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Pricing and calibration of Libor Market Model

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I would like to express my sincere gratitude to my thesis supervisor **Ing. Marek Kolman, Ph.D.** for his guidance and thoughtful insights.
I would like to thank my parents for their support during my studies.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Oceňování a kalibrace Libor Market Modelu

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Abstrakt: Tato práce se zabývá problematikou modelů úrokových měr, speciálně LIBOR tržním modelem. Práce prezentuje základnou teorii k úrokové míře, k derivátům úrokových měr a nejdůležitější modely úrokových měr. Je provedena kalibrace LIBOR tržního modelu na reálné tržní data. Pomocí tohto modelu a Hull-Whitova modelu je oceněná Evropská swapce a výsledky jsou porovnány. Dále je oceněná bermudská swapce, což si vyžaduje poněkud jiný přístup, výsledky jsou znovu porovnány.

Klíčová slova: LIBOR tržní model, kalibrace, ocenění swapce, evropská swapce, bermudská swapce, Hull-Whitův model

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Abstract: This thesis focuses on models of interest rate, especially LIBOR market model. It presents the interest rate theory, interest rate derivatives and most important models of the interest rate. The LIBOR market model is calibrated to real market data. A pricing of a European swaption is performed according to Hull-White model and according to LIBOR market model, the results are compared. A Bermudan swaption is priced with the same models, which requires a different approach and the results are compared again.

Keywords: LIBOR market model, calibration, swaption pricing, European swaption, Bermudan swaption, Hull-White model

Contents

Introduction

Interest rate is one of the most important variables in finance. Its future evolution is unknown and that has provoked a need for models of interest rate.

Interest rate modeling has begun with short rate models. Many short rate models appeared, each with different features. These models work with a theoretical instantaneous rate, which is highly unpractical for the real world. In the 1990s, a new framework of *market models* is introduced to solve this and other problems. LIBOR market model belongs to the class of market models and it has become the industry standard over the years.

There have been many extentions proposed to the LIBOR market model. This thesis focuses on the lognormal LIBOR market model.

The goal of this thesis is to calibrate the LIBOR market model and then use it to price a financial derivative.

It is organized as follows:

The **first chapter** introduces the theoretical background. It presents basic notions of the interest rates and is also devoted to the description of the phenomenon of negative interest rates that has been present on markets for several years.

The **second chapter** is devoted to the presentation of the interest rate derivatives as a whole and also presenting the most important interest rate derivatives for this thesis, mainly swaptions.

The **third chapter** discusses models of the interest rate. At first, some notions as the Brownian motion or the change of numeraire are introduced, so that the background of the models is understandable and then the most important models are introduced, as they appeared in time.

In the **fourth chapter**, the calibration of the LIBOR market model is performed. It is done with market data taken in April 2018 and the results of the calibration are presented.

The **fifth chapter** uses the same data to calibrate the Hull-White model, which is a simpler model of interest rates that uses different assumptions. It is done to compare the two models.

In the **sixth chapter**, a pricing of a swaption will be performed with given parameters and to do this, we use both LIBOR market model and Hull-White model. We then compare the results.

The **conclusion** summarizes the results and we propose further steps that can be done.

All the computations are performed in the Matlab software.

A significant problem that has to be mentioned are the negative interest rates that are present on markets. This new phenomenon is observable since 2014. Several models of interest rates have the capability to model negative interest rate. Until now, it has been treated as an imperfection, but recent market development puts it in a different light and even models unable to model negative interest rate need to be enhanced in some ways to get this virtue.

Chapter 1

Theoretical background

In this section, we will present some of the basic notions of financial mathematics, an overview of interest rate derivatives that the thesis focuses on, an overview of how modeling of the interest rate evolved over time and some problems associated with modeling the interest rate at the moment.
This chapter relies mainly on the theory described by Brigo, Mercurio and Rebonato. The notation that is used should correspond to the notation used by Brigo and Mercurio. The list of important abbreviations can be found at the end of this thesis.

1.1 Zero-coupon bond

Before defining a zero-coupon bond, we will establish the foundation of the money-market account $B(t)$. The initial value of the money-market account is 1, $B(0) = 1$.

$$dB(t) = r_t B(t) dt. \tag{1.1}$$

in this equation, r_t is the instantaneous rate (also called short rate) and is a positive function of time.

The discount factor is defined as a value at time t of a single unit that will be paid at time T. The instantaneous rate r_t being stochastic, the price of the zero-coupon bond can be expressed as the expected value of the discount factor:

$$P(t,T) = E[e^{\int_t^T r(s)ds}] \tag{1.2}$$

A zero-coupon bond is one of the basic notions in financial mathematics. It is a financial instrument that pays one unit of currency at maturity T. $P(t,T)$ represents the price of a zero-coupon bond at time $t < T$. It does not pay any coupon and no intermediate payments associated with zero-coupon bond. At time t, the contract pays the present value of one unit of currency.

The price of the zero coupon bond is defined as:

$$P(t,T) = \frac{1}{1 + L(t,T)\tau(t,T)}, \tag{1.3}$$

where $\tau(t,T)$ is the difference between t and T, therefore $\tau(t,T) = T - t$ and is in years.

From this equation, we can define the spot interest rate as:

$$L(t,T) = \frac{1 - P(t,T)}{\tau(t,T)P(t,T)} \tag{1.4}$$

The price of the zero-coupon bond is the discounted value of its face value. A zero coupon bond is usually used as an example of a risk-free investment. The value of the zero-coupon bond at time t is important because of its relationship with interest rates.

1.2 Interest rates

An interest rate is the amount of money defined as a proportion of the principal that the borrower promises to pay the lender. There is a possibility, that the borrower will not repay the loan and that is a risk for the borrower. The level of the interest rate is dependent on the risk that the lender enters. The other important factor for the level of interest rate is maturity.
The level of interest rate is different for currencies, and even in a single economy, there are many interest rates.

These types can be listed as: *Government interest rate* (known as Treasury rate in the US) is the rate at which bonds are issued. *Interbank interest rate* is the rate at which banks borrow money to each other. The most important interbank rate is LIBOR, which wil will be presented later in detail. A specific type of interest rate is the *key interest rate*, which is defined by the central bank in the economy.
A simply compounded spot interest rate will be noted as $L(t,T)$.

Another difference in the interest rate theory lies in the interest rate *compounding*. The frequency of the compounding can make a significant difference in the calculations.

The **continous compounding** is used mainly in theory. It is defined as:

$$R(t,T) = -\frac{lnP(t,T)}{\tau(t,T)} \tag{1.5}$$

and therefore

$$1 = e^{R(t,T)\tau(t,T)}P(t,T) \tag{1.6}$$

and

$$P(t,T) = e^{-R(t,T)\tau(t,T)} \tag{1.7}$$

The **annual compounding** represents the situation where we invest an amount A today and we get $A = 1 * (1 + Y)$, with the interest rate being Y exactly one year from today.

Formally:

$$Y(t,T) = \frac{1}{P(t,T)^{1/\tau(t,T)}} - 1 \tag{1.8}$$

The **k-times per year compounding** is defined as:

$$Y^k(t,T) = \frac{k}{P(t,T)^{1/k\tau(t,T)}} - k \tag{1.9}$$

1.3 Forward interest rate

From now on, we will have to differentiate between the *spot* and the $forward$ interest rate.

Forward interest rates are associated to Forward rate agreements (FRAs). Forward rate agreement is a contract fixed at time t, that fixes the interest rate for both parties at time T, which is the FRA expiry. The contract matures at time S. A FRA allows to fix the interest rate between time T and time S.
At time S, the payoff of the contract is:

$$N\tau(T,S)(K - L(T,S)).$$

At time t, the value of the contract corresponds to:

$$FRA(t, T, S, \tau(T, S), N, K) = N[P(t, S)\tau(T, S)K - P(t, T) + P(t, S)],$$

where K is the fixed rate and N is the nominal value of the contract.

In general, forward rate is the rate at which money can be borrowed in the future, but is fixed today. Forward rate is therefore characterized by three moments in time: present time t, the time T in the future where the loan begins T and the time S in the future, when the loan ends.

A forward interest rate is defined as:

$$F(t, T, S) := \frac{1}{\tau(T, S)} \left(\frac{P(t, T)}{P(t, S)} - 1 \right), \tag{1.10}$$

where $\tau(T, S)$ is analogous to the definition of $\tau(t, T)$ and is the difference of time between T and S.

1.4 Term structure of interest rates

The term structure of interest rate, also called zero-coupon curve or the yield curve [2] is probably the most famous curve in finance. It is a graphical representation of several interest rates varying in their maturities and it can be obtained from the market. One could assume that the term structure of interest rate is always increasing in time, however, that is not always the case and the yield curve can have many shapes.

Figure 1.1 and Figure 1.2 are a representation of several shapes of the yield curve. Figure 1.1 represents the term structure from March 28, 2018 for the euro-area. It can be found on the website of European Central Bank [20]. We can see two separate curves. The solid line is derived from a selection of AAA government bonds (which means that only the most trustworthy bonds were selected). On the other hand, the dashed line represents the yields for all government bonds. It is only logical that the compensation for these bonds is higher, because they represent a more risky investment in comparison to the selection of AAA bonds. The graph of the spot rate is a representation of L, we can see spot rates with different maturities, the shortest being 3M (three months from today) and the longest being 30Y (30 years from today). Figure 1.2 is a representation of instantaneous forward rates corresponding to different residual maturities. According to the methodology of the European Central Bank, "the instantaneous forward rate contracted at time t for duration d measures the short-term interest rate that investors can lock-in at time t in order to be received at time $t + d$." [21]. Analogically to the Figure 1.1., the solid line represents the yield of only AAA bonds and the dashed line is the yield of all bonds for the instantaneous forward rate.

We can see, that the spot yield, as well as the instantaneous forward yield start below zero. The curve of the spot rate attains zero for bonds with maturities of 4 to 5 years and the curve of the instantaneous forward rate attains zero for bonds with maturities of more than 2 years maturities (for all bonds). This phenomenon of negative interest rates will be discussed later.

