observed one period before the actual payment date, both counterparties know at any given time of the contract the exact amount of the next payment to be made. The other option would be a swap where the LIBOR is observed at the end of the period, such a swap is called “in-arrears”.

The fixed rate payments should be theoretically always the same, but in practice, they can be slightly different, depending on a day convention that is applied. There is usually a netting principle involved, so that only one entity pays the difference between the cashflows. As mentioned before, there is no

[5] An interest rate at which highly rated banks are offering to deposit money. LIBOR (London interbank offered rate) rates are quoted for seven maturities and five currencies by the Intercontinental Exchange (ICE).
[6] Note that the example is considerably simplified for illustration purposes.

exchange of the notional principal amount of 100 mil EUR, however even if there was such an exchange, the net cash flow would not change.

1.2. Using Interest Rate Swap to Transform an Interest Rate

From the table above, the "transformation function" of the swap is visible. One entity transforms fixed rate payments to floating rate payments and the other entity does the opposite. Let us illustrate the usefulness of the transaction on a few examples. When investors expect a rise in interest rates, they will demand floating rate instruments, e.g. floating rate bonds. Companies on the other side would like to issue bonds that provide fixed rate coupon, so that they can better calculate the cost of their debt. For such a company, it is quite often cheaper to issue a floating rate coupon bond (and take on interest rate risk) and enter and interest rate swap contract (and minimize the interest rate risk), than just issue fixed rate bond with higher fixed coupon. In a similar way companies might want to transform their commercial loan payments or transform an asset earning a fixed rate into an asset earning a floating rate. The transformation function is illustrated in the diagram below. The diagram shows how company A transforms floating rate payments of LIBOR + 50bps into fixed rate payments of 2.99% + 50bps. Company B transforms fixed rate payment of 3% into floating rate payments of LIBOR + 0.03%. A financial intermediary is also involved.

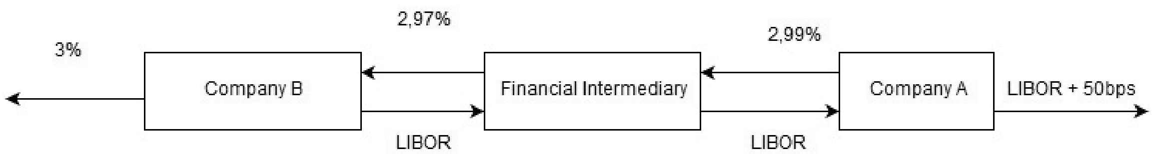

Figure 1.4 Illustration of the transformation function

Financial intermediaries (swap dealers), most commonly banks, are involved because companies do not usually interact directly with each other. Such an intermediary then collects a fee, that among others, reflects the risk that a swap party might not honor its obligation (credit counterparty risk). In our diagram, the bank collects a fee of 0.02% of the notional principal amount (under the simplifying assumption that the swaps´ notional principal amounts with both companies are the same). When an intermediary is involved, the entities are in contract effectively with the bank and not with the other firm. The bank can hedge its position with one company by entering a contract with another company.

The banks can act either as brokers or more commonly as market dealers. When a bank acts like a brokerage, it merely arranges the swap between two counterparties. However, most of the time, banks act like market makers. They are ready to enter a swap transaction under certain conditions at any time. Such a bank than usually quotes its bid and ask fixed rates for various maturities. In our example these quotes are 2.97% and 2.99% respectively. The difference between the bid and ask rates is the spread, i.e. the profit that the bank makes. The average of the bid and ask rates is called the mid-swap rate, that is 2.98%. Entering a swap contract in a position of a market maker means taking on more risk and this creates the need for the bank to hedge its transaction either by entering an offsetting swap transaction or by using other financial instruments.

Financial intermediary (a bank) is exposed to credit risk arising from the swap contract. This is the risk that a counterparty will default and will not repay its obligation. This would however harm the financial institution only if the value of the swap from the bank´s perspective is positive. If the value of the swap from the bank´s perspective is negative (the present value of future cashflows that it will have to pay to the counterparty), this might theoretically not harm the bank. As a reminder, the initial value of the swap is usually set to zero.

1.3. Comparative Advantage Argument

Comparative advantage argument is used to explain the economic efficiency of swapping floating rate to fixed rate or vice versa. When a company has a comparative advantage on one of the fixed/floating interest rate markets, it might make sense for the firm to always borrow at such a market. If a company wants to borrow at the other market, it will use an interest rate swap to transform the interest rate. In general, companies achieve different interest rates because their creditworthiness also varies. It is intuitive that banks offer lower rates to companies with a better rating.

Let us illustrate this also on an example. There are two companies, AAA and BBB, the first has much higher creditworthiness than the other and can in general achieve a lower interest rate on its loans in both markets. Both companies want to borrow 100mn EUR. The interest rates that banks offer to both companies are summarized in the table below:

Company	Fixed rate market	Floating rate market
AAA	2%	6M-LIBOR + 0.01%
BBB	3.2%	6M-LIBOR + 0.10%

Table 1.2 Offered interest rates

Let us assume that BBB wants to borrow at the fixed rate market and AAA at the floating rate market. While AAA achieves a better rate at both markets, the difference between the rates in respective markets is different. AAA achieves a better rate than BBB at the fixed market by 1.2% but only by 0.09% at the floating rate market. AAA has a comparative advantage at fixed rate market and BBB has a comparative advantage at the floating rate market. When the companies enter an interest rate swap agreement, they can achieve a better interest rate. The schema of the swap agreement is depicted below. BBB swapped an interest payment of 6M Libor + 10bps to 2.5% + 10bps. AAA swapped a 2% payment into 6M LIBOR – 0.5%. Both companies achieve better rates with the use of the interest rate swap then they would do if they went directly to the desired markets.

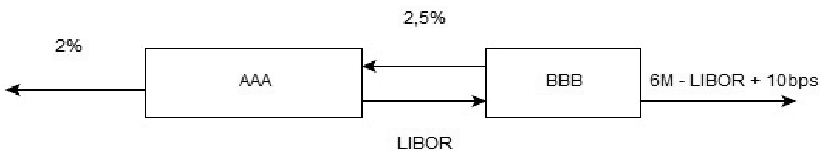

Figure 1.5 The comparative advantage argument schema

Financial intermediary, that would charge fees for its services, could be also depicted in the diagram, the principle would be the same. The comparative advantage argument is quite often illustrated on the cross-currency swap market, where it uses the fact that there are usually different interest rate levels in two different countries. Other explanations of the transformation function, the comparative advantage argument and the criticism of this argument are provided by Martellini (2003) or Hull (2012).

1.4. Swap as a Portfolio of Forward Rate Agreements

When hedging interest rate risk, one could use a portfolio of interest rate forwards. In such a case, fixed rate payer buys a set of forwards with maturities that match the maturities of the assumed interest rate swap payments. That means, that buyer will pay a predetermined rate for each forward contract. When at a time of expiration of each forward contract, the actual interest rate is above the forward rate, the fixed rate payer makes a profit. Value of such a portfolio would equal the sum of values of individual forwards. If we assume that the value of an interest rate swap at its initiation is zero, then the sum of the values of the forward contracts should also equal zero. This does not mean that the price of every forward must equal zero. That would be the case only if the forward rate curve was flat. If it is upward sloping, then the forwards with lower maturity have negative values and those with longer maturities have positive values. When the forward rate curve is downward sloping, the opposite applies. This approach is also known as the forward projection method (Martellini 2003).

Entering an interest rate swap agreement has many advantages over setting up a portfolio of forwards. Firstly, the maturities of forwards do not go as long into the future as the maturities of interest rate swaps. Secondly, the interest rate swap market is nowadays more liquid than the interest rate forward market. Thirdly, interest rate swap is simply more efficient, it is one contract whereas the forward approach would require entering many forward contracts. However, forwards can be still used to hedge interest rates when a firm cannot find a swap partner at a certain market (for example for a certain maturity).

1.5. Swap as a Portfolio of Bonds

Interest rate swap can be also viewed as a portfolio of two bonds, one that pays a fixed coupon and the other that pays floating rate coupon. Depending on whether the investor wants to pay or receive the fixed rate he takes a short position in one bond and a long position in the other bond. The bonds have the same nominal value, so that the cash outflow (for purchasing the bond) at the beginning offsets the cash inflow for selling the other bond. This is necessary, because in the interest rate swap agreement, there is no exchange of nominal values and it is assumed that the notional principal amount is same for both legs of the swap. Short position could be realized through issuing a bond. If we assumed that at the maturity of the bonds both the coupon and the principal notional amount are paid/received, it would not change the value of the swap. The value of the swap from the fixed rate payer´s perspective is then calculated as a difference between the present value of the fixed rate bond and the present value of the floating rate bond:

$$V = B_{fl} - B_{fix} \tag{1.2}$$

and is the opposite from the floating rate payer´s perspective

$$V = B_{fix} - B_{fl} \tag{1.3}$$

The bonds are priced as a present value of future cash flows.

$$B_{fix} = \sum_{t=1}^{n} \frac{C_{fix}}{(1+r_t)^t} + \frac{F}{(1+r_n)^n} \tag{1.4}$$

$$B_{float} = \sum_{t=1}^{n} \frac{C_{fl}}{(1+r_t)^t} + \frac{F}{(1+r_n)^n} \tag{1.5}$$

F is the notional principle amount of each bond
n denotes the time to maturity
t denotes the time of each payment
C_{fix} and C_{float} denote the fixed and floating coupons
r_t and r_n denote the discount interest rates

Fairly valued swap has its value at inception equal to zero. Changes of market interest rates during the life of a swap cause the swap´s value to be further either positive or negative. If the floating interest rate rises, the value of the swap becomes positive from the fixed rate payer´s perspective. Theoretically, a floating rate bond´s value at inception or just upon any reset is equal to its notional value. To illustrate this, one can imagine that the coupon rate and the discount rate are the same and therefore the present value of all the future interest payments equals the bond´s notional. This method is sometimes called the zero-coupon method (Martellini 2003). That is because the value of a swap is viewed as a difference between the value of a zero-coupon bond maturing at the next swap floating rate payment date and a coupon bearing bond maturing at the maturity of the swap. Both bonds have the same principal. It can be theoretically shown that forward projection method and zero-coupon method give the same results.

It would be theoretically also possible to hedge against the rise of interest rates on a hypothetical obligation simply by buying a bond with the same maturity as the obligation and with the same floating interest rate as is the floating interest rate on the obligation. The advantage of interest rate swap is that it is cheaper and in practice more feasible. Both approaches would however have the same sensitivity to interest rate changes.

1.6. Interest Rate Swap Conventions

Some of the commonly used conventions and terminology is described in this chapter. Confirmation is a short document that outlines the most important information of a derivatives contract such as all the important dates, interest rates, notional amount etc. The trade date is the date when the contract is transacted and it precedes the effective date. The interest on the swap is calculated from the effective date onward. The payment date is when the payments are to be made, but period start/end days mark the start and end of each interest period which is the period for which a specific interest rate is applicable and when the interest accrues. Reset date is the date when the next floating interest rate is determined and is usually at the start of an interest rate period. Termination date is the date when the contract ends, it is usually also when the last payment occurs. The applied floating rate is then known by both entities during the entire interest period. Day conventions are also specified in the contract and they might differ for one leg of the swap from the other leg. Frequency of the payments must also be agreed upon and can also differ for both legs, e.g. fixed payments might be paid annually while the floating rate payments might be paid semiannually.

Day Count Conventions

The day count conventions that are used vary depending mostly on the regional market. For the fixed rate actual/365, act/360 or 30/360 conventions are usually used, whereas the floating rate might be quoted on an actual/360 basis for euro or dollar denominated swaps. Swaps denominated in sterling use actual/365 basis.

This means that the rates are not directly comparable with each other and to make them comparable, they should be transformed so that they both use the same conventions. Another important thing that needs to be declared is which calendar is to be used for determination of holidays. Swap parties might for example decide that USA calendar will be used. When the payment date is on holiday, the payment is usually made on next business day after the holiday. When this day is in a next month, the payment is made on the immediately preceding business day (this principle is known as following business day convention). ISDA defines many parameters and terminology used in swap agreements and provides freely much of its documentation on its website.

2. OTHER TYPES OF SWAPS

There are many types of interest rate swaps that are regularly transacted. The variety of interest rate swaps is limited only by imagination of financial engineers. Interest rate swaps can be also combined with credit default swaps to create even more complicated instruments. Only the most important types of interest rate swaps are described in this chapter.

2.1. Overnight Indexed Swaps (OIS)

Overnight indexed swaps are short-term interest rate swaps. The maturity of these swaps is up to one year, but a 2-year OIS were also traded. Because of their short maturity, they are often considered to be money market instruments. OIS contracts have different names, depending on the market where they are traded. There is Eonia – Eurocurrency OIS or SONIA – Sterling overnight interest rate average swaps, other OIS contracts are traded on all major currency markets.

OIS swaps are almost solely used by banks. Commercial banks manage their liquidity at the end of each day by depositing money at other commercial banks or at a central bank. The interest rate that is applied on these deposits can be the target of a central bank´s monetary policy. In the USA this rate is called the federal funds rate.[7] An OIS contract is a swap contract that exchanges a fixed interest rate for the geometric average of overnight interest rates. When a bank wants to borrow at the overnight loan market for a longer period, it must roll the loan forward each day. Therefore, when a bank borrows money at the overnight rate, the effective rate for the whole period is the geometric average of the interest rates during that period. The overnight interest rates are recorded during the life of the OIS contract. The exchange of interests takes place at the maturity of the contract. The OIS then effectively swaps overnight lending or borrowing at an uncertain rate for overnight lending or borrowing at a certain fixed rate. This fixed rate is also called the overnight indexed swap rate. A bank can utilize the OIS in the following way:

1) Borrow 100mn USD at the overnight market. Bank rolls the loan forward each day and on each day, it is debited with different interest rate, thus the loan has effectively floating interest rate.
2) Enter the OIS contract to secure a fixed rate on the overnight borrowings.
3) Lend the 100mn USD for USD LIBOR to another Bank.

The result of the transaction is such, that the bank receives LIBOR and pays the (fixed) overnight indexed swap rate. The bank got rid of the risk that the overnight interest rate will unexpectedly rise. There is however still a risk that the other bank will default and not repay its LIBOR loan. The overnight indexed swap rate is generally lower than the LIBOR rate, because the bank wants some compensation for the risk of the other bank´s default.

[7] Federal funds rate is a rate at which depository institutions in the US trade reserve balances held overnight at Federal Reserve Banks with each other.

The difference between LIBOR and the overnight indexed swap rate (known as LIBOR-OIS spread) can be used as a measure of stress in the financial markets. When this difference rises sharply or becomes more volatile it is a sign that banks require bigger compensation for lending to other banks. This was most evident in the recent financial crises as discussed in chapter 3.

2.2. Currency Swaps

Currency swaps and interest rate swaps are similar, the difference is that the principals of a currency swap are not denominated in the same currency but in different currencies. The principals are usually exchanged at maturity, but it is also possible to exchange only the interest payments. Another option is to set the principals in such a way, that their values are the same at the beginning of the contract using an exchange rate set at the beginning. As the exchange rate develops in some direction during the life of the contract, the principal notional amounts might be quite different at the maturity of the contract. When both interest rates are fixed (fixed-for-fixed currency swap), the swap hedges mainly the exchange rate risk. When one or both interest rates are floating, the swap hedges effectively also against the interest rate risk. Such a swap can be for example used by a multinational company that wants to transform a loan denominated in one currency into another currency. The exchange rate is another unknown variable. The motivation for using currency swaps can be also illustrated using the arbitrage argument. The interest rates are generally not same in various currencies.

The value of a currency swap can be also viewed as a portfolio of bonds or forward contracts. When we value the swap as a difference between two bonds, the value (from the fixed payer´s perspective) is calculated as difference between the present value of the fixed rate bond and the present value of the floating rate bond adjusted by the spot exchange rate:

$$V = B_{fl} - S \times B_{fix} \tag{2.1}$$

And is the opposite from the floating rate payer´s perspective

$$V = S \times B_{fix} - B_{fl} \tag{2.2}$$

2.3. Amortizing and Accreting Swaps

In a standard plain vanilla interest rate swap that was assumed in previous text, the notional amount did not change during the life of the swap. It is possible to have the notional value of the swap decreasing or increasing during its life. In an accreting step (also known as a step-up swap) the notional amount increases over time. As a result, when calculating the present value of the cashflow, the future cashflows take a bigger part of the entire “pie”. It is intuitive to think that an accreting swap´s value will more likely depart from its initial value than in the case of a plain vanilla interest rate swap.

An amortizing swap is the opposite of an accreting swap, i.e. in amortizing swap the notional amount decreases over time. This can be used for example to hedge an amortizing loan or to hedge against a rise in interest rates when a company issues a floating rate bond with amortizing notional. Pricing of an amortizing swap is further illustrated on a real-life swap contract in chapter 7.

If the interest rate swap´s notional amount decreases during some time of its life and then increases in its other part, such a swap is known as a roller-coaster swap.

2.4. Basis Swaps

Basis swap is an agreement to exchange two floating rate cashflows. Each floating leg uses different floating rate. This can be used when an entity pays one interest rate on its debts, but receives interest payments from its asset calculated from different interest rate. Both floating rates can be denominated in the same currency, but there are also cross currency basis swaps (also known as differential swaps). Such a swap might for example exchange USD LIBOR for GBP LIBOR.

2.5. Other Swaps

In practice, one interest rate swap contract usually contains more features of the aforementioned swap types.

Commodity Swaps

Commodity swaps are instruments that exchange cashflows derived from a price of a commodity. These instruments are used to hedge the future prices of commodities, for example the price of oil. In such a swap one party makes a series of fixed rate payments (e.g. 70 USD per barrel) and the other makes payments based on current oil price. Notional amount is than specified in barrels.

Swaptions

Swaptions are financial options to enter a swap contract. A swaption gives it´s buyer the right to enter in the future an interest rate swap contract with specified parameters. The swaptions can be theoretically treated as an option on a fixed coupon bond, that has its strike equal to the notional.

Interest Rate Caps and Floors

An interest rate cap is a derivative that pays the difference between a specified reference floating rate and a specified fixed interest rate (strike price). If the floating rate exceeds in a certain period the strike price, the buyer of the swap receives the difference times the notional. Interest rate floor is the opposite of interest rate cap. The buyer of an interest rate floor receives payoff if the interest rate declines below a certain strike price. Interest rate caps and floors can be combined to form an interest rate collar. Interest rate collar is viewed as a synthetized instrument that hedges against rises of interest rates above cap´s strike price and below the floor´s strike price. Cap´s and Floor´s strike prices thus mark a range where the interest rate can oscillate without any payments being made.

3. THE INTEREST RATES

When talking about derivatives, one must consider various interest rates. As there are many different interest rates in the economy, it is suitable to briefly describe the concepts of various interest rates. This might help to better understand more complicated concepts. Each interest rate reflects various factors but all interest rates reflect above all the time value of money. One interest rate can then reflect also credit risk and many other "premiums" that compensate for other types of risk.

Interest rate on government bonds is usually considered to be risk free, and is therefore a good measure of the time value of money in the economy. Investors believe that government bonds (especially those of highly rated countries[8]) bear almost no credit risk because it is highly unlikely that these countries will default on their obligations. US market participants might therefore work with interest rates on US government bonds (the Treasury bonds), while for example European market participants might follow the German government bond rates. These government rates are also known as the benchmarks, a measure against which all other rates/yields are compared. The interest rates charged by central banks on their deposits are sometimes also known as the base interest rates, other interest rates are derived from them. When a central bank lowers its base rate (as we have seen in recent years at many central banks around the world) other interest rates in the economy tend to decrease as well, but are still higher by a certain premium that compensates investors for different types of risk. Corporate bonds are riskier. This can be pictured for example by the risk premium that corporate bonds bear in addition to a benchmark rate, therefore

Yield of a bond = benchmark rate (government bond) + risk premium (also known as spread) (8)

There are many different types of spreads, the most commonly used are Z-Spread, G-Spread, I-Spread, OAS-Spread or T-Spread. For example, a Z-spread (or zero volatility spread), is a spread that is added to the spot rate of government bond yield curve to make the price of a security equal to the present value of its cashflows.

The bond yield is quite often also expressed against a mid-swap rate, i.e.: *yield = MS + x bps*. Mid-Swap rate is an average of ask and bid of a quoted interest rate swap rate. Risk premium can be explained by a term to maturity, type of issuer, type of instrument, credit rating of an issuer, instrument characteristics or general market conditions, among others. A risk premium indicates the difference in risk between government and non-government securities. More interest rates are thought to be risk free. Every situation might demand a use of different interest rates. It has been argued if the Treasury rates implied by the government bonds should be used as risk free rates or not. There is a strong argument that these rates are artificially low for several reasons. Firstly, government bonds must be purchased by many institutions for regulatory reasons and thus there is

[8] There are three major globally recognized rating agencies (Moody´s, Standard and Poor´s, Fitch) that assign ratings to financial instruments, countries and other entities. Each agency has its own rating scale and methodology.

artificial demand that drives their prices up and their yields down. This has been seen especially in a few countries where central banks began asset purchases after the financial crises.[9] Secondly, many government bonds are given a more favorable tax treatment compared to other instruments, this is true at least in the US. Due to these reasons, many analysts used LIBOR rates as the risk-free rates. This approach was however also problematic during the financial crisis of 2007 – 08, when banks grew cautious of lending to other banks. LIBOR rates are not totally risk free (there is a risk of default of a counterparty bank), and thus during the crisis many analysts used overnight indexed swap rates as risk-free rates.

Discount Rate

This is a rate that reflects the time value of money. Any future cashflow that a person receives should be discounted with an appropriate interest rate to reflect the time value of money. When the future value is discounted, we get the present value. It is debatable if the discount rate should reflect only the time value of money or whether it should reflect also any other financial risk. For simplification purposes, let us assume that the desired discount rate should reflect only the time value of money, i.e. a risk-free interest rate can be used as a discount rate. Discount rates will be needed to discount the interest rate swaps´ future cashflows to express their present value. Discount factor (DF_t) is a term that discounts a future value to get the present value and can be calculated in the following way:

$$DF_t = 1 * e^{-(r_t \times t)} \tag{3.1}$$

where r_t is the spot rate applicable at time t.

3.1. Interest Rate Term Structure

Term structure of interest rates shows the relationship between the time to maturity of a security and its yield to maturity. Graphical depiction of the term structure is called the yield curve. This curve can be upward sloping (most common shape) or flat or downward sloping (inverted). The left side of the curve (shorter time to maturity) is referred to as the short end of the curve, while the right side (longer time to maturity) is referred to as the long end of the curve. The short end of the curve is usually steeper than the long end.

[9] For example, quantitative easing in the USA and Asset Purchase Program in the Eurozone

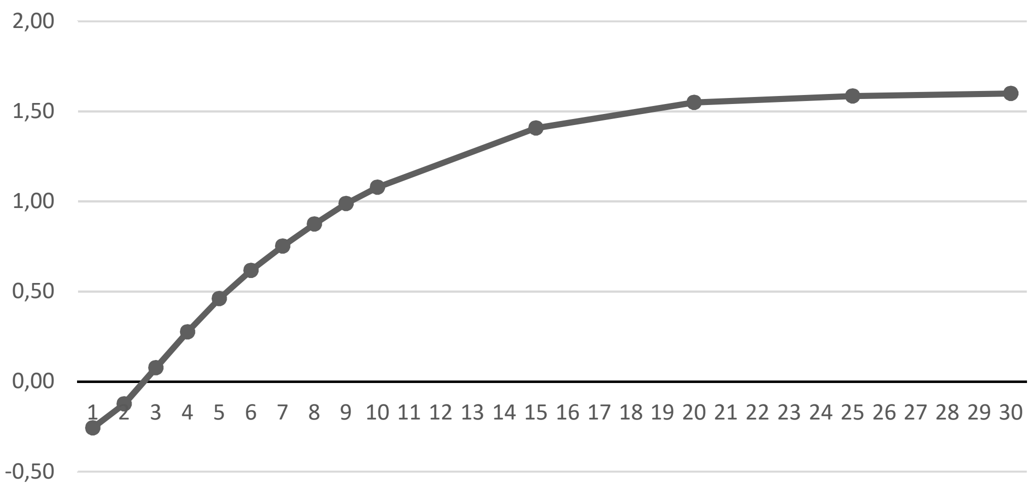

Figure 3.1 The term structure of EUR swap rates (fix - float)

Source: Bloomberg, 7.3.2018

Shape of the Term Structure

There are numerous theories that help to explain the shape of the term structure, most significant ones are the expectation theory and market segmentation theory. Explanations of both theories can be found in Hull (2012). The most influential factors of the term structure are the expectations of future interest rates and bond risk premiums. The shape of the yield curve can be viewed as an expectation of forward interest rates. When the yield curve is upward sloping, it might indicate that the market expects the interest rate to be higher in the future. Detailed view on how the shape of the term structure indicates future interest rates is found in Ilmanem (1995). Risk premium is the difference between the return of a fixed income instrument such as of a bond and the return of the risk-free rate. Positive risk premium makes the yield curve slope upward.

Measuring Interest Rates

Interest rates are usually quoted per annum (p.a.)[10]. Interest rates can have various frequencies of compounding. For example, USD interest rate swap rates are compounded semiannually. To compare these rates with other rates that are compounded only once a year, a certain transformation is necessary:

$$r_y - 1 = (1 + r_c \times \frac{1}{m})^m \qquad (3.2)$$

Where r_y is the desired interest rate compounded once a year and r_c is the interest rate compounded *m* times per year.

[10] Other possibilities are: per semester (p.s), per quartale (p.q.) etc.

3.2. Forward Versus Spot Rates

Spot Rates

Spot rates are the rates that are applied from now (time *t*) until time *T*, where $t < T$. The forward rates are those that are applicable from time t_2 until *T*, where $t < t_2 < T$. Forward rates are therefore effectively future spot rates. Hence, determining forward rates can help us predict the future spot rates.

Let us illustrate on an example how to determine forward rates. Investor might have two options on how to invest money for a two-year period. He can either buy a two-year zero-coupon government bond or he might buy a one-year zero-coupon government bond and on its maturity day, buy another one-year government bond. The investor knows the yield of the first one-year bond as well as that of the two-year bond. The investor however does not know the yield of the "second" one-year bond that will be quoted in one year from now.

Forward Rates

The yield on the second one-year bond is the forward rate and can be theoretically determined from the other two bonds. An investor should be theoretically indifferent between both options. Therefore, the following relationship should hold true:

$$\frac{100}{(1+\frac{r_2}{2})^2} = \frac{100}{(1+r_1)(1+f)} \tag{3.3}$$

r_2 denotes two-year spot rate *(T_0, T_2)*
r_1 denotes one-year spot rate *(T_0, T_1)*
f denotes one-year forward rate *(T_1, T_2)*
100 denotes the bond´s notional value

If there are following spot rates; r_2 = -0.124% and r_1 = -0.2575% then we can calculate the unknown forward rate *f (T_1, T_2)* as

$$f = \frac{(1+\frac{r_2}{2})^2}{(1+r_1)} - 1 \tag{3.4}$$

f = 0.1339%. Its worthy to note a simple observation (that will be explained later), the two spot rates were increasing with maturity and the forward rate for the second period was higher than the spot rate with the same maturity. It is intuitive that the spot rate and the "forward rate" for the first period are the same.

The relationship can be generalized for any periods. The concept of implied forward rates shows that an investor can theoretically combine bonds with varying maturities to guarantee a rate of return that starts at a future point in time. This concept can be used to forecast future interest rates. Even though this method seems quite simple in theory, in practice it is much more difficult to find the appropriate forward rates. This is because there are various day conventions and interest rate periods implied by various market instruments do not have always the same length and do not start where the previous interest rate period ends. Consequently, a lot of adjusting of interest rates and interpolation is needed. It is obvious that if an analyst knows one of the two sets of forward or spot rates, he can calculate the other set. This means that, when an analyst knows spot rates up to 10 years, he can also calculate the forward rates up to that time.

Par Rates

Par yield curve is constructed from yields to maturity of hypothetical bonds that sell at par. A bond sells at par if its yield to maturity is equal to its coupon rate. Par yield is a single discount rate that can be used to set the present value of a bond to be equal to its par value. The par yield for a certain maturity can be therefore thought of as an average of all the spot rates leading up to the maturity of a hypothetical bond. That leads to the conclusion that if the spot yield curve is increasing, the par yield curve must lie below it. When calculating the par yield, it is usually assumed that bonds bear a semiannual coupon. Illustration of computing the par yield follows. If there are the following spot rates; $r_2(T_0, T_{0.5})$ = -0.2769% and r_1 (T_0, T_1) = -0.2575% then we can calculate the unknown par rate $r_p(T_0, T_1)$.

$$\frac{c}{2}e^{-0.5\times -0.2769\%} + \left(100 + \frac{c}{2}\right)e^{-1\times -0.2575\%} = 100 \qquad (3.5)$$

$$c = \left(\frac{100 - \ 100\ \times\ e^{-1\times -0.2575\%}}{e^{-1\times -0.2575\%} +\ e^{-0.5\times -0.2769\%}}\right) \times 2$$

From this equation it easy to determine c = -0.25732, that means that the actual par yield is -0.2573% p.a. with semiannual compounding. This interest rate can be of course further transformed to continuously compounded rate. The relationship can be generalized for longer periods adding more terms.