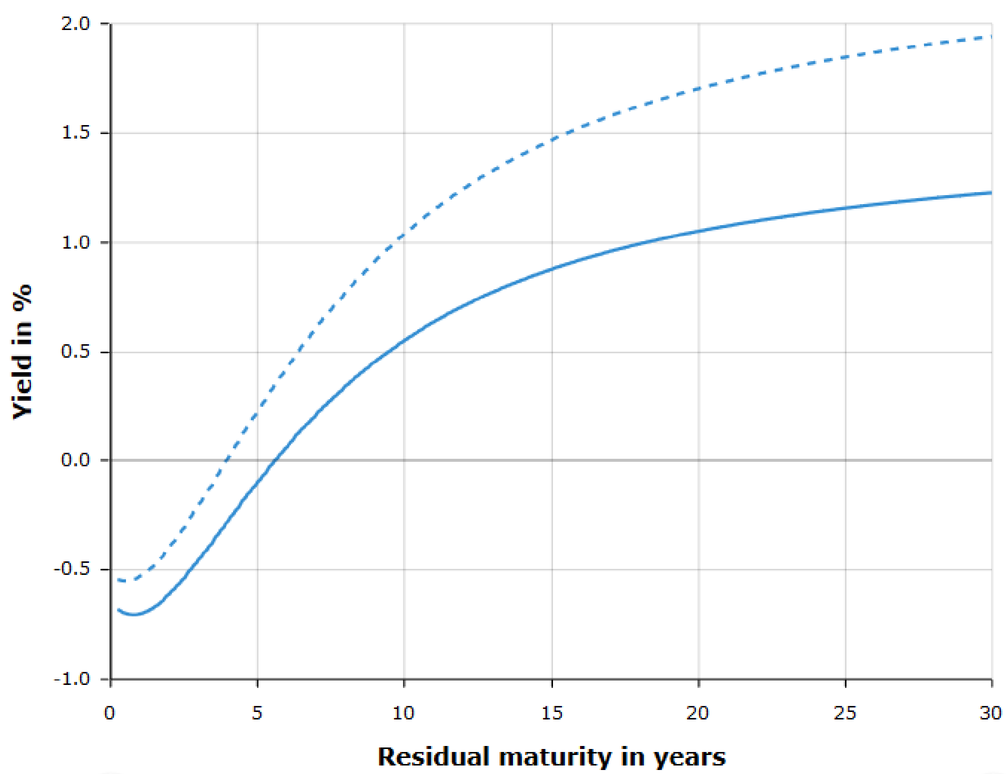

Figure 1.1: Euro area yield curves - Spot rates. Source: European Central Bank

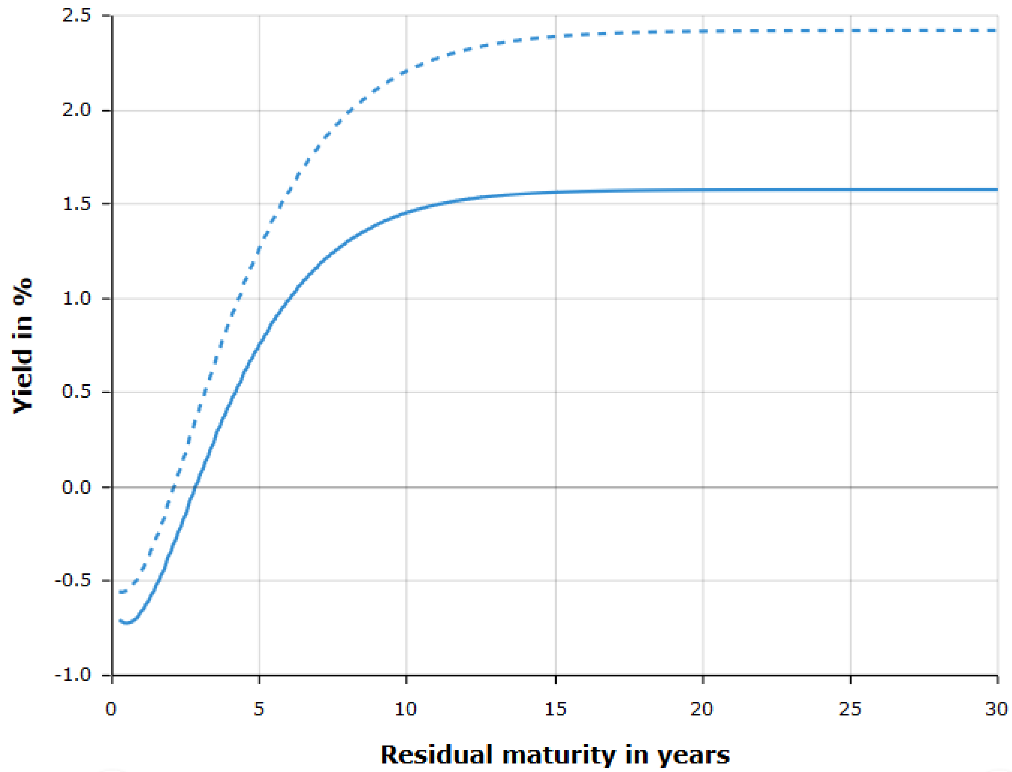

Figure 1.2: Euro area yield curves - Instantaneous forward rates. Source: European Central Bank

1.5 LIBOR

As already stated, interbank rates are the interest rates, which is used by banks for lending money to other banks. These rates are known as IBOR. The most common and used example is LIBOR, this case stands for London Inter-Bank Offered Rate. LIBOR was officially introduced in 1986. LIBOR is the average interest rate that most important banks are willing to lend money at the London inter-bank market. LIBOR is settled for several currencies and for different time periods. Alongside LIBOR, there are rates settled on other markets, we can mention EURIBOR (European Inter-bank Offered Rates), which consists of an agreement of more than 20 european banks, or PRIBOR (Prague Inter-bank Offered Rates). All of them are an example of a simply compounded spot interest rate.

LIBOR rate serves as a benchmark for other short rates and as a reference rate for many derivatives, including swaptions.

The simply compounded forward LIBOR rate at time t for maturity pair T_1, T_2

is denoted as $F(t, T_1, T_2)$.

1.6 Negative interest rates

"The fact that lending money must be rewarded somehow, so that receiving a given amount of money tomorrow is not equivalent to receiving exactly the same amount today, in indeed common knowledge and wisdom." [2]

In the 1990s, the interest rates in Europe have been very low and the earnings on bonds have been very low as well. The decline of interest rates on markets has been visible since 1990s. After the financial crisis of 2008, countries have experimented with different policies for stimulating their economies. Several central banks in Europe have tried negative interest rates experiment in the name of reducing the borrowing costs for firms and households. This policy is an interesting turn of events, because it turned out that zero is not the lowest point for interest rates.

A negative interest rate basically means, that the lender pays to the borrower. Interest rates below zero deny the theory of time value of money (that assumes that money is more valuable now than in the future, and thus creating the notion of a (positive) interest rate, that the borrower has to pay) and were considered very unusual. The goal of this policy is to induce banks to borrow money, instead of keeping it.

European central bank (ECB) has adopted the policy of negative overnight rates in mid-2014. Target inflation in the eurozone is set by ECB to 2 %. However, the inflation has been very low (it has dropped below zero several times since 2014), the nominal interest rates were low as well and ECB felt the need to intervene. As a reaction to this policy, Swiss central bank was next to do the same thing. In the first half of 2016, the inflation seems to get back on its track. The rates on the deposit facility have been growing lower since 2008 (with the highest value in 2008 of 3.25%). Currently (April 2018), ECB still holds the policy of negative interest rates. The overnight credit for banks (deposit rate) is now -0.40 percent p.a., which was set in 2016. The ECB's refinancing rate has been zero for quite some time now.

ECB was not the only institution to adopt such an unconventional policy. Central banks in Sweden, Denmark, Switzerland and Japan have also resorted to this.

For decades, the general assumption was that zero is the lowest interest rate that is possible. It is a difficult to assess, whether the negative interest rate policy had the desired effect. However, it has certainly shifted the paradigm on markets.

It was also a twist for theoretical modeling of the interest rate. Several models have the ’ability’ to model a negative interest rate, but until now, it has been treated as a disadvantage and an imperfection of the model.

Chapter 2

Interest Rate Derivatives

A financial derivative is a product with an underlying asset. The price of the derivative is closely linked with the price of the asset. An interest rate derivative is a financial instrument whose payoff depends on the level of interest rate and the expected trend in the future. Interest rate derivatives are a product with a finite lifetime. Derivatives can be traded on exchange or off exchange (OTC market). Trading of interest rate derivatives became very popular and has multiplied in volume in the 1980s and 1990s.

As it is apparent from Figure 2.1, interest rate derivatives represent the most important group among OTC derivatives. In the first half of 2017, Bank of International Settlements (BIS) claims that interest rate derivatives to make approximately 67 % of the total value of OTC derivatives. The most important interest rate derivatives (also relevant for this thesis) are caps, floors, caplets, floorlets and swaptions. This thesis focuses on swaptions and for this reason, swaptions will be discussed in detail.

2.1 Interest rate swaps

An interest rate swap is a simple contract between two parties. A specified amount of money N is settled at the beginning, it is known as the notional. The notional is never exchanged. Counterparties only exchange the interest rates on this amount. The floating leg is fixed on a reference interest rate (in most cases LIBOR), the fixed rate is fixed in the contract. There are predefined time moments T_i for the exchange of payments. Interest rate swaps are popular among corporations and companies, which use this derivative to protect themselves against interest rate exposure on financial markets.

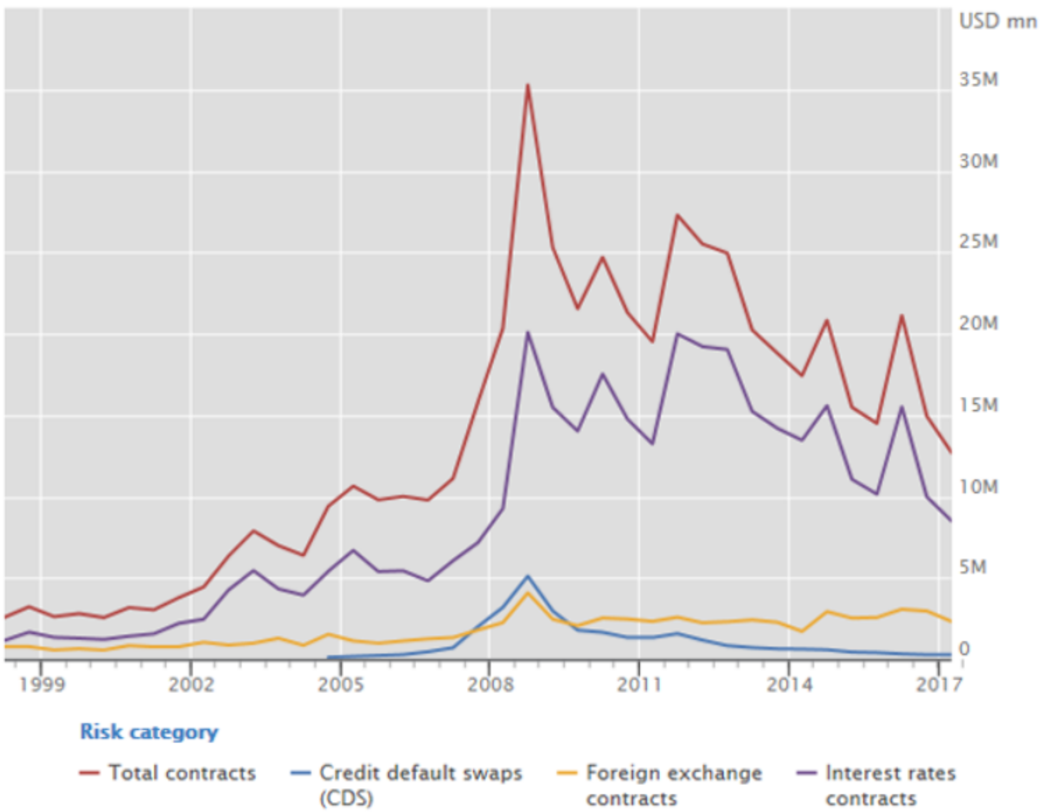

Figure 2.1: Market value of OTC derivatives from 1997 to 2017. Source: Bank of International Settlements

T_α is the time of the swap beginning. At each time T_i that belongs to the time series from $T_{\alpha+1}$ to T_β, the fixed leg pays $N\tau_i K$, K being the fixed rate, and the floating leg pays $N\tau_i F(T_{i-1}, T_i)$, so N only serves as a base amount of money.

The net cash flow of a payer swap at time T_i is defined as:

$$N\tau P(t, T_i)(F(t, T_{i-1}, T_i) - K) \tag{2.1}$$

and the value of the **payer swap** at time t is

$$N\tau \sum_{i=\alpha+1}^{n} P(t, T_i)(F(t; T_{i-1}, T_i) - K) \tag{2.2}$$

The value of a **receiver swap** at time t is defined as the same expression multiplied by -1:

$$N\tau \sum_{i=\alpha+1}^{n} P(t, T_i)(K - F(t; T_{i-1}, T_i)) \tag{2.3}$$

If we set the price of the swap to be 0 at time t, and therefore the price of the swap to be *fair* for both parties of the contract, we come to the rate K and to the forward swap rate. We define the forward swap rate $S_{\alpha,\beta}(t)$ at time t for times T as

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} \tag{2.4}$$

or in terms of forward rates

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{ß} \frac{1}{1+\tau_j F_j(t)}} \tag{2.5}$$

2.2 Caps and Floors

Caps and floors are traded among the main interest rate derivatives. The two parties agree on a strike price and the buyer of the cap receives payments, if the interest rate is above this strike price K. A floor is analogous and the buyer receives payments, if the interest rate is lower than the strike price. The starting payment in the contract happens at time T_α and the payments happen until T_β. These derivatives are used to protect against the interest rate changes.

A cap can be decomposed to a series of caplets. A single caplet is a call option on interest rate.

The cash flow of a **caplet** at time T_i is:

$$\tau(L(T_{i-1}, T_i) - K)^+ \tag{2.6}$$

We denote $Cpl(t; T_{i-1}, T_i)$ the price of the caplet at time t and therefore the price of a **cap** at time t is:

$$Cp(t) = \sum_{i=\alpha+1}^{n} Cpl(t; T_{i-1}, T_i) \tag{2.7}$$

The cash flow at time T_i of a **floorlet** is:

$$\tau(K - L(T_{i-1}, T_i))^+ \tag{2.8}$$

and analogously, the price of the **floor** is defined with the price of floorlets denoted $Fll(t; T_{i-1}, T_i)$:

$$Fl(t) = \sum_{i=\alpha+1}^{n} Fll(t; T_{i-1}, T_i) \tag{2.9}$$

2.3 Swaptions

A swaption is an OTC derivative, that combines an option and an interest rate swap. Swaptions have been present on markets since 1980s and are usually used by banks and hedge funds. They provide the guarantee that the interest rate on a loan will not exceed a given level and are basically used as a protection against raising interest rates. A swaption gives the right to enter in an interest rate swap at a certain date in the future.

We distinguish between a payer and a receiver swaption. A (European) payer swaption gives the holder the option to enter a swap at a time T_α, where the payer would pay the fixed rate and get the floating rate. T_α is the time of the swaption maturity and it can be the first payment date. A receiver swaption represents the opposite – receiver pays the floating rate payments.

The parameters of a swaption are:

- The notional amount
- The frequency of payments
- The length of the option period

- The fixed rate and the rate it will be compared to (which is usually LIBOR)

There are several types of swaptions: European, American and Bermudan. The differentiation is analogous to the one of options.

With **European** swaptions, the holder is allowed to enter the swap only at the expiration date T_α. European swaptions are considered to be a vanilla derivative.

An **American** swaption represent a interest rate derivative, where the holder can enter the swap at any time during the maturity of the swaption.

A **Bermudan** swaption gives the holder the right to enter the swap at predefined dates until the maturity of the swaption T_α. Bermudan swaptions are considered to be exotic derivatives.

In the fixed leg, there is a fixed interest rate K and the owner gets $K\tau_j$.

The maturity of a swaption corresponds to T_α. A tenor of a swaption is $T_\beta - T_\alpha$. At time T_β, the interest rate swap expires.

The value of a payer swaption payoff at maturity T_α is:

$$N \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)$$

The discounted value of a payer swaption is given by:

$$ND(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^+$$

At time 0, the value of a payer swaption priced with Black's formula is:

$$PS^{Black}(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) = NBl(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, 1) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i), \quad (2.10)$$

where τ are the payment dates and $\sigma_{\alpha,\beta}$ is the volatility of the swaption price.

Swaptions will be used later to perform the calibration of the LIBOR market model. In the last part, swaptions will be priced, the pricing will be presented later in detail.

Chapter 3

Modeling the interest rate in theory

In this chapter, we will present some important notions necessary to understand the modeling of the interest rate and a brief history of important models of interest rate from their introduction until now.

The forward interest rate is crucial for pricing interest rate derivatives. Forward interest rate, as already mentioned, is the interest rate fixed at time t for a period in future, with the expiry at time T_1 and the maturity at time T_2, where $t \leq T_1 \leq T_2$.

3.1 Brownian motion

Randomness is an important notion in interest rate models. The interest rate models are based on the fact that financial processes they have a component of randomness (a stochastic process). This randomness is usually captured by a Lévy process, specifically the Brownian motion.
Brownian motion is a concept that has its origins in biology. It was first described in 1828 by a Scottish Robert Brown, when he discovered that pollen suspended in water were moving when the water itself was still. This movements were explained by a physicist Jean Perrin in 1909 as collisions of water molecules. Brownian motion was later mathematically formulated by Norbert Wiener and it was used by Bachelier, who used this process in finance to price options.

It has found its significance in the theory of randomness in mathematics or physics. It is the key object of probability theory and is widely used in finance, for example with solving stochastic differential equations. Today, the term Brownian motion serves as a synonym for random movements.

In 1930s, Kolmogorov defined the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and that contributed to the usage of the language of probability theory to finance.

The Brownian motion satisfies the conditions of:

- a martingale, meaning that today's value is the best estimate of the future value
- a Markov process, meaning it has no memory and the historical development does not affect future development

A **standard Brownian motion**, also known as Wiener process, is a time continuous stochastic process $W_t : t \geq 0$ that will be denoted as W_t. [12]

The properties of the Standard Brownian motion are:

1. W_t has independent increments for all times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$
2. W_t has stationary increments
3. W_t has continuous sample paths
4. W_t starts at 0, $W_0 = 0$

A process follows a **geometric Brownian motion**, if it can be described by the following equation:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{3.1}$$

where μ represents the drift parameter and σ the volatility parameter, for the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{3.2}$$

3.2 Change of numeraire technique

The assumption in financial models is that there is no opportunity of arbitrage. Risk neutral measure is heavily used in finance. It is a probability measure, that allows the expression of the price of an asset/numeraire to be equal to its discounted price as of today. This allows the price process of the asset to become a martingale (a process without a drift).

The Girsanov theorem is closely related to this approach. If the underlying probability measure changes, Girsanov theorem defines the changes in the stochastic differential equation (SDE) that is used to describe the evolution of a specific process. It helps to modify the drift coefficient and is used for the transition to risk neutral measure. It allows the changing of a Wiener process with a drift to a Wiener process without a drift (which will be a martingale).

A particular probability measure $\mathbb{P}$ is defined in the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$. We will consider the second measure $\mathbb{P}^*$ on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$.

$\mathbb{P}^* \backsim \mathbb{P}$ is a sign of the equivalence of the two measures. The conditions of the equivalence can be found in [2]. When two measures are equivalent:

$$\mathbb{P}^*(A) = \int_A \rho_t(\omega) \mathrm{d}\mathbb{P}(\omega), \qquad A \in \mathcal{F}_t$$

or we can simplify:

$$\left.\frac{\mathbb{P}^*}{\mathbb{P}}\right|_{\mathcal{F}_t} = \rho_t,$$

where ρ_t is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, and it is the Radon Nikodym derivative of $\mathbb{P}^*$ with respect to $\mathbb{P}$ restricted to $\mathcal{F}_t$.

The expected value becomes:

$$\mathbb{E}^*[X] = \int_\Omega X(\omega) \mathrm{d}\mathbb{P}(\omega) = \int_\Omega X(\omega) \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}(\omega) \mathrm{d}\mathbb{P}(\omega) = \mathbb{E}\left[X \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right],$$

the expected values with respect to the probability measures $\mathbb{P}, \mathbb{P}^*$ are indicated as $\mathbb{E}$, $\mathbb{E}^*$.