When the spot yield curve is decreasing the par yield curve lies above it. Furthermore, when the spot rate is increasing the forward rate will lie above it, when the spot rate is decreasing the forward rate will lie below it. When the spot rate curve is flat, the forward, par and spot rates are equal. To visualize these relationships, the following term structure of interest rates was constructed. All the spot rates were simply determined using only the EUR swap rates with maturities from 1 to 10 years and 15, 20, 25 and 30 years, this is totally sufficient for the illustration purposes. Practitioners could however argue that it is more suitable to model the short end of the term structure with Libor rates and futures rates (as discussed later). These spot rates were further interpolated to get a smoother curve and finally the par and forward rates were calculated using equations 3.4. and 3.5. The par rate in the illustration is only slightly lower than the spot rate.

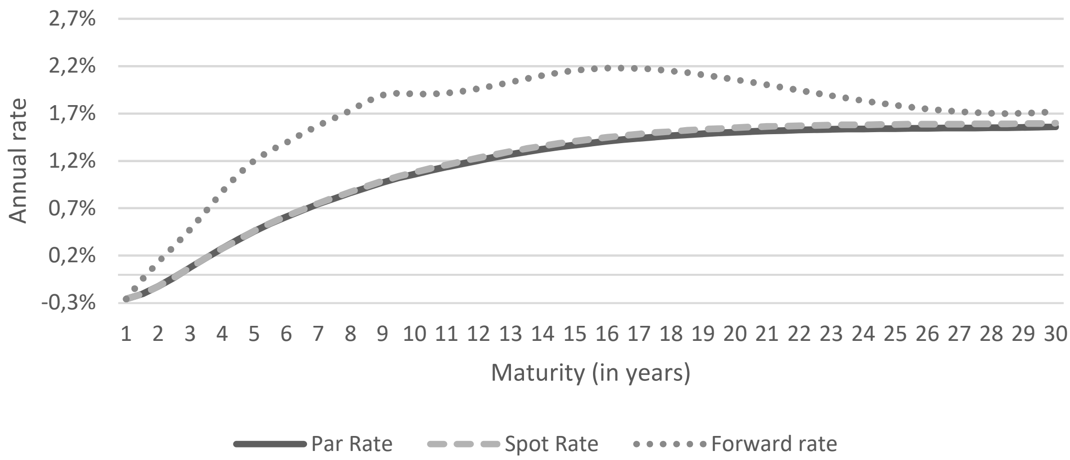

Figure 3.2 Terms structure of the spot, par and forward EUR rates observed on 7.3.2018

Source: Bloomberg

3.3. Using Eurodollar Futures to Determine the Forward Rate

Futures are exchange traded contracts with daily settlement. The Eurodollar futures that are traded on the CME[11] are one of the most liquid futures contracts in the world and are commonly used to derive the future interest rates. The contract unit is a Eurodollar deposit with 1mn USD principal value and three months to maturity.[12] One basis point is worth 25 USD (1 000 000 USD x 0.0001 x 90/360 = 25 USD). Eurodollar futures contract with settlement on 30. 6. can be used to lock in the interest rate for the period from 30. 6. to 30. 9. of the respective year, spot settlement is on 3rd Wednesday of the contract month. The length of the three months period varies depending on the number of days in each month, additionally the settlement date must be a business day both in London and New York. The three months period can be therefore as short as 86 days and as long as 95 days. Trader can lock in the interest rate for this period when he buys the contract. In our case, the trader would buy 100 contracts to hedge 100 000 000 USD. Even though the futures rate is not equal to the forward rate (see the convexity adjustment), it can be used as a good proxy of the forward rate. The Eurodollar futures prices are quoted by the exchange in such a way, that *price = (100 – implied futures rate)*. That means that if futures price is 96.5, the implied futures rate is 3.5%.

3.4. Convexity Adjustment

Using the Eurodollar futures quoted on the CME to derive the zero-coupon curve is quite advantageous, the Eurodollar futures are very liquid, and so we can consider them to be priced fairly

[11] Chicago Mercantile Exchange
[12] Detailed specification can be found on the CME website: http://www.cmegroup.com/trading/interest-rates/stir/eurodollar_contract_specifications.html

and their market to be highly efficient. In practice, these futures are however used only as proxies for the forward rates.

For our purposes, the futures are very similar to forwards, with one important distinction: futures are settled daily. Because of this daily resettlement feature the implied futures rate is likely to be different from the forward rate (futures rate is likely to be higher). This is because if we assume that both negative and positive changes to the futures price happen with the same probability, the short position is advantageous for the investors. This is explained in the following example. Assume there is a Eurodollar contract that is trading at 96 USD (implied futures rate of 4%). There are both short and long positions for this contract. When the price rises to 98, short positions lose 5,000 (value of one basis point is 25 USD) and the long positions will gain the same amount. Short positions finance this loss at 4% and the long positions earn on the profit also 4%. If instead, the Futures price declines to 94 USD, the long positions will show a loss of 5,000 USD that is financed at 6%. So, if the buyer wants to finance his loss he can borrow now only at much higher rate than the short seller could at the first scenario. And the short seller of course makes a profit of 5,000 USD and earns 6% on his profit. The profits and losses are debited / credited to the margin account of the traders and do earn an interest rate. To summarize this, when futures price declines, the short position´s profit equals the long position´s loss, but the short position´s profit can be reinvested at relatively higher rate, than if the long position would make a profit. Because market participants are aware of this asymmetry, the long position traders require a compensation, that drives the futures price lower. Since the price is lower, the implied futures rate is higher than it would otherwise be.

The more daily settlements there are during the life of the contract, the more obvious this asymmetry becomes. The longer the maturity of the futures contract, the worse proxy it is and the bigger adjustment needs to be made. Another factor that affects this asymmetry is the price sensitivity of the asset underlying the futures contract, with bigger price sensitivity there is also bigger asymmetry. To adjust for this asymmetry, so that the implied futures rate is a better proxy of the forward rate, we use the convexity adjustment. There are more approaches to the convexity adjustment. A comparison of various convexity approaches and a case study performed on real life interest rate swap contract is provided by Witzany (2009). One popular simple adjustment is:

$$convexity\ adjustment = \frac{1}{2} \times \sigma^2 \times T_1 \times T_2 \qquad (3.6)$$

T_1 denotes the time to start of the futures interest rate period (three months before its maturity)
T_2 denotes the maturity of the futures contract
σ denotes annualized standard deviation of the change in the short-term interest rate. There are several approaches to estimating the σ, one popular approach is using the GARCH (1, 1) model. From the futures rate and convexity adjustment, it is possible to calculate the adjusted forward rate as:

$$forward\ rate = futures\ rate - convexity\ Adjsutment \qquad (3.7)$$

3.5. Zero-coupon Curve

This chapter describes how to construct the zero-coupon curve, sometimes referred to as the spot rate curve. Each interest rate on this curve reflects the interest rate an investor receives from now until that point on the curve (p.a.). This interest rate curve is used to derive the discount factors and possibly to predict the forward interest rates. Other implicit curves can be then derived from the zero-coupon curve, namely the forward rate curve, instantaneous forward yield curve and the par yield curve.

There are various techniques how the zero-coupon curve can be obtained. BIS (2005) published a technical documentation on various methods of estimating the zero-coupon curve by central banks. Many of them rely on Nelson and Siegel (1987) model or the more advanced Svensson model (1994). These are parametric models and are defined by a function that is used to calculate the curve over the entire maturity spectrum, only the parameters need to be estimated. These approaches can be used to derive in general a term structure of any interest rates.

A different approach is described in this thesis. This different approach derives the zero-coupon curve from several observable market rates. In this “market” approach, the first step an analyst should make, is to observe the market and to decide which instruments can be used to construct the term structure, different instruments are available at different markets. The main idea is that zero-coupon yield curve should best reflect the time value of money. The curve should be therefore constructed from risk-free instruments that are fairly priced. This means that the instruments that we use to construct the curve should have no or minimum risk, liquidity or any other premium. The instruments should also not bear any coupons during their life because their yield to maturity reflects not only the time value of money but also the compensation that an investor receives through coupons. The government securities seem therefore quite suitable for the task. Government zero-coupon bonds could be theoretically used to construct the curve. Zero-coupon bonds are also known as discount bonds or in the USA as Strips. Strips are US Treasury bonds, that are “striped” of their coupons. After the “striping process” the coupons are traded separately and this creates in effect a zero-coupon bond. One problem with this approach is that the liquidity of the stripped coupon treasuries is not as big as that of their coupon counterparts. This means that the striped bonds might in theory bear a liquidity premium and might be therefore mispriced.

There are also not Strips issued for every maturity, that an analyst might need. Therefore, an investor often uses several methods to construct a zero-coupon curve that can be used to discount any cashflow during the swap´s life. Some of these methods are described in the following chapters.

3.6. Bootstrapping

The concept of bootstrapping plays an important role in constructing a zero-coupon curve. Bootstrapping uses data about several coupon bonds to construct the yield curve. The first step is to collect information about bonds that are available at the market and fulfil the desired criteria. The bonds should be as close to risk free instruments as possible, liquid (illiquid bonds might be mispriced) and without options (options are difficult to price).

The main idea of this concept is following: each coupon bond can be thought of as a set of zero-coupon bonds whose notional amounts are equal to the coupons of the coupon bearing bond and their maturities are on the coupon dates. If we worked with coupon bearing bonds, the interpretation of the yield curve would be problematic as the bonds´ yields would also reflect the compensation of an investor through the bond´s coupons.

The implied future rates of return (the forward rates) can be derived from coupon bearing bonds using the bootstrap method. Bootstrapping is a process whose goal is to create a theoretical zero-coupon curve when we do not have data on yields of zero-coupon bonds with any maturity we want but we have data on coupon bearing bonds. The process is illustrated on an example of US Treasuries.

1) Let us assume that we have the following data:

Notional	**Tenor**	**Semiannual coupon**	**Bond price**
100	0.5	0	98.5
100	1	0	94.6
100	1.5	2	97.0
100	2	4	101.5

Table 3.1 Coupon bond data

The first two bonds pay no coupon, the zero rates can be therefore easily determined by calculating the effective annual rate. For the 6-month bond it follows that:

$$100 = 98.5e^{r \times 0.5} \tag{3.8}$$

which reduces to

$$r = -\frac{ln\left(\frac{98.5}{100}\right)}{0.5} \cong 0.0302 \tag{3.9}$$

and using the same procedure, it can be shown that one-year continuously compounded rate *r* = 0.0555 or 5.55%. These rates can be further used for discounting securities with time to maturity of 0.5 and 1 year. This reflects reality, as short-term government zero bonds are suitable for deriving the short end of zero-coupon curve (another approach uses the interbank deposit rates). Now we can use interpolation techniques to derive the zero-coupon yield curve for maturities up to one year. Next step is to construct the zero-coupon curve behind the one-year horizon. The third bond bears a coupon, that we must account for. The equation to price the bond can be set up as follows:

$$2 * e^{-0.0302*0.5} + 2 * e^{-0.0555*1} + (100 + 2) * e^{-\mathrm{r}*1.5} = 97.0 \tag{3.10}$$

This is the general equation for deriving the net present value of future cashflow, where the cashflows are the bond´s coupons (2 USD) and its notional value (100 USD). Solving the equation, we get, r = 0.0394[13]. Using the same principle, it is easy to determine the forth zero rate, that is 0.0806. We could repeat this method to get the whole zero-coupon curve, the only condition is that we have enough bonds, with appropriate maturities. While the bootstrapping method is quite simple in theory, it is much harder to apply it in practice, as there are number of problems that need to be solved. In the example it was assumed that the maturity of the 1.5-year bond falls directly on the third coupon date of the 2-year bond. It is hard to imagine that in practice there would be enough government bonds whose coupon dates would match directly the maturities of other bonds. While the volume of the government bond market is usually relatively big there are usually only limited number of bond issues with varying maturities. This technique alone is therefore not sufficient to construct the zero-coupon curve, but its principle can be used nonetheless. The bootstrapping method needs to be supplemented by interpolation techniques.

While bootstrapping the bonds, we thought of a coupon bond as a "package of a several zero-coupon bonds, that have different maturities and their face values equal the respective coupons of the coupon bond. In the above example the bootstrapping method was used to calculate the interest rates. It can be similarly used to calculate the discount factors. Once we have one of the following sets: zero rates, discount factors, forward rates, we can easily calculate the remaining two sets.

3.7. Interpolation Techniques

Interpolation techniques are used to derive an interest rate t_2 when we know the interest rate for time t_1 and time t_3, but do not know the interest rate for time t_2, where $t_1 < t_2 < t_3$. This is very simple but useful technique as there are hardly ever bonds with the exact maturity that an analyst needs. In principle interpolation techniques make discontinuous term structure acquired by bootstrapping continuous. This is an important technique because in practice we might not have a bond with maturity exactly equal to the swap payment date. In that case, an analyst usually interpolates between the bond price data.

Linear Interpolation

Any point of the zero-coupon yield curve can be obtained using the linear interpolation by solving following equation:

$$R_{(0,t)} = \frac{(T_2 - t)R_{(0,T_1)} + (t - T_1)R_{(0,T_2)}}{T_2 - T_1} \tag{3.11}$$

The interest rates $R_{(0,T_1)}$ and $R_{(0,T_2)}$ at times T_1 and T_2 are known and the interest rate $R_{(0,t)}$ at time t is unknown. Any interest rate interpolated from two other interest rates lies on a straight line connecting the two interest rates.

[13] Continuous compounding is generally used when pricing derivatives and is thus also used in this context. Continuous compounding is in many situations approximately equal to daily compounding.

Cubic Interpolation

Using cubic interpolation makes it possible to obtain a curve that is concave or convex. The curve might be even concave at one segment and convex at other one. Any point of the zero-coupon yield curve can be obtained using the cubic interpolation by solving following three-order polynomial equation:

$$R(0,t_d) = a \times t_d^3 + b \times t_d^2 + c \times t_d + d \tag{3.12}$$

t_d denotes the desired time to maturity of interpolated interest rate
$R(0,t_d)$ denotes the interpolated interest rate at time t_d

Coefficients *a, b, c* and *d* are obtained solving the following system of equations (3.13)

$$R_{(0,T_1)} = at_1{}^3 + bt_1{}^2 + ct_1 + d$$

$$R_{(0,T_2)} = at_2{}^3 + bt_2{}^2 + ct_2 + d$$

$$R_{(0,T_3)} = at_3{}^3 + bt_3{}^2 + ct_3 + d$$

$$R_{(0,T_4)} = at_4{}^3 + bt_4{}^2 + ct_4 + d$$

The system of the equations can be simply solved using matrixes (3.14)

$$\begin{pmatrix} R_{(0,T_1)} \\ R_{(0,T_2)} \\ R_{(0,T_3)} \\ R_{(0,T_4)} \end{pmatrix} = \begin{pmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_3^2 & t_3 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Using matrix algebra, we get (3.15)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_3^2 & t_3 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} R_{(0,T_1)} \\ R_{(0,T_2)} \\ R_{(0,T_3)} \\ R_{(0,T_4)} \end{pmatrix}$$

An example of both types of interpolation follows. The following interest rates are given:

Interest rate maturity (in years)	**1**	**2**	**3**	**4**
Interest rate	0.10%	0.40%	2.00%	2.30%

Table 3.2 Given interest rates

We can interpolate between the interest rates as follows:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0.001 \\ 0.004 \\ 0.020 \\ 0.023 \end{pmatrix}$$

Calculating this matrix, we get a = -0.00433, b = 0.03250, c = -0.06416, d = 0.03700. Using the equation 3.12 we can get any point on the curve that lies between 1 and 4 years of maturity.

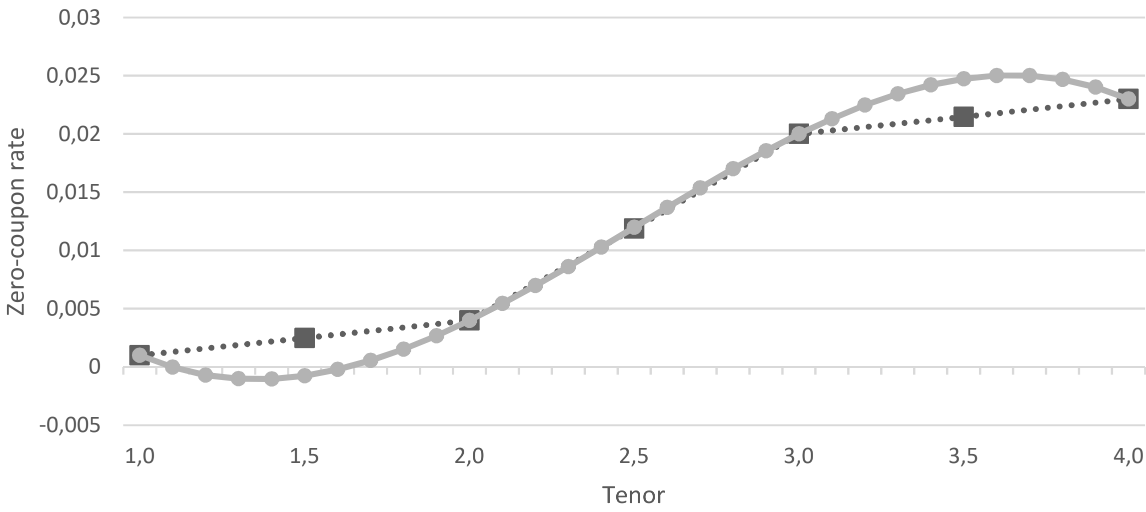

Figure 3.3 Comparison of linear and cubic interpolations

The dashed line in the picture represents the linear interpolation, with key maturities highlighted by a square. The solid line represents the cubic interpolation, with a few points highlighted, but is never straight between any two of the highlighted points. Cubic interpolation gives a smoother curve, as can be seen from the graph. It is however disputable if the results from the cubic interpolation are always better than those from the linear interpolation. In our example the cubic interpolated curve is negative around the 1.5 years of maturity which might be not the case in practice. In praxis, it is often sufficient to use the linear interpolation.

Other Interpolation Techniques

It is important to note that there is not one single approach that could be considered correct and the other false when it comes to interpolation and extrapolation. Each approach has advantages and disadvantages under certain conditions. There are just some market conventions and fundamental concepts that should be followed. The final zero-coupon curve should be smooth and lie on the observed market rates. On the other hand, the curve should not be over-smoothed as this might cause misinterpreting the observed data. There are numerous regression techniques that under certain conditions can generally fit any data points, however overfitting and misspecification is a problem as the derived curve might be smooth and look “visually pleasant” to an unskilled observer, but fatally misinterpret the reality.

The are many indirect methods, that fit the data to a formula describing the zero-coupon curve. The procedure can be generalized into three steps, developing an appropriate formula, fitting the parameters and verification. Popular method is using polynomial splines introduced by McCulloch

(1971), this is an advanced regression technique that uses polynomials to derive a smoother curve than the basic linear regression. Third order polynomial (cubic polynomial) has the following form:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_2 x_i^3 + \varepsilon_i \tag{3.16}$$

Terms can be added or removed to create a polynomial of greater or smaller order. Adding too many terms is however not advised as it leads to more complexity without no real interpretation or justification. Significance tests can be applied to find the right order of the polynomial. Other procedures for verifying the calculated curve and measuring its error are described in many econometrics textbooks, e.g. Greene (2002). Polynomials can be modeled using either standard spline model, more advanced B-spline basis or exponential splines introduced by Vasicek and Fong (1982).

Nelson and Siegel (1987) and more advanced Svensson (1994) are another advanced method used for modeling the term structure of interest rates. These models are widely used even by many central banks mostly to model future behavior of interest rates and model their characteristics. These models are quite complicated and fit the market data only loosely, causing over-smoothing and eliminating market pricing information. Therefore, these models are generally not appropriate for market pricing. When constructing the zero-coupon curve, the market convention has been to mainly use the linear interpolation or cubic splines.

3.8. Constructing the Zero-coupon Curve

Practitioners construct the zero-coupon curve from observed yields of several instruments and interpolate between them. When choosing the instruments several factors should be considered. One of them is risk free characteristics of the instrument, the less risk there is, the better. Other important factor is liquidity, illiquid instruments might be mispriced. Liquidity characteristics can also depend on the life of a bond. Newly issued instruments (so called on-the-run instruments) are generally more liquid than instruments that have been already on the market for some while (off-the-run instruments). High price volatility of an instrument and therefore also of its yield is also not a desirable characteristic. Different instruments are chosen for the construction of the short end, middle area and the long end of the curve. The method used for constructing the curve is also currency-specific. Practical guide on constructing the curve was published by the Bank of Canada (2000). The day count conventions used for market instruments usually also differ by currency.

Short End of the Zero-coupon Curve

Practitioners usually use overnight, one-month and three-month deposit rates (also known as cash rates) to construct the short end of the curve. As explained above, Libor rates and other money market deposit rates can be considered as good proxies for risk-free rates. On the other side, they are quoted only in few currencies and only in several maturities leading up to one year. These rates are in effect zero-coupon. Because various quoted market rates differ by convention, compounding frequency and payment frequency used (see table below), it is useful to convert these rates to continuously compounded rates. Following equation can be used to convert these rates:

$$r_c = \frac{n_y}{n_m} \times \ln\left[1 + \frac{r_d}{\frac{n_y}{n_m}}\right] \tag{3.17}$$

n_y is the number of days in a year according to a convention used (360 for USD deposit rates)
n_m is the number of days until maturity of the interest rate
r_d is the quoted deposit rate
r_c is the continuously compounded rate

Interest rate	Payment frequency	Compounding frequency	Day count convention
USD deposit rates	S/A	S/A	ACT/360
USD swap rates	S/A	S/A	30/360

Table 3.3 Compounding frequencies and day count conventions of USD rates

Source: Bloomberg

Middle Area of the Zero-coupon Curve

Eurodollar Futures or FRAs are used for the middle of the curve. They are relatively liquid and still a good proxy of forward rates. The convexity adjustment is applied to implied futures rates so that these rates are better proxies of the forward rates. Futures rates are usually used to extend the zero-coupon curve to 2 years and when the futures market is liquid, they could be used to extend the curve up to 5 years. Eurodollar futures contracts on CME are quoted only up to 10 years.

Implied futures forward rates are first adjusted by the convexity adjustment (equation 3.6). Afterwards the quarterly compounded rates are converted to continuously compounded rates (equation 3.17), where n_m equals the interest rate´s accrual period. The resulting continuously compounded forward rate is then converted to the continuously compounded zero (spot) rate with the following equation.

$$r_2 = \frac{r_f\,(T_2 - T_1) + r_1 T_1}{T_2} \tag{3.18}$$

r_f denotes the forward rate
T_2, T_1 denote the start and end of the forward rate accrual period
r_2, r_1 denote the spot interest rate in periods T_1 and T_2

Long End of the Zero-coupon Curve

For the long end, swap rates can be used as they are quoted up to 50 years from now. When the swap market is not sufficiently liquid at longer maturities, government bonds can also be used. The USD fixed-float interest rate swap rates are quoted as par rates with semiannual compounding.

That means that a swap rate quoted for *n* years to maturity is effectively a coupon rate on a hypothetical bond that pays a semiannual coupon and has a value of 100 (par). Bootstrapping method is used to derive zero-coupon interest rates from the swap rates quoted by the market.

4. INTEREST RATE SWAP VALUATION PROCEDURE

Plain vanilla interest rate swap, where one party exchanges floating rate cashflow for fixed rate cashflow is used in this chapter to illustrate the valuation procedure. The payments are assumed to be in the same currency. The valuation technique can be then easily generalized for other types of swaps. As mentioned above, the value of a swap is determined as the difference between the present value of fixed and floating cashflows. The swap value is always negative to one party and positive to the other one (unless the value is zero to both parties). Understanding the valuation method of an interest rate swap is useful for determining the swap rate that will make the initial value of an interest rate swap equal to zero. Once the interest rate swap is initiated the value of the swap can depart from zero, the same valuation method is then used to value the swap during its life.

To value an interest rate swap, both swap legs must be valued separately and there are several steps that lead to valuation of each respective leg. These steps are:

1) Constructing the zero-coupon curve, forward rate curve and calculating the discount factors
2) Valuing the floating leg´s cashflows
3) Valuing the fixed leg´s cashflow
4) Discounting all the cashflows

For illustration purposes, the following interest rate swap is assumed:

1) The notional amount is 100mn USD
2) Payments are on quarterly basis for both legs
3) Actual/360 convention is used[14]
4) Reference rate is 3-month LIBOR
5) The duration of the swap is two years.
6) The swap starts on 1.1.2018

4.1. Valuing the Floating Leg

As stated before, the first floating rate payment is usually known at the settlement date, all other payments depend on future development of the floating reference rate and are unknown at the initiation. The payments are usually known at the beginning of each period (at the reset date) but made at the end of each period (payments made in arrears). If on 1. 1.2018 the LIBOR was 2%, then the cashflow on 31. 3. is determined as follows:

$$Payment = Notional\ amount \times 3M\ LIBOR \times \frac{number\ of\ days\ in\ period}{360} \quad (4.1)$$

[14] Every year has 360 days. To compute the number of days in the interest period, the actual number of days is used.

That means that the buyer of the hypothetical swap will receive on 31.3.2018

$$500\,000\,USD = 100\,000\,000\,USD \times 2\% \times \frac{90}{360}$$

While the amount of the first payment was derived with certainty, it is not possible to do so with the other remaining payments. The next payment will be determined by a 3M LIBOR on 1.4.2018. An analyst´s task is to forecast this rate using forward rates. Forward rates can be derived from a zero-coupon curve. Construction of a zero-coupon curve is described in chapter 3. Assuming the forward rates are known, the payments at each period can be calculated as:

$$Payment = Notional\ amount \times forward\ rate \times \frac{number\ of\ days\ in\ period}{360} \tag{4.2}$$

To determine the present value of the floating leg, we need to discount each future cashflow with appropriate spot rate.

$$Present\ Value = \frac{Cash\ Flow\ at\ time\ T}{(1+spot\ rate\ for\ period\ T)^T} \tag{4.3}$$

When the cash flow is equal 1 USD, the present value of such a cashflow discounted by the spot rate is called the discount factor. The present value can be also calculated with the use of forward rates in the following way:

$$Present\ Value = \frac{Cash\ Flow}{(1+f_{(t_2-t_1)})\times(1+f_{(t_3-t_2)})\times(1+f_{(T-t_3)})\times\ldots} \tag{4.4}$$

, where $t_1 < t_2 < t_3 < T$.

If the cash flow is 1 USD, the present value of such a cashflow discounted only by the appropriate forward rate is called forward discount factor. The discount factor is thus determined with spot rates, whereas the forward discount factor is determined with the forward rates. Let us define another term, period forward rate. That is the rate that is applicable to the appropriate period, i.e. it is adjusted by the number of days in the respective period.

$$Period\ forward\ rate = forward\ rate\ (p.a.) \times \frac{number\ of\ days\ in\ period}{360} \tag{4.5}$$

The discount factors for successive periods are calculated as:

$$discount\ factor_t = \frac{Discount\ factor_{t-1}}{period\ forward\ rate_t} \tag{4.6}$$

In the given example:

$$1.\ period\ discount\ factor = \frac{1}{1+0.02} = 0.9803$$

$$2.\ period\ discount\ factor = \frac{1}{(1+0.02)\times(1+0.0196)} = 0.9615$$

Forward discount factors are used to calculate the present value of each cash flow. The following table illustrates the payments to be received from the floating leg.

Year	Start of Period	End of Period	Days in Period	Forward Rate	Period Forward Rate	Discount factor	Payment	PV of the Payment
0.25	1.1.2018	31.3.2018	90	2.00%	0.5000%	0.99502	500 000	497 512
0.5	1.4.2018	30.6.2018	91	1.96%	0.4954%	0.99012	495 444	490 549
0.75	1.7.2018	30.9.2018	92	1.97%	0.5034%	0.98516	503 444	495 973
1	1.10.2018	31.12.2018	92	1.98%	0.5060%	0.98020	506 000	495 981
1.25	1.1.2019	31.3.2019	90	1.99%	0.4975%	0.97535	497 500	485 235
1.5	1.4.2019	30.6.2019	91	2.00%	0.5056%	0.97044	505 556	490 612
1.75	1.7.2019	30.9.2019	92	2.01%	0.5137%	0.96548	513 667	495 936
2	1.10.2019	31.12.2019	92	2.02%	0.5162%	0.96052	516 222	495 844
						Sum of the present values:		3 947 643

Table 4.1 Floating leg cashflows from a hypothetical interest rate swap

When the present value of each payment is known, it is easy to sum all payments up and get the present value of the entire floating leg.

4.2. Valuing the Fixed Leg

The frequency of the fixed rate cashflows does not have to be the same as the frequency of the floating rate payments. The frequency is of course specified in the contract´s confirmation. For illustration purposes, it is assumed that the fixed rate cashflows are made quarterly. The cashflows are determined in the following way:

$$Payment = \ Notional\ amount \times swap\ rate\ \times \frac{number\ of\ days\ in\ period}{360} \qquad (4.7)$$

As mentioned before, the swap rate is constant during the entire life of the swap contract. Nevertheless, the cashflows are not necessarily the same because there are different number of days in each period. Information needed to calculate the fixed rate payments of the assumed interest rate swap is depicted in the following table. The swap rate is assumed to be 2%.