For the conditional expectation:

$$\frac{\mathbb{E}\left[X \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} | \mathcal{F}_t\right]}{\rho_t} = \mathbb{E}^*[X | \mathcal{F}_t],$$

Let the asset price dynamics be described by a SDE under a real world probability measure P:

$$\mathrm{d}X_t(\omega) = \mu X_t(\omega) \mathrm{d}t + \sigma X_t(\omega) \mathrm{d}W_t(\omega)$$

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}(\omega)\right|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t - \frac{\mu - r}{\sigma} W_t(\omega)\right\}.$$

with the risk neutral measure, the process will become:

$$\mathrm{d}X_t(\omega) = r X_t(\omega) \mathrm{d}t + \sigma X_t(\omega) \mathrm{d}W_t^*(\omega),$$

where $dW_t^*(\omega)$ is the Brownian motion under $\mathbb{P}^*$.

As defined by Brigo and Mercurio, a numeraire is "any positive non-dividend paying asset" [2]. The choice of the numeraire depends on several factors, most common is the money market account or a bond. The approach of changing the numeraire was used by Merton in 1973.

3.3 Before LIBOR market model

With the need to model the future development of the interest rate, interest rate models appeared. Interest rate models are also crucial to price derivatives. In this field, the work of Black, Scholes and Merton was related to the precious research of Bachelier and remained very significant until today. **Black-Scholes formula** was introduced in 1973. It determines the theoretical price of a European call option, thus it is not designed to model the interest rate directly. The disadvantage of this approach was that it needs a constant volatility of the underlying, but the volatility of a bond is not constant during its life.

The formula for calculating the price of a call option (that pays no dividend) in the Black-Scholes model is:

$$C = SN(d_1) - N(d_2)Ke^{-r(T-t)}, \tag{3.3}$$

where N is the CDF of the standardized normal distribution, S is the price of the underlying, K is the strike price of the option and d_1 and d_2 are given by:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)\right) \tag{3.4}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{3.5}$$

where σ is the volatility parameter.

Even if the model is not designed to determine the interest rate directly, it is very closely linked to Black's formula, presented in 1976.

3.3.1 Black's model

In 1976, Black's model (alternatively called Black's formula, Black-76 model) was introduced. It is a variation of previously introduced Black-Scholes formula for option pricing.

It assumes a lognormal distribution of the future price of the underlying asset. Apart from pricing stocks, it is also used for valuing swaptions and caps.

Black's model, however popular, has several limitations: there is no way for this approach to take into account the correlation between forward prices of different assets (even if they belong to different assets, the correlation is still present) and it assumes that the volatility in the model is a constant number, however in reality, it varies over time.

Black's model is not able to take into account negative interest rates. For this reason, there has been a proposition of a shifted Black's model. It is also known as the displaced diffusion model. In shifted models, the forwart rate F_t is replaced with a forward rate with a shift $F_t + s$.

Black's model is practical for pricing interest rate derivatives, mainly swaptions, and it has become very popular for practitioners.

As already stated, the price of a payer swaption according to this model is:

$$PS^{Black}(0, \tau, N, K, \sigma_{\alpha,\beta}) = NBl(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, 1) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i). \tag{3.6}$$

There are two reason why we mention Black's model in detail: first, this model is the basic approach to price swaptions. Second, it represents a foundation that LIBOR market model is build on.

3.3.2 Short rate models

The first generation of interest rate models appeared in the late 70's. They are now called the "short rate models" and they assume a instantaneous interest rate, that is a theoretical value. The hypothetical short rate approximation is the overnight rate. It was was the first generation of models to model the interest rate coherently. In these models, there is a perfect correlation among all forward rates. The models that will be mentioned were described by Vašíček, Cox-Ingersoll-Ross, Ho-Lee and Hull-White. All of the interest rate dynamics are under risk-neutral measure. The short rate models describe the possible movement of the interest rate with a stochastic differential equation. It is not difficult to derive the price of a zero-coupon bond $P(t, T)$ and it can be found in the literature, for example in [7].

Vašíček's model was introduced in 1977. The dynamics of the process is described by:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \tag{3.7}$$

where $k[\theta - r(t)]$ is the drift parameter and $r(0) = r_0$.

Vašíček's model has a property of mean reversion. These models are now used mainly in the academic sphere. In the Vašíček's model, the modeled interest rate can go below zero.

Cox, Ingersol and Ross model (CIR) is a one factor short rate model. The difference with the previous model is that CIR does not allow the interest rates to be negative. The CIR process dynamics is defined as:

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)dW(t)} \tag{3.8}$$

and $k[\theta - r(t)]$ is the same drift parameter as in Vašíček's model.

A no-arbitrage model is porposed by **Ho and Lee** in 1986.

In 1986, **Ho and Lee** have proposed a model, that fits the initial (today's) term structure of interest rates and therefore eliminates one of the inconveniences of the previous models. The class of models that have this ability is called no-arbitrage. The dynamics of the model is presented as:

$$dr(t) = \theta(t)dt + \sigma dW(t) \tag{3.9}$$

In this case, the element $\theta(t)$ is not a constant, but a function of time and it is designed to fit the initial term structure. It is expressed as

$$\theta(t) = F_t(t, T) + \sigma^2 t \tag{3.10}$$

Hull and White combined the features of Vasicek and Ho and Lee model in a sense that it contains a mean reversion a, but is in the class of no-arbitrage models.

$$dr(t) = [\vartheta(t) - a(t)r(t)]dt + \sigma(t)dW(t), \tag{3.11}$$

where ϑ is a function that allows to fit the initial term structure of r and a is the mean reversion rate. It has a form

$$\vartheta(t) = F_t(t, T) + aF(t, T) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{3.12}$$

In these models, there is exactly one source of uncertainty (randomness) and because of this property, they are called single factor models. Another class of models to appear was **two-factor interest rate models**. These models were enhanced to include another source of uncertainty. The other stochastic process modeled is often volatility. In a single factor version, the volatility is a constant. Two-factor models give us the possibility to include a correlation structure, where we can include the information that interest rates with maturities closer together have a higher correlation than interest rates with maturities further apart.

The two-factor Gaussian-Vašíček model is described by:

$$\begin{aligned} r(t) &= x(t) + y(t) \\ dx(t) &= k_x(\theta_x - x(t))dt) + \sigma_x dW_1(t) \\ dy(t) &= k_y(\theta_y - y(t))dt) + \sigma_y dW_2(t) \end{aligned} \tag{3.13}$$

The two Brownian motions $dW_1(t)$ and $dW_1(t)$ are correlated, the instantaneous correlation is defined $\rho dt = dW_1(t)dW_2(t)$.

In the beginning of the 90's, exotic derivatives were really gaining importance and new types of derivatives were appearing. There was an increasing need to model the whole term structure of the interest rate.

The second generation of models, mainly represented by **Heath-Jarrow-Morton** (HJM) model the whole forward curve. HJM also belongs to the class of no-arbitrage model that is capable to fit today's term structure. Some of the principles of short rate models were kept (for example mean reversion).

We begin with the simple equation

$$dr(t) = \theta dt + \sigma dW(t), r_0. \tag{3.14}$$

and for the instantaneous forward rate, we can get

$$df(t,T) = \sigma^2(T-t)dt + \sigma dW(t) \tag{3.15}$$

and in general

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f^M(0,T) \end{aligned} \tag{3.16}$$

where $T \to f^M(0,T)$ is the curve of instantaneous forward rate at time t, α is an adapted process, $\sigma(t,T)$ is a vector of adapted processes, $W = (W_1, ..., W_N)$ is a N-dimensional Brownian motion.

If we want to attain a risk-neutral measure for this dynamics, than

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds \tag{3.17}$$

and

$$f(t,T) = f(0,T) + \sum_{i=1}^{N}\int_0^t \sigma_i(u,T)\int_u^T \sigma_i(u,s)dsdu + \sum_{i=1}^{N}\int_0^t \sigma_i(s,T)dW_i(s) \tag{3.18}$$

3.4 Market models - LIBOR market model

Traditional short rate models offer only a framework, where forward rates are perfectly correlated $\rho_{i,j}(t) = 1$. The need to price interest rate derivatives that have several payments scheduled in the future (swaptions) has been rising and short rate models were unable to properly fulfil this task. This has lead to the introduction of more complex models, one of them being the LIBOR market model. They offer much more suitable framework for pricing such derivatives.

“LIBOR market model is not a model; rather, it is a set of no-arbitrage conditions among forward rates (or discount bonds).” [8]

Market models were introduced in the 1990s and became the third generation of interest rate models. They are much more advanced then the previous generations. It happened after the theory of numeraire and change of measure was introduced. Market models are very popular in practice. It was a breakthrough and “ (it) creates an environment which makes calibration of a model relatively straightforward compared with models from arising alternative frameworks.” [1], mainly because LIBOR forward rates are observable directly on markets and the model does not predict a theoretical short rate. Also, market models were the first models, that were compatible with Black’s formula for pricing caps. LIBOR in the title does not refer specifically to LIBOR, but it can indicate another interbank rate, for example EURIBOR.

The framework of LIBOR market model was created in 1994 by Brace, Gatarek and Musiela and has been developing since. It is an interest rate model based on the development of LIBOR forward rates. It has become popular because it is consistent with practice and it has become the industry standard.

The model predicts the evolution of forward rates, based on the instantaneous volatilities of forward rates and correlations among these rates. It assumes a lognormal distribution of interest rates and a correct change of measure needs to be implemented and each forward rate is driftless under a specific measure.

As already said, the model is consistent with the standard market approach of Black formula for pricing caps. In fact, it is a collection of Black models under a single measure. The inputs to the model are: the volatility structure of interest rates and the correlation of stochastic processes. There are several methods to determine this volatility:

1. from historical data
2. bootstrapped directly from correlation-sensitive market-quoted instruments, for example european swaptions
3. the analyst predicts them based on his assumptions about the markets in the future

Usually, the first two are preferred.

The drift of the process naturally depends on the choice of numeraire. We can chose a forward numeraire (a zero coupon bond) or a spot numeraire (rolling bank account).

LIBOR market model is an improvement in comparison to Black's formula: it captures the correlation structure among the forward rates, Black's formula does not have the ability. It is therefore used in pricing fixed income derivatives products that can be decomposed as a set of forward rates, such as caps (collection of caplets), floors or swaptions. LIBOR market model is compatible with Black's formula for caps. However, it is not compatible with Black's formula for swaptions. For this reason, numerical approximations appeared and it can be dealt with.