Year	Start of Period	End of Period	Days in Period	Payment	Discount factor	PV of the Payment
0.25	1.1.2018	31.3.2018	90	500 000	0.99502	497 512
0.5	1.4.2018	30.6.2018	91	505 556	0.99012	500 560
0.75	1.7.2018	30.9.2018	92	511 111	0.98516	503 526
1	1.10.2018	31.12.2018	92	511 111	0.98020	500 991
1.25	1.1.2019	31.3.2019	90	500 000	0.97535	487 674
1.5	1.4.2019	30.6.2019	91	505 556	0.97044	490 612
1.75	1.7.2019	30.9.2019	92	511 111	0.96548	493 469
2	1.10.2019	31.12.2019	92	511 111	0.96052	490 934
				Sum of the present values:		3 965 278

Table 4.2 Fixed leg cashflows from a hypothetical interest rate swap

The same discount factors are used to discount the fixed rate leg payments as are used to discount the floating leg payments.

4.3. Determining the Swap Rate

As mentioned before, the values of floating and fixed legs must be equal at the initiation of the swap, so that its value is zero. In such a scenario there is no exchange of payments at the swap´s initiation. This equivalence principle is the key principle used for calculating the swap rate.

The equivalence principle can be written as:

$$PV\ of\ floating\ rate\ payments\ = PV\ of\ fixed\ rate\ payments \tag{4.8}$$

More precisely:

$$N\ \times \sum_{j=1}^{m} V_{j-1}\left(\frac{T_j - T_{j-1}}{360}\right) F(t,T_j)\ = N\ \times \sum_{i=1}^{n} S\left(\frac{T_i - T_{i-1}}{360}\right) F(t,T_i) \tag{4.9}$$

N denotes the notional amount,
m denotes the number of floating leg´s cash flows, with $j = 1,\ldots..,m$
V_{j-1} denotes the floating rate (e.g. LIBOR) determined at T_{j-1} but made at T_j,
S denotes the fixed rate known as the swap rate
n denotes the number of fixed leg´s cashflows to be made at time T_i but determined at time T_{i-1} with $i = 1,\ldots., n$.
$T_j - T_{j-1}$ and $T_i - T_{i-1}$ denote the number of days between the two payments. Actual/360 convention is assumed.
$F(t,T_i)$ and $F(t,T_j)$ denote the respective forward discount factor

When solving for the swap rate, we get:

$$Swap\ rate\ = \frac{N \times \sum_{j=1}^{m} V_{j-1}\left(\frac{T_j - T_{j-1}}{360}\right) F(t,T_j)}{N \times \sum_{i=1}^{n}\left(\frac{T_i - T_{i-1}}{360}\right) F(t,T_i)} \tag{4.10}$$

When pricing the swap, V_{j-1} is replaced by the implied forward rate. In the above example the swap rate is equal to 2%. From the swap rate, one can determine the swap spread, i.e. the spread over a benchmark rate. A government bond with the same maturity as the swap is used by convention. If such a government bond yield is 1 % then the swap spread is 100bps. This swap spread can be viewed as a risk premium over a government bond.

This approach of determining the swap rate is purely theoretical, in practice a financial intermediary must account for market risk and possibly also incorporate some "marketing" approach for determining the swap rate that it will offer. The intermediary´s goal is to set the swap rate such that it is competitive to rates offered by other intermediaries on the market but it still makes a profit.

4.4. Valuing a Swap During its Life

In the previous text, it was shown that the value of a swap is zero at the initiation. During life of a swap, its price can however deviate considerably from zero and revaluing the swap is necessary.

Fixed rate bonds tend to have a longer duration than floating rate bonds. Fixed rate bonds are therefore more sensitive to the changes of interest rates. As a result, when interest rates rise the value of the fixed leg decreases more than the value of the floating leg. That means that the floating rate payer benefits from the decrease of interest rates during the life of the swap while the fixed rate payer´s position loses value. If the interest rates increase, the opposite applies. Valuing the interest rate swap is described in chapter 7 on an example of real life contract.

5. Risk Analysis and Hedging of an Interest Rate Swap

Several approaches can be used to describe the market risk of an interest rate swap. Interest rate swap entities might wish to know the market value of the swap under different scenarios. Due to regulatory and accounting standards, interest rate swaps are usually valued using “mark-to-market” approach. This represents the amount of money; one entity would need to make to cancel the swap and can be in general calculated as a difference between net present values of fixed and floating legs. Additionally, many entities need to report the value of the swap adjusted by its credit risk.

When an entity buys an interest rate swap as a fixed rate payer, it knows with certainty the exact payments it will make in the future, because these payments are made on the base of a fixed rate that is defined in the contract. However, even though the company knows at any time during the swap´s life the exact nominal amount of money it will have to pay, it does not know the net present value, i.e. the future cashflow, discounted by appropriate interest rate. The net present value of a swap is zero at the initiation, but if interest rates move unexpectedly and the company does not use any other hedging instruments, the net present value changes. So, there is still a risk of changes in the “discount rates”, i.e. changes of the term structure of interest rates. More generally, interest rate swap can be viewed as an insurance product against the risk of unfavorable development of interest rates. However, same as in classical insurance business, entering an interest rate swap pays off only when the future development is unfavorable.

5.1. Duration and Convexity Approach

Interest rate swaps can be used to hedge a portfolio of bonds. The value of a swap from the fixed rate payer´s perspective can be viewed as a bond with coupon *C* maturing in time T_m and notional amount *N* minus the notional amount *N*. This can be expressed as

$$Swap = N \times \left(\sum_{j=1}^{m} C\left(\frac{T_j - T_{j-1}}{360}\right) \times F(t, T_j) + F(t, T_m)\right) - N \qquad (5.1)$$

As mentioned earlier, the advantage of using an interest rate swap for hedging interest rate risk is that using the interest rate swap is cheaper than hedging with bonds. Both approaches have the same duration, i.e. sensitivity to changes in interest rates. A hypothetical portfolio *P* of bonds can be hedged with interest rate swaps using duration and convexity.

Duration

Duration measures the price sensitivity of a bond or a portfolio to changes in interest rates. In general, a present value of a portfolio *P* can be expressed as a sum of future cashflows CF_i discounted by y_t:

$$P = \sum_{i=1}^{n} \frac{CF_i}{(1+y_t)^{t_i - t}} \qquad (5.2)$$

where y_t is the yield to maturity, which can be considered as the only source of risk. Duration attempts to measure the change in value of *P* that is caused by small changes in yield *y*. To quantify this, Taylor series expansion is used to approximate the change of *P*. Taylor series expansion can be written as:

$$f(a+h) = f(a) + f'(a)dh + \frac{f''(a)dh^2}{2!} + \cdots + \frac{f^{(n)}(a)dh^n}{n!} \tag{5.3}$$

$$df(h) = f(a+h) - f(a) = f'(a)dh + \frac{f''(a)dh^2}{2!} + \cdots + \frac{f^{(n)}(a)dh^n}{n!}$$

which, given a yield *i*, can be rewritten as

$$dP(i) = P'(i)di + \frac{P''(i)di^2}{2!} + \cdots + \frac{P^{(n)}(i)di^n}{n!} \tag{5.4}$$

Taylor series expansion consists of *n* polynomials, where each succeeding polynomial adds less approximative value than the preceding one. In finance, using only the first or first and second polynomials is generally considered to be a sufficient approximation and the other terms are considered negligible. When using only the first order approximation, we can derive the dollar duration:

$$dP(i) = P'(i)di \cong \$Dur(di) \tag{5.5}$$

Dollar duration is sensitivity to a change in yield, and can be expressed as first partial derivation of the bond´s price with respect to yield. *$Dur* is always negative, because of the inverse relationship between price change and interest rate change. When determining the dollar duration of a swap, simple relationship holds true:

$$\$Dur \textit{ of a swap} = \$Dur \textit{ of the fixed rate leg} - \$Dur \textit{ of the floating rate leg} \tag{5.6}$$

Each leg´s duration can be derived in a similar way as normal bond´s duration.

Convexity

The fact that the relationship between interest rates and the bond prices is not linear is called convexity. Second order Taylor expansion is used to include also convexity, which gives a better approximation of price change with respect to yield. Dollar convexity is second order derivation of price with respect to yield.

$$dP(i) = P'(i)di + \frac{P''(i)di^2}{2} = \$dur(di) + \frac{\$Conv(di)}{2} \tag{5.7}$$

All bonds that have no embedded options have positive convexity. This means that given a yield increase, their price decline is smaller than the price rise if the yield falls by the same amount. Long term bonds tend to exhibit higher convexity than short term bonds. Each bond exhibits a different convexity characteristics and different impact on the shape of the yield curve. Duration can be used to hedge against small changes in interest rates, but is not effective when hedging against big changes in interest rates. This is because the relationship between price of a bond and its yield is not linear. Consequently, when an interest rate changes, the duration changes as well. When determining the dollar convexity of a swap, simple relationship holds true:

$$\$\text{Conv of a swap} = \$\text{Conv of the fixed rate leg} - \$\text{Conv of the floating rate leg} \tag{5.8}$$

Each leg´s convexity can be derived in a similar way as normal bond´s convexity. The following figure shows he relationship between price and yield of two bonds who have the same price and duration but different convexity. Bond B has higher convexity than bond A. When interest rates change bond B´s price changes by a bigger amount than that of a bond A.

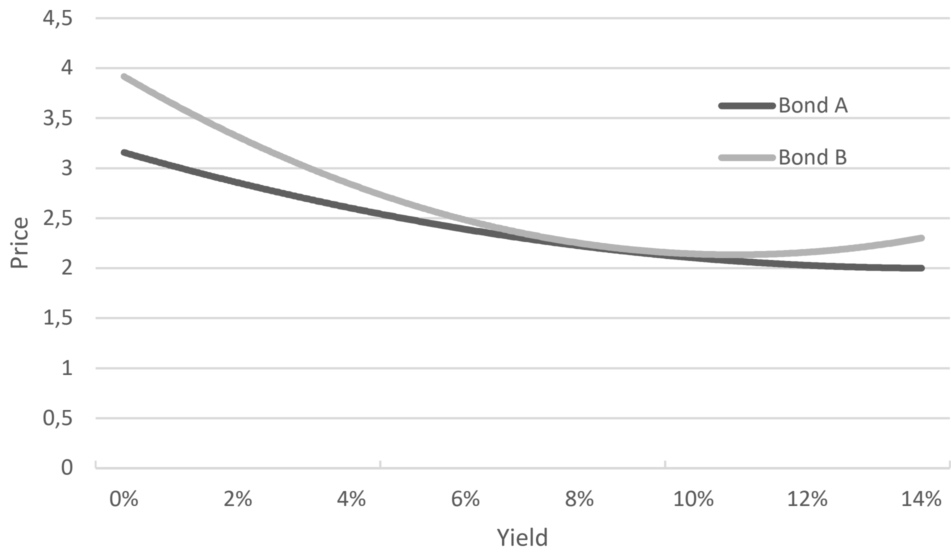

Figure 5.1 Bonds with the same duration but different convexity

Duration Hedging

The idea used for hedging is that a duration of a bond portfolio *P* and a duration of an interest rate swap *S* used for hedging should equal. This is expressed in equation:

$$\Delta = \frac{Dur_p}{Dur_S} \tag{5.9}$$

Δ denotes the number of interest rate swaps that hedge the portfolio *P*. This number is also known as the hedge ratio. Dur_p and Dur_s denote the duration of the portfolio and of the swap. When the structure of interest rates is flat, the duration hedge proves to be a good tool in hedging against small change in interest rates. When the changes in interest rates are non-parallel or relatively bigger the duration as well as the hedge ratio also change. The solution would be the so called dynamic hedging, that means restructuring portfolio accordingly to changes of the hedge ratio. This approach is however not feasible, as the restructuring of the portfolio would need to be theoretically continuous and thus very costly. The approach would also not be proper when there are big price jumps. When the change of interest rates is expected to be bigger, it is better to use duration and convexity hedge.

Duration and Convexity Hedging

Convexity can be expressed as second partial derivation of the bond´s price with respect to interest rates. It uses the second-order Taylor expansion. Convexity is a hedge against changes in slope of interest rate structure. When hedging duration as well as convexity, two assets need to be used, so that:

$$\begin{gathered} \Delta_1 Dur_{S_1} + \Delta_2 Dur_{S_2} = -Dur_P \\ \Delta_1 Conv_{S_1} + \Delta_2 Conv_{S_2} = -Conv_P \end{gathered} \tag{5.10}$$

Where Δ_1 and Δ_2 denote the amounts of the two interest rate swaps that are needed to hedge the portfolio *P*. Dur_{s1}, Dur_{s2} and Dur_P denote the durations of the two swaps and the bond portfolio. $Conv_{S1}$, $Conv_{S2}$ and $Conv_P$ denote the convexity of the swaps and the bond portfolio. The two equations can be expressed as matrixes:

$$\begin{pmatrix} Dur_{S1} & Dur_{S2} \\ Conv_{S1} & Conv_{S2} \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} Dur_P \\ Conv_P \end{pmatrix} \tag{5.11}$$

which can be solved as

$$\begin{pmatrix} Dur_{S1} & Dur_{S2} \\ Conv_{S1} & Conv_{S2} \end{pmatrix}^{-1} \begin{pmatrix} Dur_P \\ Conv_P \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \tag{5.12}$$

Martellini (2003) proposes a three-factor hedge as even more precise hedging method. The idea of this method, is to make the portfolio immune against changes in the level, slope and curvature of the interest rates. Principal component analysis shows that these three factors can explain most of the changes in the term structure. Three-facto hedge uses three interest rate swaps to hedge against each of these three factors.

5.2. Other Approaches

There are other approaches commonly used to describe the risk of a change of net present value of an interest rate swap. It is important to realize that interest rate swaps are instruments that can be used to hedge only against the risk of uncertain future cashflows that are tied to an uncertain future reference rate. A buyer of a swap knows at the time of initiation the exact nominal value of all the fixed payments that he will have to make during the life of the swap. However, because he does not know with certainty the discount factors (the buyer only estimates them from the current term structure), he also does not know the exact real present value of all his fixed payments. This means that, if interest rates unexpectedly rise, the discount factors decrease and the present value of future fixed payments will be smaller. The buyer of the swap also does not know the value of the floating rate payments that he is going to receive. Practitioners thus developed a few methods how to evaluate the risk of unexpected interest rate changes. The word "unexpected" is important here. When constructing the zero-coupon curve an analyst makes a prediction of where the rates will be in the future. If the forward rate curve is upward sloping, it means that an analyst makes a prediction that the interest rates will be higher in the future. If this prediction comes true, this information is already priced in. However, if the prediction does not come true and the interest rates change by a smaller or bigger amount than initially envisaged, the present value of an interest rate swap deviates from zero.

Scenario Analysis

One very simple, easily interpretable and powerful approach is scenario analysis. An analyst might propose various scenarios how the interest rates might change. For example, an analyst might add or subtract 50bps to each point on the zero-coupon curve and then calculate again the net present value of the swap. Using scenario analysis, analysts can simulate any change of the shape of the curve. PCA analysis is very helpful here, as it can advise on the ratio of changes between interest rates that vary in maturity. It might for example advise that if one-year interest rate changes by one basis point than the two-year rate changes by x basis points. PCA analysis is explained in more detail in chapter 7.

DV01 and PV01

DV01 or dollar value of 01 basis point expresses how a value of an interest rate sensitive security changes if there is one basis point parallel shift in underlying interest rate. To calculate the DV01, one must first calculate the interest rate swap´s value using the observed zero-coupon curve and then calculate the swaps value again using the same curve shifted by one basis point. The difference is the value of DV01. This risk measure is commonly used also with other interest rate derivatives or fixed income instruments, even though an assumption of parallel shift is rather unrealistic. Another disadvantage is that DV01 is not constant due to convexity. When interest rates rise DV01 value falls. PV01 is the present value of a change of the coupon rate by one basis point and its calculation is like that of DV01, but it shifts only the instrument´s coupon rate (the swaps fixed rate).

Credit Risk Analysis and CVA

Credit risk is inherent in every interest rate swap transaction. It is the risk that a counterparty defaults. It is of outmost interest to the buyer of the swap to closely follow the credit rating of its counterparties and enter a transaction only with highly rated intermediary. Effective and common way of diversifying the credit risk is entering a several interest rate swap transactions with several counterparties. The swap agreements might also contain downgrade protection and other financial covenants that protect one entity from the default of the other one. For example, if a counterparty does not have a rating within a certain range or above a certain threshold, it might be obliged to post collateral.

CVA (credit value adjustment) is a measure that expresses the risk of counterparty´s default. CVA became important after the financial crises with introduction of stricter regulatory requirements, especially the Basel III and new accounting standards. The standard introduced by Basel Committee on Banking Supervision in Basel III was further advanced when BIS (2015) issued Review of the Credit

Valuation Adjustment Risk Framework. Under these new regulations, entities involved in derivatives trading must in many situations account for the counterparty risk, which can be expressed by the CVA. This measure can be defined as the difference between the value of a portfolio without any counterparty risk and a value of a portfolio adjusted by the credit risk. There are relatively extensive methodologies of how to calculate the CVA, a simplification of the calculation can be expressed in the following way:

$$CVA = \sum_{t=1}^{T} LGD_t \times PD_t \times DF_t \times EE_t \tag{5.13}$$

LGD_t denotes the loss given default, which is also calculated as *1 – recovery rate*
PD_t denotes the probability of default at time t, under risk neutral measure
DF_t denotes the discount factor at time t
EE_t denotes the expected exposure or the value of an interest rate swap, under the assumption of risk free environment

Recovery rate expresses the percentage of the derivative´s value an institution would be able to recover from the contract if it´s counterparty defaulted, the value ranges from zero to one.
When calculating the credit risk, an analyst must consider that the price of the swap is generally zero at the initiation but deviates considerably during the swap´s life and becomes zero again after the termination. There are no analytical methods to calculating the credit risk, institutions calculate the CVA through Monte Carlo method simulating various scenarios. These institutions use either a standard approach, where some parameters are strictly given by the respective authorities or they can use the internal model approach, where they estimate some of the parameters themselves. Perhaps the most demanding is the estimation of probability of default and loss given default.

Institutions developed also other valuation adjustments commonly known as XVA´s, for example debit valuation adjustment (DVA), funding valuation adjustment (FVA) or liquidity valuation adjustment (LVA). Reader interested in these adjustments is referred to the Bank for International Settlements and the International Valuations Standards Council, an institution that prepares documentation describing various valuation methods.

Basis Risk

Basis risk is a risk that the received floating reference rate will not fully offset the floating rate paid on a liability. This can be caused by different day conventions, tax reasons, receiving the floating rate payments from the swap on different day than the payments on a liability are due or by paying a different rate on a liability than is received from an interest rate swap. This risk is considered significant in the case of NY State Swaps described in chapter 7, as the reference rate is only 65% of USD Libor.

6. ANALYSIS OF THE TERM STRUCTURE OF INTEREST RATES

Analyzing the term structure is necessary to get a clear picture of how interest rates have developed in the past, where they can go in the future and how they interact with each other. Simple descriptive and graphical analysis of interest rates is a powerful tool that can be further supplemented by more sophisticated principal component analysis.

6.1. Descriptive Analysis of the Swap Rates

The following graph shows the development of the EUR Swap curves (fix-float) since 2010. Its apparent, that the swap rates are highly correlated. Other noticeable fact is, that long term swap rates are more volatile, this is especially noticeable, when the general interest rates in the economy decline. In such a time, the long-term swap rates fall more than the short-term swap rates.

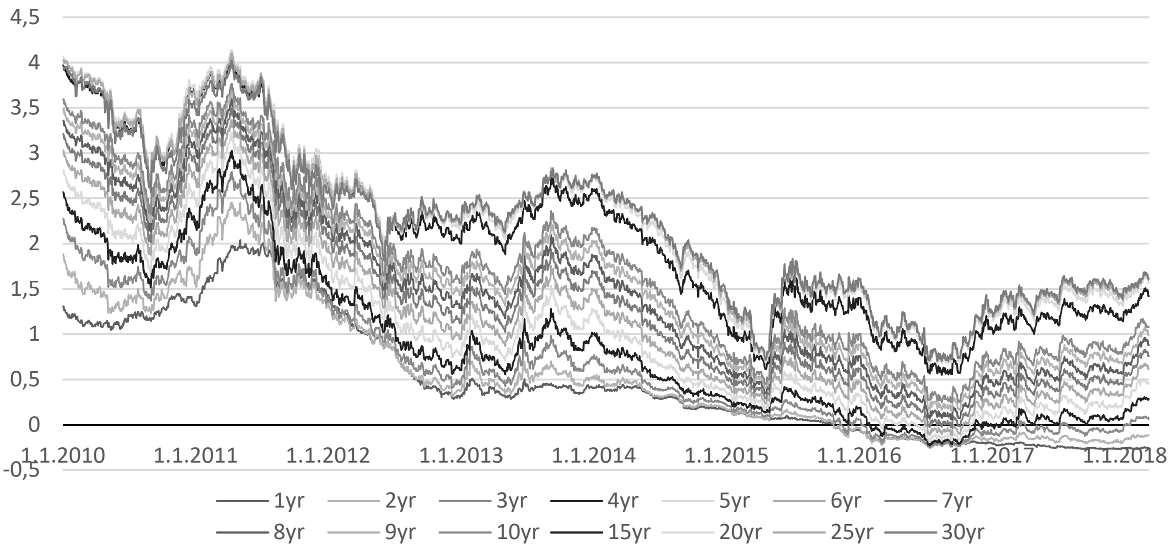

Figure 6.1 Historical development of EUR swap rates with maturity of one to thirty years

Source: Bloomberg

It is not surprising that lowest interest rate within the observed period was the one with shortest maturity.

	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr	15yr	20yr	25yr	30yr
Variance	**0.44**	0.52	0.62	0.71	0.79	0.85	0.89	0.90	**0.91**	**0.91**	0.90	0.86	0.80	0.73
Minimum	**-0.27**	-0.25	-0.25	-0.23	-0.18	-0.11	-0.03	0.06	0.16	0.25	0.55	0.68	0.70	0.70
Maximum	2.04	2.45	2.79	3.03	3.23	3.37	3.50	3.60	3.70	3.77	4.07	**4.14**	4.11	4.01
Mean	**0.49**	0.57	0.71	0.86	1.03	1.18	1.33	1.47	1.59	1.70	2.06	2.19	**2.21**	2.19

Table 6.1 Descriptive statistic of USD swap rates

Source: Bloomberg

6.2. Principal Component Analysis (PCA)

Principal component analysis is a method that can be used to analyze the risk of changes in the term structure of interest rates. The goal is to identify factors that influence the term structure. There are many risk factors that influence the shape of the entire interest rate term structure. Each interest rate on the term structure can be viewed as one risk factor, i.e. a reader can imagine that there is a certain risk that only n-year spot interest rate will change (and all the other interest rates will stay the same). A risk manager then wants to quantify this risk. When considering a discrete term structure, there are at least as many risk factors as there are interest rates. In practice, it is inconceivable that only one interest rate would move and the others would remain the same. It is much more likely that there is a move in the whole term structure, i.e. all the interest rates rise or decline (because interest rates are highly correlated). The general rise of all interest rates can be then also viewed as a risk factor. There are many factors that influence the shape of the structure of interest rates, the idea of PCA is to explain the behavior of interest rate structure with fewer variables without losing much information. According to Levander (2010), Dauwe (2009) and other studies, the PCA finds that most of the risk of change in interest rates is caused by only two or three factors often interpreted as a change of level, slope and curvature. The analysis then suggests that a multi-factor approach should be used when hedging an interest rate risk. This means that only Duration using the first-order Taylor expansion is insufficient in most cases and the investor should also hedge against changes in slope and possibly curvature.

Data on the term structure of interest rates can be structured in a matrix, where each column represents a time series of one n^{th}-year interest rate and each row represents a specific term structure of interest rates on an observed date. In our context, each swap rate (matrix´s column) represents one dimension. PCA is a method of highlighting certain patterns in the data of high dimensions. Two or three-dimensional data can be easily analyzed using graphical analyses, this is not possible with multidimensional data. PCA then reduces the number of dimensions, without losing too much information value. PCA is also regarded as a dimension-reduction tool. It transforms multi-dimensional data of highly correlated variables into a dimensionally reduced data of uncorrelated variables also known as principal components. The principal components can be then used to model a development of interest rates (for example zero coupon rates).

Explanation of the Method

PCA transforms a set of variables into a smaller set of orthogonal variables. The steps of the procedure are explained in the following text. A following made-up two-dimensional dataset is considered in the explanation of the procedure. The dataset is only two-dimensional so that graphical visualization of certain steps is also possible.

One-year rate	Two-year rate
0.5	0.7
0.8	0.9
0.7	0.8
1.0	1.1
1.1	1.2
1.2	1.3
1.3	1.4
1.4	1.6
1.6	1.7

Table 6.2 PCA input interest rates

1) Firstly, a matrix is constructed, where each column represents a time series of n^{th}-year interest rate and each row represents a specific term structure of interest rates observed on a certain date.
2) The data are transformed to be stationary. It is possible to do this by simply taking the first difference. Some authors transform the data as a logarithmic return over one day, this approach is especially used when working with stock prices that are more volatile. Logarithmic return is then defined as:

$$X_n = LogS_n - \ logS_{n-1} \tag{6.1}$$

   Interest rates are quite often taken only as absolute differences. It can be empirically shown that the difference in results from these two approaches is negligible.
3) Data are further transformed by subtracting the mean of each column (data dimension) from each value. The mean of each column (and the whole dataset) is afterwards zero.
4) Calculating the covariance matrix (some approaches also use correlation matrix).
5) The eigenvectors and eigenvalues[15] of the covariance matrix are calculated (most software will give unit eigenvectors). Each eigenvalue corresponds to one eigenvector (and to infinite amount of its multiples). The eigenvector with the highest eigenvalue explains most of the variance in the data. For our data, we get:

   Eigenvalues: 0.2451543727, 0.0009567384

   And corresponding eigenvectors:

   Vector 1: (-0.7127149 ; -0.7014538)
   Vector 2: (-0.7014538 ; -0.7127149)

6) The eigenvectors are then sorted by their eigenvalues in descending order (this is usually done automatically by a software.) Now the components are sorted by their relevance. The result is shown below. The points represent the transformed data. The lines represent the vectors. The dashed vector is the one with a higher eigenvalue. It is obvious that the dashed

[15] The idea behind the method is, that a square matrix C can be represented by its eigenvector and eigenvalues, when the following equation is satisfied $C * v = \lambda * v$, where v is an eigenvector and λ is an eigenvalue of the matrix C.

vector explains most of the variability of the data. The most visible pattern in the data is the one in the direction of the dashed vector (-0.7127 , -0.7014). The numbers indicate that when the observed variable changes by -0.7127 on axis x, then it also changes by -0.7014 on axis y. The vectors can be of course arbitrarily scaled without changing their direction. It is also visible that the vectors are perpendicular to each other and therefore also orthogonal.

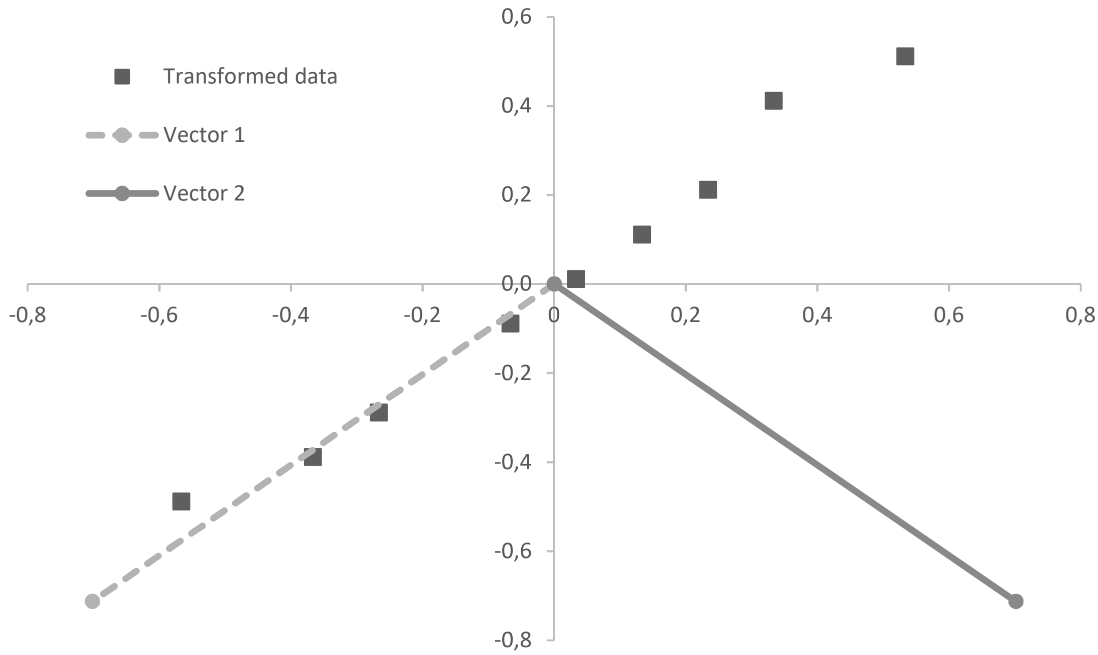

Figure 6.2 Dataset with highlighted eigenvectors

7) When the eigenvectors are sorted, it is possible to leave out irrelevant factors. The irrelevant factors are those that have the lowest eigenvalue, but the informational value also decreases. So, it is possible to leave out only so many factors that the sum of their eigenvalues is larger than x percent of the sum of all original components.