LIBOR market model does not produce perfectly correlated forward rates, which is an improvement in comparison to short rate models. The real data however shows, that the level of correlation between forward rates (especially with two consecutive rates).

"In the most basic setting of the LIBOR market model, the only source of randomness in the market is a d-dimensional standard Brownian motion. (...). There exists a spot martingale measure Q, under which all bonds discounted with the money market account B are martingales." [16]

In most cases, the term LIBOR market model refers to a lognormal LIBOR market model with deterministic volatility. Over time, several extensions of LIBOR market model were added to deal with some of its imperfections. The extensions are described in more detail in the section 3.5, and we distinguish:

- Standard lognormal LMM with deterministic volatility - assumes that the evolution of a forward interest rate has a lognormal distribution
- LMM with stochastic volatility
- LMM with Local Volatility Extension
  I. Enhanced with a constant elasticity of variance (CEV), proposed by Andersen and Andreason
  II. Enhanced with displaced diffusion (DD), when the lognormal process is replaced by displaced diffusion, proposed by Joshi and Rebonato

Brigo and Mercurio distinguish the lognormal forward-LIBOR market model and the lognormal swap market model.

Considering the negative interest rates observable on markets, they produce a problem for LMM, since "The main assumption underlying these models is that each rate is log-normal under the corresponding forward measure." [3]

3.4.1 LIBOR market model formula

We will follow the definition of Brigo and Mercurio. Let $t = 0$ be the present. We will consider a set of moments in time $\mathcal{T} := \{T_0, ..., T_M\}$ with pairs of dates that signify the time expiry-maturity (T_{i-1}, T_i). Let τ_i be the year fraction, that corresponds to a certain pair of expiry-maturity (T_{i-1}, T_i) and τ_0 being the time from settlement to T_0. We set T_{-1} :=0 and that gives that T_i will be the time period in years.

We assume a general forward rate $F_k(t) = F(t, T_{k-1}, T_k), k = 1, ..., M$, that is defined until time T_{k-1}, when it corresponds to the spot rate $F_k(T_{k-1}) = L(T_{k-1}, T_k)$.

We consider the price of the bond with maturity in the same time as the maturity of a forward rate to be $P(\cdot, T_k)$. We associate to this price a probability measure Q^k.

Under this forward measure Q^k, $F_k(t)$ has the dynamics:

$$dF_k(t) = \underline{\sigma}_k(t) F_k(t) dW^k(t), t \leq T_{k-1}, \tag{3.19}$$

where $F_k(t)$ is a martingale (without a drift) and where $W^k(t)$ represents a M-dimensional Brownian motion with the covariance $\rho = \rho_{i,j},\ \ i, j = 1, ..., M$,

$\rho dt = dW^k(t)dW^k(t)^T$ and $\underline{\sigma}_k(t)$ is the horizontal M-dimensional vector of the volatility of the forward rate $F_k(t)$.

Each vector $\underline{\sigma}_j(t)$ has on j-th place only one non-dead element $\sigma_j(t)$.

We can transform the previous equation:

$$dF_k(t) = \sigma_k(t)F_k(t)dW_k^k(t), t \leq T_{k-1}. \tag{3.20}$$

where $W_k^k(t)$ represents the k-th element of the M-dimensional Brownian motion W^k.

We can simulate the forward rates with the following:

$$dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{j,k}\tau_j\sigma_k(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)W_k(t), \tag{3.21}$$

where $\rho_{j,k}$ is the correlation between forward rates F_j and F_k.

We consider a forward rate $F_k(t)$, with a probability measure Q^k that is different from the measure Q^i, we will get the following forward-measure dynamics for the LIBOR market model:

$$\begin{aligned}
&i < k, t \leq T_i : dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dW_k(t),\\
&i = k, t \leq T_{k-1} : dF_k(t) = \sigma_k(t)F_k(t)dW_k(t),\\
&i > k, t \leq T_{k-1} : dF_k(t) = -\sigma_k(t)F_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dW_k(t)
\end{aligned} \tag{3.22}$$

where W_k is a standard Brownian motion under the measure Q^i.

For $i = k$, the distribution of the forward rates is lognormal, and therefore we denote this model as *lognormal* LIBOR market model. For $i \neq k$, the distribution is not lognormal and is not known.

Another way is to use the spot-measure dynamics.

The numeraire in this case is B_d, which is a bank account that is balanced at times defined in the tenor structure and is called the spot LIBOR measure. It has the following form:

$$B_d(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=1}^{\beta(t)-1} P(T_{j-1}, T_j)(3.23)}$$

and leads to the following equation

$$dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=\beta(t)}^{k}\frac{\tau_j\rho_{j,k}\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dW_k^d(t) \tag{3.24}$$

3.4.2 Overview of the LIBOR market model - Strengths and weaknesses

Market models can be seen as a breaktrough in the field of interest rate modeling and thus the pricing of interest rate derivatives. The long term use of Black's model for swaption valuation could be replaced with a model that is theoretically valid. Since the 1990s, the LIBOR market model has become a market standard for several reasons:

- it is the first model that is consistent with Black's formula
- it models rates that can be directly observed on markets
- it is relatively easy to calibrate

It still has several imperfections. It cannnot properly model the volatility smile. It assumes the lognormal distribution of forward rates, and this assumption is very oversimplifying if compared to real market data.

3.5 Dealing with smile in LMM

"In 1998, after Russian crisis, cap and swaption markets started to show evident volatility smile and skew"[5]. The problem with the standard lognormal LMM is that it does not cover the phenomenon of volatility smile/skew observable on markets and the volatility structure produced in the LIBOR market model is flat. However, it can be dealt with in several ways. The possible solutions are:

- The shifted lognormal model
- The constant elasticity of variance model
- Stochastic volatility LIBOR market model
- SABR-LIBOR market model

The **shifted lognormal** model lets the forward rate evolve under:

$$dF_j(t) = \beta(t)(F_j(t) - \alpha)dW_t \tag{3.25}$$

The distribution becomes shifted lognormal and the density of this distribution can be found in [2]. The α parameter produces a difference in the volatility sturcture. The shifted lognormal model has an interesting feature of non-zero probability of the negative interest rates to be below zero, that today does not seem as such an inconvenience.

The **CEV model** lets F_j follow:

$$dF_j(t) = \sigma_j(t)[F_j(t)]^\gamma dW_t, F_j = 0 \tag{3.26}$$

with γ lying between 0 and 1.

The **stochastic volatility models** assume that the ϑ parameter in the following function is a stochastic function:

$$dF(t, T_1, T_2) = \vartheta(t, F(t, T_1, T_2))dW_t \tag{3.27}$$

SABR is a simple model designed to describe the price evolution of one asset. It is able to capture the volatility smile on the markets. It belongs to a class of constant elasticity of variance models with stochastic volatility. As proposed by Rebonato, the LMM-SABR combines the advantages of both models. The existence of a developed market for traded derivatives is necessary for this calibration.

The SABR model dynamics:

$$dL(t) = \sigma(t)C(L(t))dW_1(t), d\sigma(t) = \alpha\sigma dW_2(t) \tag{3.28}$$

Chapter 4

LMM calibration

Once the model has been chosen, the parameters need to be set. Following the work of Myska [13], we can distinguish between a static and a dynamic method of calibration. The calibration performed with the help of historical values is considered to be *dynamic*, and the calibration to current market data is called a *static* method of calibration. The calibration can be also done by estimating the parameters from experience. The difference between the predicted price and the price observed on markets is minimized using a numerical optimization algorithm. The number of factors (parameters) needs to be specified at the beginning. The ideal calibration should indeed estimate the parameters of the model as precisely as possible. However, in general, the more precise is the model, the more demanding it is in terms of computational cost.

LIBOR market model is specified when correlation functions and instatnaneous volatility is set.

4.1 Correlation matrix

The instantaneous correlations in this this model are a representation of the correlation between the changes of the forward rates. Therefore for example, the correlation between $F_2(t)$ and $F_3(t)$ is:

$$\rho_{2,3} = \frac{dF_2(t)dF_3(t)}{Std(dF_2(t))Std(dF_3(t))} \tag{4.1}$$

The whole correlation structure is given in a form of a matrix.

In general, a correlation matrix is:

1. symmetrical: $\rho_{i,j} = \rho_{j,i}$ for all i, j

2. normalized: $\rho_{i,j} \leq 1$ for all i, j

3. positive semidefinite: $x'\rho x \leq 0$ for all x

4. maximum correlation for maximum dependance: $\rho_{i,i} = 1$ for all i

   The correlation matrix has $M(M-1)/2$ values. In LIBOR market model, the matrix can produce a problem with too many values, that have excessive demanding regarding the computational cost. Either the matrix is used in the full form, or we can perform some kind of a reduction, that tries to use less values, but at the same time to minimize the information loss.

   In LIBOR market model, the correlation of the forward rates is positive. The correlation matrix can be both an input and an output to the model.

   Instantaneous correlation matrix is the representation of the correlation between various forward rates. On of the reason we do not use the full rank correlation matrix is, that the the structure of LIBOR forward rates shows a high level of correlation. This fact is used in the model and the correlation matrix is usually used in a reduced rank form.

   Rebonato proposes a reduction of parameters. We re-write the correlation matrix as

$$\rho = PHP',$$

   with P being an orthogonal matrix, $PP' = I_M$ and H being a diagonal matrix with the eigenvalues of ρ. We denote Λ the square roots of components of H and $A = P\Lambda$ and we get:

$$AA' = \rho, A'A = H$$

   The correlation ρ corresponds to $\rho^B = BB'$ defined below.

   The i-th row component is suggested to be for $i = 1, ..., M$:

$$b_{i,1} = cos\theta_{i,1}$$
$$b_{i,k} = cos\theta_{i,k}sin\theta_{i,1}...sin\theta_{i,k-1} \text{ for } 1 < k < n$$
$$b_{i,n} = sin\theta_{i,1}...sin\theta_{i,n-1}$$

In case of $n = 2$, the matrix B will have the following form:

$$\mathrm{B} = \begin{pmatrix} cos\theta_1 & sin\theta_1 \\ cos\theta_2 & sin\theta_2 \\ \vdots & \vdots \\ cos\theta_M & sin\theta_M \end{pmatrix}$$

and the matrix ρ^B will have the elements

$$\rho^B_{i,j} = cos(\theta_i - \theta_j)$$

and that leaves us with a correlation matrix of M parameters.