The total variance is defined as

$$\sum_{k=1}^{n} \lambda_k \tag{6.2}$$

where, λ denotes the eigenvalues.

$$p \rightarrow \frac{\sum_{k=1}^{i} \lambda_k}{\sum_{k=1}^{n} \lambda_k} \tag{6.3}$$

where p denotes the percentage of variance explained by the first i components. In the performed example, the solid vector could be left out as it explains less of the variability in the data.

6.3. PCA Performed on EUR Swap Rates

The steps of the procedure are described in previous chapter. The procedure was performed on EUR swap rates.

EUR Swap Rates

The dataset involves data on EUR interest rate swaps, with maturity in 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25 and 30 years. The data were observed from 1.1.2010 to 7.3.2018.

1	2	3	4	5	6	7	8	9
0.867	0.938	0.956	0.969	0.976	0.983	0.986	0.99	0.992

Table 6.3 Cumulative sum of the first nine principal components variances.

Source: Bloomberg, own calculations

The first three components explain more than 95% of the variance in the data. The change of level is accountable for almost 87% percent of variance in the data. Following figure shows the first three principal components (PC1, PC2, PC3).

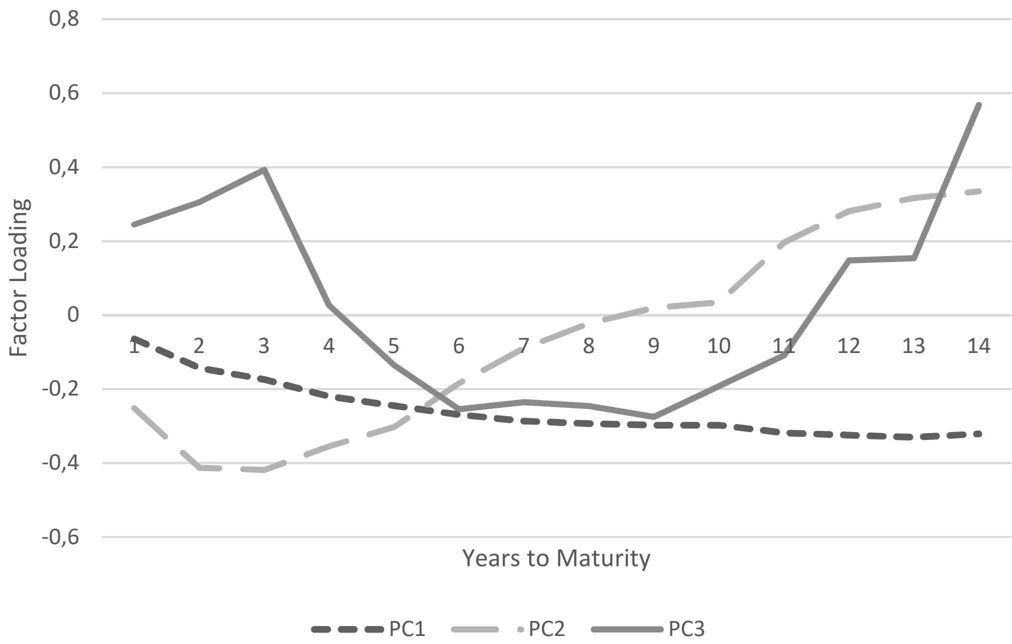

Figure 6.3 Factor loadings

The 14 Principal components are shown in the following table. PC1 (first principal component) represents a roughly parallel shift in interest rates, PC2 represents a change in slope and the PC3 represents a change in curvature. It follows from the table that if a 1-year interest rates changes by a 0.06 basis points than a 10-year interest rate changes by 0.30 basis points in the same direction. That means that a longer maturity interest rates have bigger variance, which is also reflected by the descriptive statistics. The second principal factor represents the change of the slope or steepness of the curve. The interest rates with maturity of up to eight years move in one direction while the interest rates with longer maturities move in opposite direction. The third principal factor represents a change in curvature, first four interest rates move in one direction, the other seven interest rates in opposite direction and the last three interest rates move in the same direction as the first four. The individual values are called factor loadings.

	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9	PC10	PC11	PC12	PC13	PC14
1yr	-0.06	-0.25	0.24	-0.11	-0.06	0.43	-0.32	0.18	0.20	0.61	-0.34	-0.08	-0.01	-0.01
2yr	-0.14	-0.41	0.31	-0.20	0.00	0.37	-0.17	0.22	-0.07	-0.46	0.48	0.07	0.08	0.00
3yr	-0.17	-0.42	0.39	0.09	-0.31	-0.13	0.58	-0.28	0.07	0.01	-0.19	0.24	-0.04	0.01
4yr	-0.22	-0.35	0.03	-0.15	0.38	-0.33	-0.32	-0.53	0.15	-0.07	-0.05	-0.36	-0.02	-0.02
5yr	-0.24	-0.30	-0.13	0.47	0.51	-0.07	0.16	0.33	-0.29	0.15	-0.04	0.06	0.30	-0.03
6yr	-0.27	-0.18	-0.25	-0.19	-0.10	-0.26	0.19	0.48	0.15	0.05	0.09	-0.28	-0.57	-0.11
7yr	-0.29	-0.09	-0.23	-0.15	-0.20	-0.06	-0.24	-0.02	-0.40	-0.05	-0.23	0.23	-0.10	0.67
8yr	-0.29	-0.02	-0.25	-0.12	-0.24	0.00	-0.22	-0.10	-0.28	-0.05	-0.18	0.29	0.07	-0.72
9yr	-0.30	0.02	-0.28	-0.16	-0.32	-0.10	0.09	0.06	0.37	0.11	0.20	-0.14	0.68	0.12
10yr	-0.30	0.03	-0.19	0.68	-0.24	0.37	-0.12	-0.22	0.20	-0.15	0.07	-0.19	-0.23	0.04
15yr	-0.32	0.20	-0.11	-0.15	0.40	0.14	0.08	-0.14	0.38	0.16	0.22	0.60	-0.21	0.06
20yr	-0.32	0.28	0.15	-0.18	0.03	0.22	0.28	-0.24	-0.49	0.33	0.31	-0.35	-0.04	-0.02
25yr	-0.33	0.32	0.15	-0.17	0.22	0.22	0.19	0.16	0.13	-0.45	-0.56	-0.19	0.10	0.00
30yr	-0.32	0.33	0.57	0.23	-0.12	-0.46	-0.34	0.21	0.02	0.06	0.11	0.09	0.00	-0.01

Table 6.4 Principal component factors

Source: Bloomberg, calculations done in R

According to Levander (2010), the explanatory power of the PCA analysis decreases when there are more short-term maturity interest rates in the dataset. The explanatory power increases when there is a higher data frequency.

7. PRICING NEW YORK STATE SWAP

The procedure of valuing an existing swap agreement is illustrated in this chapter on a real-life example of an interest rate exchange agreement initiated by the New York State (NY swap hereafter). The swap confirmation and other relevant information is made public on the website of the Division of Budget of the New York State. The buyer of the swap is New York State. The counterparty is Citibank. Purpose of the swap is to hedge against interest rate risk arising from an issuance of bonds by the New York State. All necessary information needed for pricing the NY swap is summarized in the following table.

Notional Amount	36 473 530 USD
Trade Date	27.3.2003
Effective Date	10.4.2003
Termination Date	1.1.2025
Party B – Fixed Rate Payer	
Fixed Rate Payer Payment Dates	Semiannually on each January 15 and July 15, commencing on July 15, 2003 and terminating on the Termination Date, subject to adjustment in accordance with the Following Business Day Convention.
Fixed Rate Payer Period End Dates	Semiannually on each January 15 and July 15, commencing on July 15, 2003 and terminating on the Termination Date. No Adjustment shall apply to Period End Dates.
Fixed Rate	3.36%
Fixed Rate Day Count Fraction	30/360
Party A – Floating Rate Payer	
Floating Rate Payer Payment Dates:	Monthly on the fifteenth of each calendar month, commencing on May 15, 2003 and terminating on the Termination Date, subject to adjustment in accordance with the Following Business Day Convention.
Floating Rate Payer Period End Dates:	Monthly on the fifteenth (15th) of each calendar month, commencing on May 15, 2003 and terminating on the Termination Date. No Adjustment shall apply to Period End Dates.
Floating Rate Option:	65% of USD-LIBOR-BBA
Designated Maturity:	One month
Floating Rate Day Count Fraction:	Actual/360

Floating Rate Reset Dates:	The Effective Date and each Wednesday thereafter. Notwithstanding the definition of USD-LIBOR-BBA set forth in the Definitions (for this purpose only), the applicable USD LIBOR-BBA for each Reset Date shall be the rate published one (1) London Business Day prior to such Reset Date.
Floating Rate Method of Averaging:	Weighted
Business Days:	New York

Table 7.1 NY State swap confirmation

Source: Budget Division of the New York State

The swap agreement contains following notional reduction amounts.

Date	Notional Reductions (USD)
15.1.2011	-
15.1.2012	-
15.1.2013	-
15.1.2014	246 195
15.1.2015	-
15.1.2016	-
15.1.2017	1 276 574
15.1.2018	2 033 398
15.1.2019	866 246
15.1.2020	939 193
15.1.2021	957 429
15.1.2022	984 786
15.1.2023	4 714 204
15.1.2024	12 975 460
1.1.2025	11 480 045

Table 7.2 NY State swap notional reduction amounts

Source: Budget Division of the New York State

General Notes

One of the first things that needs to be determined is the payment calendar. The confirmation specifies that the Following Business Day Convention is used for the payment dates and the business days are determined by the New York calendar. As mentioned before, the payment dates are the dates, when the cashflow is physically exchanged, it does not bear any relation to counting the nominal value of the payments to be made or received. The amount of the future payments to be made is affected by the period end dates, that mark the end of the accrual period and thus the number of days on which the interest accrues. Payment dates can be used to calculate the present value of the future payments, as it marks the date until which the cash can theoretically accrue interest on the bank account of the swap´s buyer. To summarize, the period end dates are used for calculating the future amount of the fixed payments to be made and of the floating payments to be received, the payment dates are used for calculating the present value of the future payments.

The payments happen semiannually on each January 15 and July 15. The last payment date is on the termination date, i.e. 1.1.2025. The following business day convention is applicable here. That means, that when 15 January or 15 July falls on a weekend or holiday, the payment is made the next business date. This convention is quite often used for apparent practical reasons, payment systems do not generally work on non-business days.

The effective date was on 10th April 2003. The date when the data was collected and the swap was valued for the purposes of this thesis was 30th April 2018. The nominal amount of the swap on 27 March 2003 was 36 473 530 USD but at the date of valuation the amount was 32 917 363 USD.

Constructing the Zero-coupon Curve and Estimating the Future Libor Rates

Firstly, it is important to note, that there are number of ways how the zero-coupon curve can be constructed. The zero-coupon curve that needs to be constructed for pricing the NY swap extends to 1.1.2025. There is no exact rule for choosing the instruments, that will be used for constructing the zero-coupon curve. As mentioned above, the zero-coupon curve consists of spot rates, that can be further used to determine the discount factors and forward rates. Libor and Swap rates can be almost (after being transformed to same compounding frequency) directly included in the yield curve as both these sets of rates are quoted as spot rates. On the other hand, futures rates are quoted as forward rates and need to be transformed to spot rates.

Short End of the Curve

USD Libor rates are used to construct the short end of the curve. These are inherently zero-coupon rates, they only need to be transformed so that their compounding frequency matches that of the other rates used for constructing the curve. When pricing derivatives, it is common to work with continuously compounded interest rates, all the LIBOR rates were thus transformed to continuous compounding. The transformation is done vie equation (3.17), where $t_y = 360$. The Libor market rates used to construct the short end of the zero-coupon curve are displayed in the following table. Interest rates quoted by the market (r_d), days until maturity and the continuous interest rates (r_c) are shown. The rates are rounded up to three decimal places.

Instrument	Rate (rd)	Maturity	Tenor (days)	Rate (rc)
ON Libor	1.704%	1.5.2018	1	1.704%
1M Libor	1.907%	30.5.2018	30	1.905%
2M Libor	2.061%	29.6.2018	60	2.058%

Table 7.3 LIBOR rates observed on 30.4.2018

Source: Bloomberg, own calculations

Middle of the Curve

The middle of the curve is constructed using the Eurodollar futures contracts quoted on CME. Implied futures rate is adjusted by the convexity adjustment (equation 3.6) to get the forward rate. Annualized standard deviation required for calculating the convexity adjustment is 0.719%, which was reported by Bloomberg on 30.4.2018. The method behind estimating the convexity adjustment is described for example by Bank of Canada (2000). The convexity adjustment does not play an important role for shorter term rates (only 0.1 basis points for October 2018 contract). For longer dated contracts the convexity adjustment gains on importance (1 basis point for March 2020 contract), but is nevertheless relatively small. If we worked with longer maturities the convexity adjustment would gain on importance. The forward rate is further transformed to continuously compounded forward rate using equation (3.17), where $t_y = 360$ for all rates. Finally, the continuously compounded forward rate was transformed to continuously compounded spot rate using equation (3.18). Linear interpolation method was used to get interest rates for the required points in time. For example, the June 2018 contract implies a forward rate that is applicable from 18.6.2018 to 18.9.2018. To derive a spot rate applicable from 30.4.2018 until 18.9.2018, an interest rate applicable for the period from 30.4.2018 until 18.6.2018 is required. This interest rate can be easily linearly interpolated using equation (3.11) from the short end of the curve, other interpolation methods could be also used. Linear Interpolation was used numerous times during the valuation process to ensure that individual forward interest rates link to each other.

Contract	Settlement	Maturity	Market Price	Convexity Adjustment	Forward Rate	Continuous Forward Rate	Continuous Spot Rate
June 2018	18.6.2018	18.9.2018	97.685	0.00000	2.3149%	2.3082%	2.1884%
July 2018	16.7.2018	16.10.2018	97.615	0.00000	2.3847%	2.3777%	2.2744%
October 2018	15.10.2018	15.1.2019	97.420	0.00001	2.5792%	2.5709%	2.3596%
December 2018	17.12.2018	17.3.2019	97.360	0.00001	2.6386%	2.6299%	2.4354%
March 2019	18.3.2019	18.6.2019	97.250	0.00003	2.7474%	2.7380%	2.4894%
June 2019	17.6.2019	17.9.2019	97.150	0.00004	2.8460%	2.8359%	2.5413%
September 2019	16.9.2019	16.12.2019	97.080	0.00006	2.9142%	2.9036%	2.5918%
December 2019	16.12.2019	16.3.2020	97.015	0.00008	2.9771%	2.9661%	2.6371%
March 2020	16.3.2020	16.6.2020	96.990	0.00010	2.9996%	2.9885%	2.6710%

Table 7.4 Continuous spot rates implied from CME Eurodollar Futures

Source: Bloomberg, own calculations

Long End of the Curve

Long end of the zero-coupon curve is constructed from swap rates observed at the market. As mentioned above, US swap rates are quoted as par rates with semiannual compounding. That means that they need to be transformed to continuously compounded spot rates. To do that, the bootstrapping principle is applied, where the coupon is equal to the swap rate, the hypothetical bond has a semiannual coupon frequency and is priced at par. Linear interpolation is again applied to derive swap rates at half-years. The swap rates can be transformed to continuously compounded spot rates using the following equation, which is derived from equations 3.9 and 3.10.

$$r_t = -\frac{\ln\left[\frac{100-\sum_{i=1}^{t-1}\left(\frac{c}{m}e^{-r_i \times t_i}\right)}{100+\frac{c}{m}}\right]}{t} \tag{7.1}$$

The swap rates observed at the market and the transformed continuously compounded spot rates are shown in following table.

Tenor	Swap Rate	Maturity	Continuous Spot Rate
3Y	2.849%	30.4.2021	2.6760%
4Y	2.880%	30.4.2022	2.7322%
5Y	2.924%	30.4.2023	2.7238%
6Y	2.930%	30.4.2024	2.7265%
7Y	2.962%	30.4.2025	2.6961%

Table 7.5 Long end of the curve

Source: Bloomberg, own calculations

	Instrument	Maturity	Zero-coupon rate
Deposit rates	ON Libor	1.5.2018	1.704%
	1M Libor	30.5.2018	1.905%
	2M Libor	29.6.2018	2.058%
Eurodollar Futures	June 2018	18.9.2018	2.188%
	July 2018	16.10.2018	2.274%
	October 2018	15.1.2019	2.360%
	December 2018	17.3.2019	2.435%
	March 2019	18.6.2019	2.489%
	June 2019	17.9.2019	2.541%
	September 2019	16.12.2019	2.592%
	December 2019	16.3.2020	2.637%
	March 2020	16.6.2020	2.671%
Swap Rates	3Y	30.4.2021	2.849%
	4Y	30.4.2022	2.880%
	5Y	30.4.2023	2.924%
	6Y	30.4.2024	2.930%
	7Y	30.4.2025	2.962%

Table 7.6 Zero-coupon rates observed on 30.4.2018

Source: Bloomberg, own calculations

These rates were further interpolated to get the whole zero-coupon curve, which is shown in the following figure.

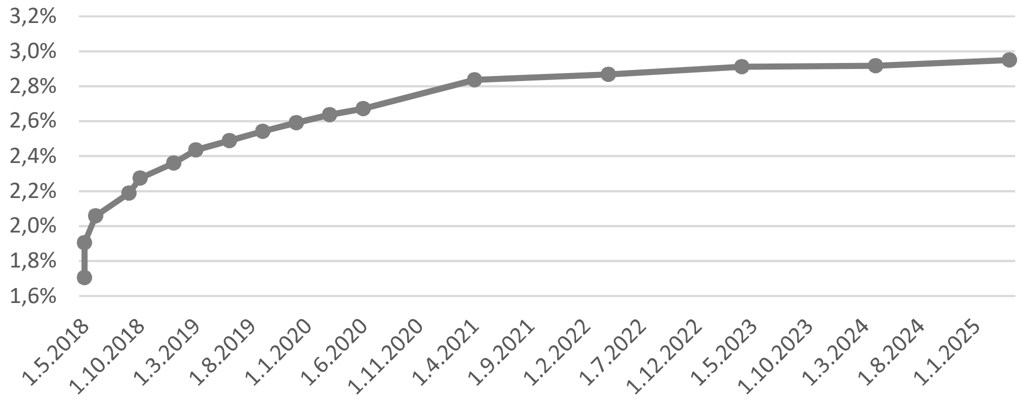

Figure 7.1 Zero-coupon curve observed on 30.4.2018

Source: Bloomberg, own calculations

Forward Libor Curve

Each interest rate on the forward interest rate curve represents a future Libor interest rate with maturity of one month. This is used to estimate future Libor interest rates. The forward Libor curve is constructed from the zero-coupon curve. To be precise, period forward rate is needed and the length of one period is a one month. The zero-coupon curve obtained in previous text was interpolated to obtain an interest rate applicable at each month. These rates were than transformed to one-month forward rates using equation 3.4. This is the estimate of future one-month LIBOR rates.

Valuing the Fixed leg

1) Determine the period end dates. The period end dates are semiannually on each January 15 and July 15. The last period end date is on the termination date, i.e. on 1.1.2025. These dates indicate the length of each interest period. It is used only for calculating interest, so that analyst knows how many days are in each period, on how many days the interest is accrued. No convention is used here, as it would not have any practical reasons, as opposed to the payment dates. Each interest period has 180 days, the last period has only 166 days. To calculate the length of each period, the 30/360 convention is used, meaning that there are 30 days in each month and 360 days in a year. ISDA provides more thorough explanation of the day count conventions. The fixed rate payments on this interest rate swap are thus the same, unless there is a reduction of the notional amount.
2) The payment amounts that are due during the swap´s life are calculated using the notional values and the fixed rate. The notional amount is reduced several times during the life of the swap (it is an amortizing swap). The sum of all reductions is equal to the original notional value.
3) The discount factors that are applicable on swap´s payment dates are calculated from the zero curve, using the equations 3.1 and 4.6. and linear interpolation.

4) The Future payment amounts are discounted using the discount factors and summed up. Now we have the present value of the fixed leg.

Selected information necessary for calculating the present value of the fixed leg is displayed in the following table.

Adj. Pay Dates	**Period End Dates**	**Remaining Notional**	**Fixed Rate Payment**	**Zero Curve**	**Discount Factor**	**Payment PV**
16.7.2018	15.7.2018	32 917 363	553 012	2.1015%	0.996	551 013.82
15.1.2019	15.1.2019	32 051 117	553 012	2.3596%	0.984	544 202.31
15.7.2019	15.7.2019	32 051 117	538 459	2.5026%	0.971	522 746.42
15.1.2020	15.1.2020	31 111 924	538 459	2.6079%	0.957	515 182.73
15.7.2020	15.7.2020	31 111 924	522 680	2.6841%	0.943	492 709.24
15.1.2021	15.1.2021	30 154 495	522 680	2.7948%	0.927	484 539.61
15.7.2021	15.7.2021	30 154 495	506 596	2.8706%	0.912	461 948.80
17.1.2022	15.1.2022	29 169 709	506 596	2.8804%	0.898	454 982.18
15.7.2022	15.7.2022	29 169 709	490 051	2.8853%	0.885	433 773.97
16.1.2023	15.1.2023	24 455 505	490 051	2.9109%	0.871	426 871.73
17.7.2023	15.7.2023	24 455 505	410 852	2.9291%	0.858	352 320.07
15.1.2024	15.1.2024	11 480 045	410 852	2.9306%	0.845	347 111.43
15.7.2024	15.7.2024	11 480 045	192 865	2.9313%	0.832	160 539.14
1.1.2025	1.1.2025	-	177 864	2.9433%	0.820	145 899.27
					Total	5 893 840.73

Table 7.7 Valuing fixed leg cashflow

Source: Bloomberg, own calculations

The net present value of the fixed leg on 30.4.2018 is 5 893 840.73 USD.

Valuing the Floating Leg

1) Period End dates are determined in a similar way as in the fixed leg, the end dates are on the 15th of each month, no adjustment is applicable and Actual/360 day count fraction is applied. The length of each interest period therefore varies and is simply calculated as a difference of each succeeding period end dates.
2) The notional amount for each period is determined. The notional reductions are the same for the floating as well as for the fixed leg.
3) The reset dates are determined. The reset date is on Wednesday every week during the swaps' life. The applicable rate for each reset rate is the USD LIBOR-BBA published one London Business Day prior to the reset date. USD Libor is now published by the ICE (Intercontinental Exchange). It used to be published by the BBA. The Libor rate is then observed on each of these reset dates, weighted average is then used to compute the rate that is applicable for the whole interest period (from 15th of one month to 15th of the next month), where weights are the number of days in the respective period. This averaging

feature adds more smoothness to the observed rates and minimizes the effect of shocks and price jumps. (These could be for example caused by a central bank meeting only few hours before the observation on the reset day). The following figure illustrates the concept of averaging floating interest rates. In the case of the NY swap, the period end dates are on the 15th of each month (15.4 and 15.5 in the figure), but reset dates are every Wednesday (11.4., 18.4, 25.4, 2.5, 9.5, 16.5). The weights are the number of days in the period (3, 7, 7, 7 and 6).

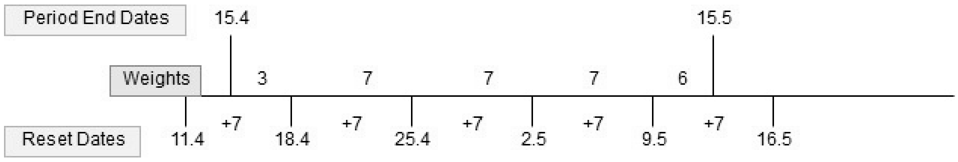

Figure 7.2 The weighted average principle

The applicable interest rate on 15.5. is than calculated as:

$$\frac{11.4.\ \times 3 + 18.4.\ \times 7 + 25.4.\ \times 7 + 2.5.\ \times 7 + 9.5.\ \times 6}{30} = 15.5 \tag{7.2}$$

where, the dates represent an interest rate observed on such a date. This method can be omitted when pricing the NY swap, without any harm on accuracy. As mentioned before, averaging is applied to limit the risk that an interest rate jumps unexpectedly for a short time on one of the reset dates due to a short-term shock. The constructed zero-coupon curve is however theoretically smooth and does not incorporate any unexpected price jumps etc. Interest periods for the floating rate are furthermore only one moth long. Forecasting an interest rate on each month will already yield quite smooth and dense forward curve. There is therefore no need to forecast the interest rate on each of these reset dates, as it would not add any real accuracy. Forecasting the future interest rate on the period end dates proves to be a good estimate of the future development.

4) The forward rates are determined from the zero-coupon curve. The zero-coupon curve is interpolated to estimate a one-month Libor rate applicable for the interest accrual period preceding each period end date.
5) The future payment amounts are estimated using the forecasted forward period interest rate and the respective notional amounts.
6) The discount factors that are applicable on swap´s payment dates are calculated from the zero-coupon curve. This step is similar as in the fixed leg, only more discount factors are needed. Linear interpolation is used.
7) The Future payment amounts are discounted and summed up to determine the present value of the floating leg.

Adj. Pay Dates	Period End Dates	Remaining Notional	Zero Curve	Forward Curve	Floating Rate Payment	Discount Factor	Payment PV
15.5.2018	15.5.2018	32 917 363	1.8069%	1.8069%	49 564.66	0.9992	49 527.36
15.6.2018	15.6.2018	32 917 363	1.9964%	2.0880%	59 186.51	0.9975	59 035.72
16.7.2018	15.7.2018	32 917 363	2.1015%	2.2627%	62 069.46	0.9955	61 790.81
15.8.2018	15.8.2018	32 917 363	2.1392%	2.2316%	63 257.03	0.9937	62 856.05
17.9.2018	15.9.2018	32 917 363	2.1811%	2.3258%	65 926.44	0.9915	65 369.02
15.10.2018	15.10.2018	32 917 363	2.2716%	2.6878%	73 730.15	0.9895	72 954.08
15.11.2018	15.11.2018	32 917 363	2.3286%	2.6373%	74 756.01	0.9872	73 801.35
17.12.2018	15.12.2018	32 917 363	2.3457%	2.4589%	67 450.92	0.9851	66 444.16
15.1.2019	15.1.2019	32 051 117	2.3596%	2.4622%	69 793.53	0.9831	68 615.57
…	…	…	…	…	…	…	…
16.9.2024	15.9.2024	11 480 045	2.9340%	3.0508%	30 158.72	0.8270	24 941.03
15.10.2024	15.10.2024	11 480 045	2.9360%	3.0880%	29 541.94	0.8249	24 370.26
15.11.2024	15.11.2024	11 480 045	2.9385%	3.1296%	30 937.92	0.8227	25 453.17
16.12.2024	15.12.2024	11 480 045	2.9414%	3.1755%	30 379.34	0.8205	24 925.36
15.1.2025	1.1.2025	-	2.9433%	3.2138%	17 422.37	0.8183	14 256.32
						Total	4 960 443.99

Table 7.8 Valuing floating leg cashflow

Source: Bloomberg, own calculations

The net present value of all the floating leg payments is 4 960 443.99 USD. The value of the interest rate swap is thus -926 734.58 USD for the New York State. This means that if the swap was to be cancelled, the New York State would have to pay 926 734.58 USD to the counterparty. The swap therefore seems to be unprofitable for the New York State and profitable for Citibank at the time of valuation.

Conclusion

In the first two chapters the interest rate swap as a financial instrument was introduced. Third chapter examined several quantitative methods that deal with transforming interest rates. Author found that each interest rate uses different compounding frequency or day count convention and they need to be thus transformed to be comparable to each other. The common practice is to transform the interest rates to continuous compounding. Author further illustrates a procedure for valuing an interest rate swap on a hypothetical example. Chapter five lists several approaches to risk management of interest rate swaps. The author finds especially the scenario analysis to be a simple but effective way of expressing the risk of changes in underlying interest rates. One of the tools that can be used to express the risks of changes in interest rates is the principal component analysis. Author performed this analysis on EUR swap rates and identified the first three principal components as a parallel shift in interest rates, change in slope and change in curvature. This is in accordance with many other studies conducted on the same issue. Author finds that the three risk factors are accountable for more than 95% of the total variance in interest rates, while the first factor identified as a parallel shift accounts for more than 86% of total variance. These results are similar to results of other studies. The thesis is concluded by an analysis of a real-life interest rate swap that is furthermore valued on 30.4.2018. The swap was initiated by the New York State. Author first discusses the individual features and specifics of the interest rate swap. The averaging method of floating interest rates is specifically elaborated and author finds that estimating the interest rate on period end dates is a good approximation of real future development. Author describes the procedure for valuing the interest rate swap and starts with construction of the zero-coupon curve. Several challenges to constructing the interest rate curve are identified. The interest rates used for construction of the zero-coupon curve were transformed to continuous compounding. Because the quoted interest rates do not link to each other, linear interpolation had to be used to derive interest rates for specific points in time. Fixed and floating legs were further valued using the described methods. The mark-to-market value of the entire interest rate swap was calculated to be negative to the New York State and positive to the counterparty (Citibank). Therefore, if New York State wanted to cancel the swap, it would have to pay for it. To conclude, the author showed several methods related to valuing an interest rate swap and then he applied them on the New York State´s swap. The thesis could be further developed in many ways. More methods could be introduced and then compared to each other. The risk management methods could be further applied to the New York State swap.

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