If the correlation matrix is an output, it is theoretically computed and therefore is very smooth. Figure 4.1 is an example of a theoretically computed correlation matrix. We can see that the depiction of the theoretical matrix is symmetrical and it is full of ones on the diagonal.

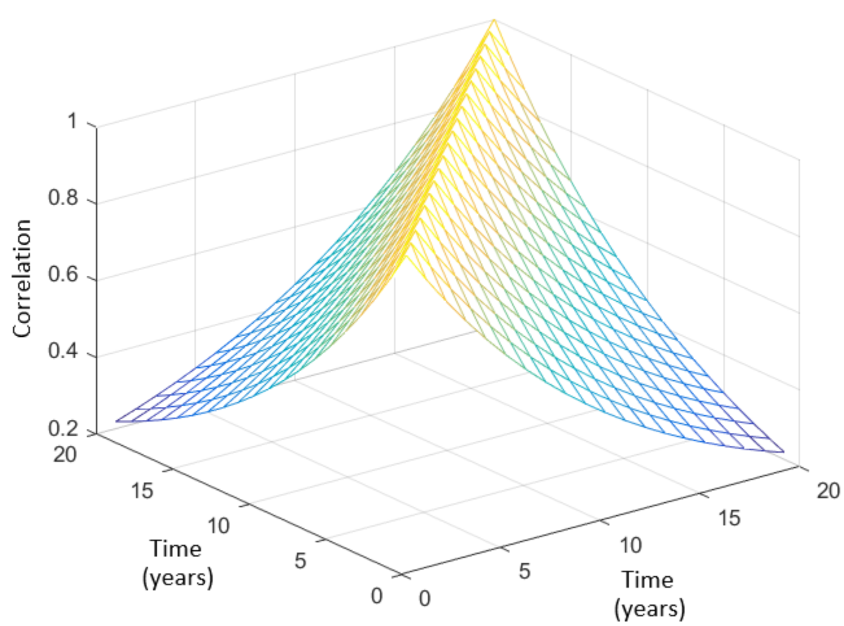

Figure 4.1: A theoretical correlation matrix. Source: Matlab

4.2 Instantaneous volatility

$\sigma_k(t)$ represents the parameter of the instantaneous volatility. We will assume that the instantaneous volatility of the forward interest rate is a continuous function by parts.

Table 4.1. shows a table of instantaneous volatilities, according to Brigo and Mercurio:

Table 4.1: Instantaneous volatilities

Instant. Vols					
Fdw. rate	Time intervals				
	$(0, T_0)$	(T_0, T_1)	(T_1, T_2)	$\dots$	(T_{M-2}, T_{M-1})
$F_1(t)$	$\sigma_{1,1}$	Dead	Dead	$\dots$	Dead
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	Dead	$\dots$	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\dots$	$\sigma_{M,M}$

We will try to reduce the high number of parameters.

Assuming that:

$$\sigma_k(t) = \sigma_{k,\beta(t)} := \eta_{k-(\beta(t)-1)}, \tag{4.2}$$

Table 4.2 is the representation of the parameters after the use of this equation.

Table 4.2: Instantanous volatilities 2

Instant. Vols					
Fdw. rate	Time intervals				
	$(0, T_0)$	(T_0, T_1)	(T_1, T_2)	$\dots$	(T_{M-2}, T_{M-1})
$F_1(t)$	η_1	Dead	Dead	$\dots$	Dead
$F_2(t)$	η_2	η_1	Dead	$\dots$	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$F_M(t)$	η_M	η_{M-1}	η_{M-2}	$\dots$	η_1

If again, we assume:

$$\sigma_k(t) = \sigma_{k,\beta(t)} := \phi_k \psi_{k-(\beta(t)-1)} \tag{4.3}$$

where $\beta(t) = k + 1$ for $T_{k-1} \leq t \leq T_k$.

From this, we get the Table 4.3.

Table 4.3: Instantaneous volatilities 3

Instant. Vols					
Fdw. rate	Time intervals				
	$(0, T_0)$	(T_0, T_1)	(T_1, T_2)	$\ldots$	(T_{M-2}, T_{M-1})
$F_1(t)$	$\phi_1\psi_1$	Dead	Dead	$\ldots$	Dead
$F_2(t)$	$\phi_2\psi_2$	$\phi_2\psi_1$	Dead	$\ldots$	Dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$F_M(t)$	$\phi_M\psi_M$	$\phi_M\psi_{M-1}$	$\phi_M\psi_{M-2}$	$\ldots$	$\phi_M\psi_1$

4.2.1 Swaption volatility matrix

Table 4.4 shows us the swaption volatility matrix, taken from Thomson Reuters on 11/04/2018. This matrix will be serving as an input for the calibration of the model. This matrix contains swaptions expiries in rows and underlying swap maturities in columns. For the calibration, we will be using swaption volatilities for at-the-money swaptions with maturities of: 1,2,3,4,5,7, and 10 years. with the underlying swap length of 1,2,3,4,5,7,10 years. This matrix contains information about the market of swaptions denominated in Euro. The prices of swaptions are included in the swaption volatility matrix taken from the market.

How accurate is the swaption volatility matrix? One of the problems of matrix can be that it contains data from both liquid and non-liquid markets. Usually, the swaptions with closer exercise dates are considered to be more liquid and therefore the price changes accurately with the market situation. However, the swaptions with exercise dates further from today do not have to be liquid and therefore their price does not necessarily reflect the situation on the market accurately. However, in this thesis, we will be not taking into account this issue, because there is no way for us to get a more accurate information about the swaptions market.

The swaption volatility is usually approximated for the purpose of computations. We will introduce **Rebonato's formula** for volatility of swaptions,

EUR	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M	0	0	333.66	100.48	68.16	54.33	46.54	41.64	37.28	34.38	26.22	23.42	22.81	22.76
3M	0	0	208.81	95.71	70.38	57.35	49.77	44.68	40.07	36.90	28.47	25.47	24.61	24.31
6M	0	0	146.01	88.71	68.86	57.71	51.10	46.39	42.15	39.17	30.81	28.00	27.18	26.92
1Y	0	181.10	97.95	76.30	64.80	56.31	50.99	46.72	42.85	40.09	32.65	30.37	29.82	29.72
2Y	142.01	78.87	67.28	59.64	53.81	49.57	46.14	43.77	41.32	39.32	33.78	32.21	32.01	32.09
3Y	84.40	62.84	56.81	52.15	48.88	45.94	43.09	41.48	39.79	38.35	33.76	32.89	32.89	33.18
4Y	70.56	56.33	51.48	48.04	44.87	42.64	40.57	39.35	38.17	37.16	33.36	32.85	32.94	33.35
5Y	60.29	50.23	47.20	44.27	42.09	40.28	38.91	37.79	36.82	36.07	33.01	32.79	32.93	33.42
7Y	47.64	42.10	40.25	38.52	37.23	36.30	35.55	35.05	34.72	34.45	32.28	32.28	32.49	32.90
10Y	36.63	34.83	34.32	33.80	33.48	33.39	33.36	33.38	33.42	33.47	32.36	32.30	32.47	32.81
15Y	32.57	32.70	33.01	33.32	33.67	34.04	34.43	34.80	35.11	35.36	34.17	33.53	33.41	34.36
20Y	35.38	36.62	37.37	38.04	38.58	38.95	39.17	39.28	39.27	39.17	36.81	35.16	35.80	34.80

Table 4.4: Swaption volatilities, Source: Thomson Reuters, 11/04/2018

which defines the approximated volatility as:

$$(v_{\alpha,\beta}^{LFM})^2 = \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\mathrm{d}t \tag{4.4}$$

This volatility $(v_{\alpha,\beta}^{LFM})^2$ can be understood as an approximation for the Black's volatility of swaptions of the swap rate $S_{\alpha,\beta}$ and where $w_i(t)$ corresponds to:

$$w_i(t) = \frac{\tau_i FP(t, T_\alpha, T_i)}{\sum_{k=\alpha+1}^{\beta} \tau_k FP(t, T_\alpha, T_k)} = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}} \tag{4.5}$$

4.3 Cascade calibration suggested by Brigo, Mercurio

Brigo and Mercurio suggest a swaption calibration and illustrate it on an example of exactly six swaptions for simplification.

The following equation combines the market swaption volatility and Rebonato's formula:

$$(V_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \frac{\omega_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2 T_\alpha} \sum_{h=0}^{\alpha} \tau_{h-1,h}\sigma_{i,h+1}\sigma_{j,h+1} \tag{4.6}$$

where $\tau_{h-1,h} = T_h - T_{h-1}$ and $T_{-1} = 0$.

Table 4.5: Table of swaption volatilities with six swaptions

Length Maturity	1 year	2 years	3 years
$T_0 = 1$ year	$V_{0,1}$, $\sigma_{1,1}$	$V_{0,2}$, $\sigma_{1,1}$ $\sigma_{2,1}$	$V_{0,3}$, $\sigma_{1,1}$ $\sigma_{2,1}$ $\sigma_{3,1}$
$T_1 = 2$ years	$V_{1,2}$, $\sigma_{2,1}$ $\sigma_{2,2}$ $\sigma_{3,1}$ $\sigma_{3,2}$	$V_{1,3}$ $\sigma_{2,1}\sigma_{2,2}$	-
$T_2 = 3$ years	$V_{2,3}$ $\sigma_{3,1}\sigma_{3,2}$ $\sigma_{3,3}$	-	-

Following equations will be stemming from the table 4.5. and from the equation 4.6.

It will lead us to the following evaluation with respect to rows:

(a) We start with the first row which leads us to evaluate sequently $\sigma_{1,1}$, $\sigma_{2,1}$ and $\sigma_{3,1}$.

i. $\sigma_{1,1}$

We begin from $V_{0,1}$ which represents a swaption that matures at time T_0 and the underlying swap ending at T_1

$$S_{0,1}(0) = w_1(0)F_1(0) \qquad \Rightarrow \qquad (V_{0,1})^2 \approx \sigma_{1,1}^2.$$

The parameter $\sigma_{1,1}$ is calibrated.

ii. $\sigma_{2,1}$

$V_{0,2}$ involves two rates $F_1,\ F_2$.

For the only unknown parameter $\sigma_{2,1}$, we assume that is a solution to an algebraic equation of second order (assumes a positive solution):

$$S_{0,2}(0)^2(V_{0,2}^2) \approx w_1(0)^2F_1(0)^2\sigma_{1,1}^2 + w_2(0)^2F_2(0)^2\sigma_{2,1}^2$$

$$+2\rho_{1,2}w_1(0)F_1(0)w_2(0)F_2(0)\sigma_{1,1}\sigma_{2,1}.$$

iii. $\sigma_{3,1}$

$V_{0,3}$ involves the rates $F_1,\ F_2$ and the rate F_3.

A solution to an algebraic second order equation

$$S_{0,3}(0)^2(V_{0,3})^2 \approx w_1(0)^2F_1(0)^2\sigma_{1,1}^2 + w_2(0)^2F_2(0)^2\sigma_{2,1}^2$$

$$+w_3(0)^2F_3(0)^2\sigma_{3,1}^2$$

$$+2\rho_{1,2}w_1(0)F_1(0)w_2(0)F_2(0)\sigma_{1,1}\sigma_{2,1}$$

$$+2\rho_{1,3}w_1(0)F_1(0)w_3(0)F_3(0)\sigma_{1,1}\sigma_{3,1}$$

$$+2\rho_{2,3}w_2(0)F_2(0)w_3(0)F_3(0)\sigma_{2,1}\sigma_{3,1}.$$

for the unknown $\sigma_{3,1}$

(b) We shift to the second row.

$V_{1,2}$, the only rate considered is F_2.

the two subintervals $[0, T_0]$ and $[T_0, T_1]$

i. $\sigma_{2,2}$

$$T_1(V_{1,2})^2 \approx \tau_0\sigma_{2,1}^2 + \tau_1\sigma_{2,2}^2,$$

where the unknown is $\sigma_{2,2}$.

ii. $\sigma_{3,2}$

we move to $V_{1,3}$.

$$T_1 S_{1,3}(0)^2(V_{1,3}^2) \approx w_2(0)^2 F_2(0)^2(\tau_0\sigma_{2,1}^2 + \tau_1\sigma_{2,2}^2)$$

$$+w_3(0)^2 F_3(0)^2(\tau_0\sigma_{3,1}^2 + \tau_1\sigma_{3,2}^2)$$

$$+2\rho_{2,3}w_2(0)F_2(0)w_3(0)F_3(0)(\tau_0\sigma_{2,1}\sigma_{3,1} + \tau_1\sigma_{2,2}\sigma_{3,2}).$$

with the only unknown $\sigma_{3,2}$.

(c) And finally the only one term left in the third row $V_{1,2}$, the only rate considered is F_2 with the three subintervals $[0, T_0]$, $[T_0, T_1]$ and $[T_1, T_2]$

i. $\sigma_{3,3}$

$$T_2(V_{2,3})^2 \approx \tau_0\sigma_{3,1}^2 + \tau_1\sigma_{3,2}^2 + \tau_2\sigma_{3,3}^2.$$

with the only unknown $\sigma_{3,3}$.

4.4 Calibration and the results

The initial interest rates can be seen in Table 4.6.

We will now perform a calibration of the LIBOR market model to market data. The model minimises the difference between the theoretical price of the swaption and the actual price of the swaption on markets. As alrady said, the data involves the swaption volatility matrix and zero rates. The source of the data is Thomson Reuters and Patria. The swaption volatility matrix involves EUR at-the-money swaptions and was taken on 11/04/2018.

The volatility functional form is given by

$$\sigma_i(t) = k_i[(a(T_i - t) + b)e^{c(T_i - t)} + d)]. \tag{4.7}$$

Table 4.6: Table of initial rates

Expiry	Interest rate
1y	-0.7003
2y	-0.6083
3y	-0.4513
4y	-0.2733
5y	-0.0917
7y	0.2120
10y	0.5507

Iteration	Func-count	f(x)	Norm of step	First-order optimality
0	6	12.5536		4.9
1	12	6.28148	0.365931	1.64
2	18	3.60959	0.586126	4.33
3	24	2.34872	0.344968	0.21
4	30	2.27303	0.123614	0.0172
5	36	2.26959	0.125555	0.00273
6	42	2.26811	0.278553	0.0108
7	48	2.26788	0.207705	0.0158
8	54	2.26787	0.0276523	0.00152

Figure 4.2: The iterations. Source: Matlab

The correlation functional form is given by

$$\rho_{i,j} = e^{-\beta|i-j|}. \tag{4.8}$$

The difference in the predicted and in market volatilities are minimized using a nonlinear least-squares algorithm.

The criteria for the parameters a, b, c and d are:

$(a+d)>0$; $c>0$; $d>0$;

The computation of the iterations can be seen on Figure 4.2.

The computed correlation matrix is displayed in the Figure 4.3.

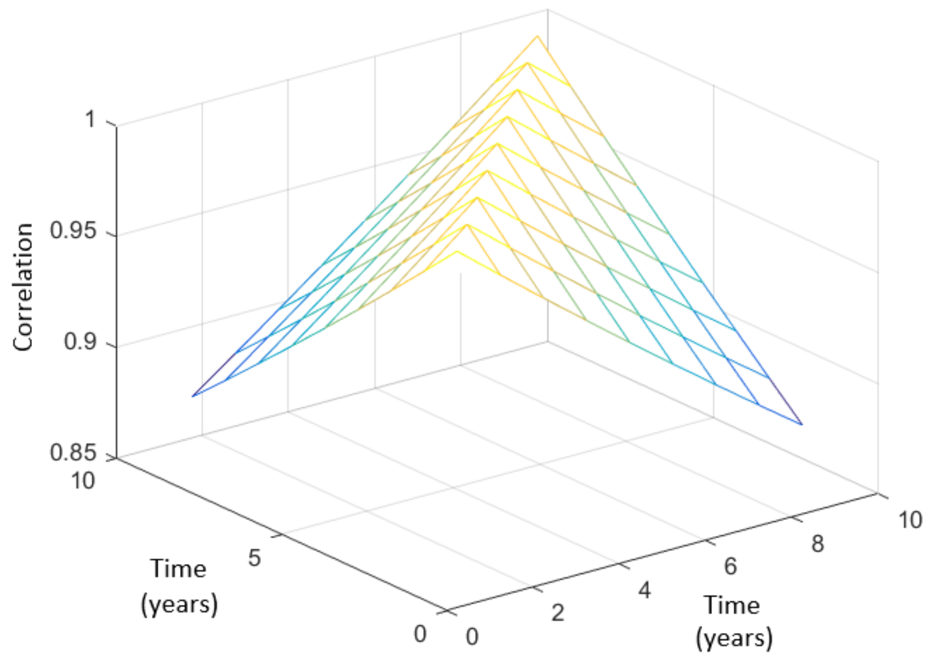

Figure 4.3: The correlation function. Source: Matlab

The quality of the fit is defined by the difference between the the theoretical price of the swaption and the actual price of the swaption. The optimization algorithm stops, after it reaches the function tolerance and thus the difference is minimized to an acceptable value. This is presented on Figure 4.4.

```
Optimization stopped because the relative sum of squares (r) is changing
by less than options.FunctionTolerance = 1.000000e-05.

Optimization Metric                                        Options
relative change r =    3.52e-06              FunctionTolerance =    1e-05  (selected)
```

Figure 4.4: LMM calibration - the optimization details. Source: Matlab

The program computes the parameters:

$$
\begin{aligned}
\mathrm{a} &= 0.9238,\\
\mathrm{b} &= 0.3635,\\
\mathrm{c} &= 0.7680,\\
\mathrm{d} &= 0.2945,\\
\beta &= 0.0158.
\end{aligned}
$$

The volatility function is presented at the Figure 4.5.

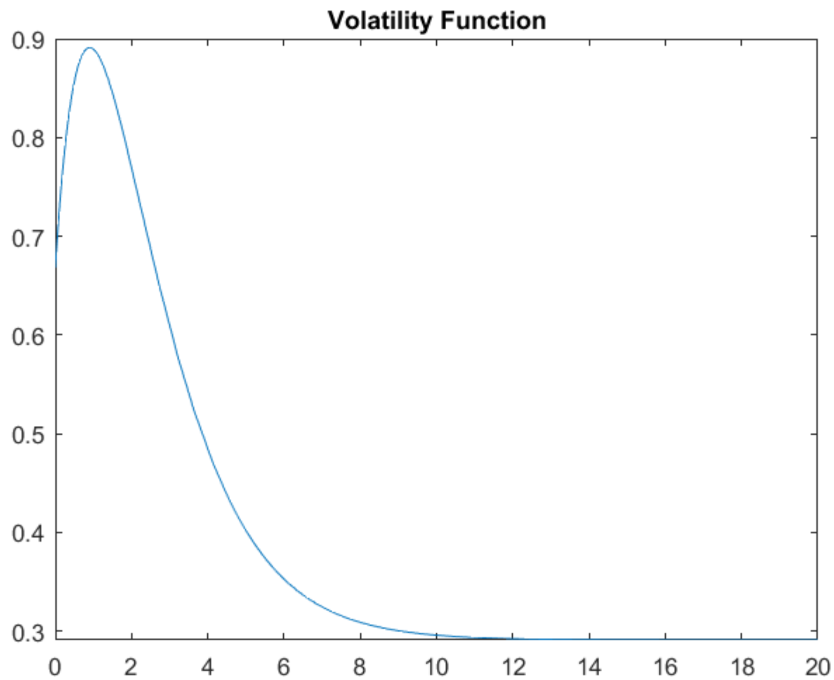

Figure 4.5: The volatility function. Source: Matlab

Figure 4.6 is a representation of the prediction of the model, which is again a three-dimensional graph, with Tenor in years on the x-axis, time on y-axis and the interest rate on the z-axis.

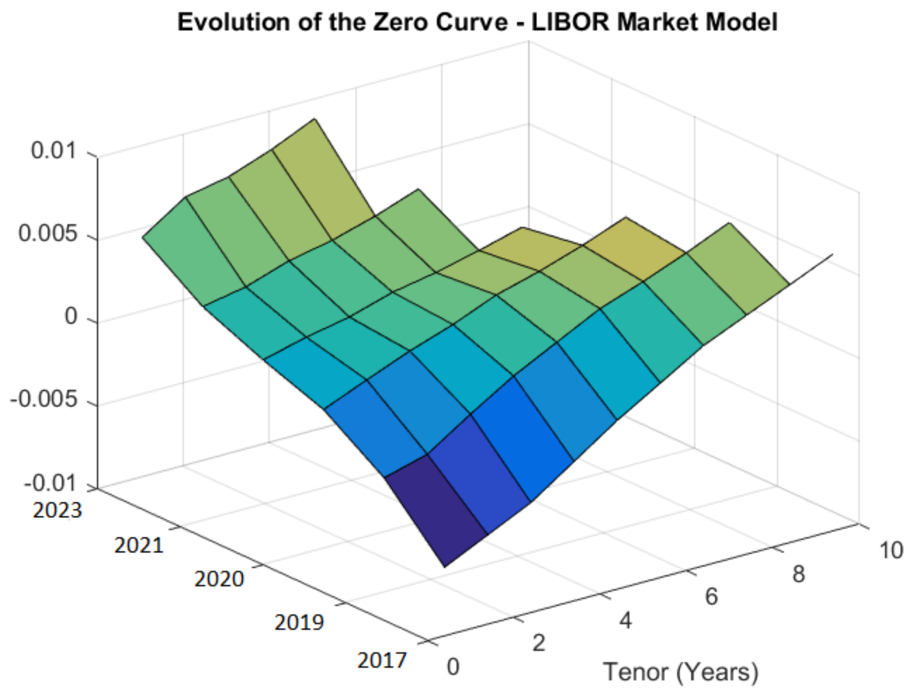

Figure 4.6: Output - the evolution of the zero curve. Source: Matlab

Chapter 5

Hull-White calibration

For comparison, we will use a simple short rate model. As already presented, the Hull-White model is a one-factor short rate model. It has one source of uncertainty and is capable of fitting the initial term structure. The process is driven by following equation:

$$dr(t) = [\vartheta(t) - a(t)r(t)]dt + \sigma(t)dW(t) \tag{5.1}$$

The input data used to calibration of the model will be identical to the data used in the previous chapter. We will proceed to a calibration of the model. The function *lsqnonlin* used to minimize the difference between the market data and modeled values.

The parameters computed were:

$$a = 1.4{*}10^{-7}$$
$$\sigma = 0.0042$$

The algorithm usually stopped after 7 iterations.

The optimization details are presented by Figure 5.1.

Figure 5.2 shows us the zero curve from April 2018. Figure 5.3 shows us the prediction of the model, which is again a three-dimensional graph, with Tenor in years on the x-axis, time on y-axis and the interest rate on the z-axis. It shows how the The Hull-White model produces the forecast.

Figure 5.3 shows how the The Hull-White model produces the forecast.

```
Optimization stopped because the relative sum of squares (r) is changing
by less than options.FunctionTolerance = 1.000000e-05.

Optimization Metric                              Options
relative change r =    7.10e-06           FunctionTolerance =    1e-05 (selected)
```

Figure 5.1: Hull White calibration - optimization details. Source: Matlab

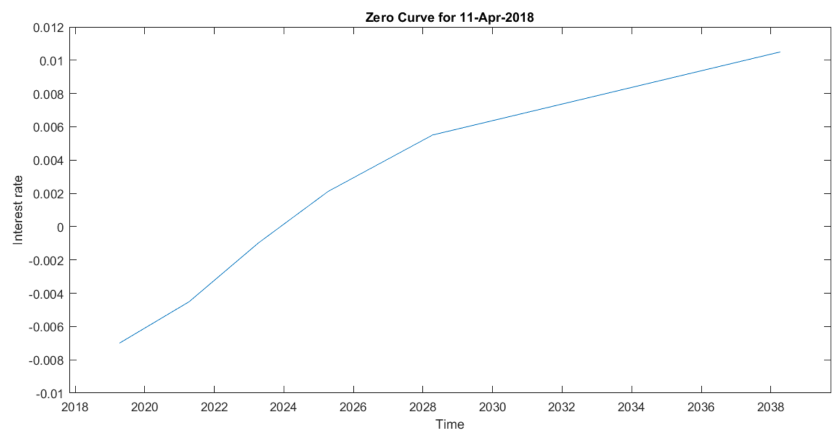

Figure 5.2: Hull White zero curve. Source: Matlab

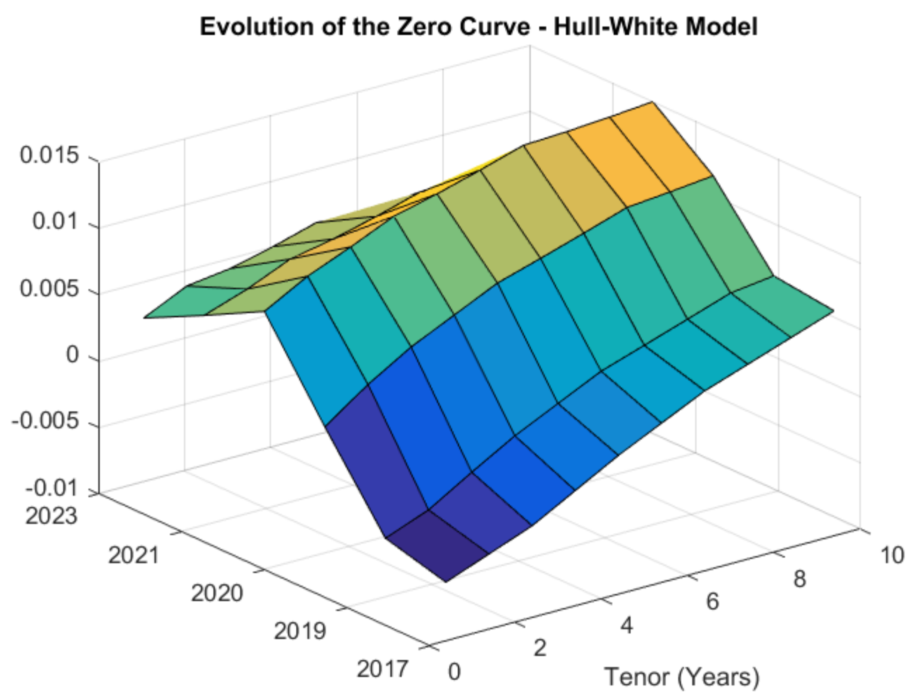

Figure 5.3: Hull White calibration. Source: Matlab

Chapter 6

Pricing a swaption

A swaption is an interest rate derivative with a finite lifetime and with the underlying asset being a swap. We will be focusing on two types of swaptions: European and Bermudan. To remind, the European (vanilla) type of swaption represents a much simpler contract than a Bermudan swaption. The European swaption has exactly one date when the holder can decide, whether to enter to a contract of a swap. On the other hand, the Bermudan type of contract has several dates that offer this possibility to the holder and is therefore more complicated.

The contract of a Bermudan swaption gives the holder another advantage in comparision to the European swaption. This advantage should be reflected by the market and should value the Bermudan swaption above the European swaption (in theory).

In this Chapter, we will focus on pricing these types of contracts with LIBOR market model and with Hull-White model and compare the results of the two.

6.1 Pricing a swaption in LIBOR market model

The aim of this chapter is to price a swaption with LIBOR market model.

„The LIBOR market model framework does not allow for an exact closed-form swaption pricing formula.“ As already stated, LIBOR market model does not provide the analytical solution to price swaptions. To solve this issue, the suggested method are numerical approximations.

In general, Monte Carlo simulation represents a class of algorithms, where pseudorandom numbers are used. They are typically used, when there are is no analytical solution for computation. The method is consuming regarding computational capacity. Monte Carlo simulations usually generate a number of scenarios. Each simulation allows to form many paths of a stochastic process.

Black's formula became the market standard for pricing swaptions for practitioners. It was not stated theoretically. LIBOR market model was developed to fill this gap.

We have already presented the forward rate dynamics in the model under the measure Q^{α} :

$$dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=\alpha+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dW_kd(t) \tag{6.1}$$

With $M = \beta - \alpha$ forward rates, we need to perform m realizations of $F_{\alpha+1}(T_\alpha), ..., F_\beta(T_\alpha)$.

By applying the Ito's formula, we can get

$$lnF_k^{\Delta t}(t+\Delta t) = lnF_k^{\Delta t}(t)+\sigma_k(t)\sum_{j=\alpha+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j^{\Delta t}(t)}{1+\tau_jF_j^{\Delta t}(t)}\Delta t-\frac{\sigma_k(t)^2}{2}\Delta t+\sigma_k(t)(W_k(t+\Delta t)-W_k(t)) \tag{6.2}$$

and that brings us to

$$E^{\alpha}\{|lnF_k^{\Delta t}(T_\alpha) - lnF_k(T_\alpha)|\} \leq c(T_\alpha)\Delta t \tag{6.3}$$

for all $\Delta t \leq \delta_0$.

A swaption is an OTC derivative and therefore does not have to be standardized. We will consider a swaption with an exercise date of five years from now and with the underlying swap with the length of five years.

The parameters of the swaption that we will be pricing are:

The starting date of the swaption: 11/04/2018

The exercise date of the swaption: 11/04/2023

The maturity of the swaption: 11/04/2028

The strike price of the instrument: 0.045

6.2 Pricing a swaption in Hull-White model

The input data used to price the swaption will be the same as the data used in the previous chapter. The swaption in this model will be priced with the aid of a trinominal tree.

The **trinominal tree** that models the evolution of the interest rate can be constructed as follows:

We denote T the time horizon, and the times $0 \leq t_0 \leq t_1 \leq ... \leq T$ and set $\Delta t_i = t_{i+1} - t_i$. The indexes i, j represent the tree nodes. The state $x_{i,j}$ is the state of the process in the node i, j. In general, we can model a diffusion process X

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{6.4}$$

where we set a constant step Δx_i and $x_{i,j} = j\Delta x_i$. At time t_i, we are at the point of $x_{i,j}$. From this point, we pass to time t_{i+1} and with the following probabilities p, we will get to the point:

with probability $p_u : x_{i+1,k+1} = (k+1)\Delta x_{i+1}$
with probability $p_m : x_{i+1,k} = k\Delta x_{i+1}$
with probability $p_d : x_{i+1,k-1} = (k-1)\Delta x_{i+1}$

At time t_{i+1}, we need to find the following mean and variance considering the probabilities:
$M_{i,j} = E\{X(t_{i+1}) \mid X(t_i) = x_{i,j}\}$
$V_{i,j}^2 = \text{Var}\{X(t_{i+1}) \mid X(t_i) = x_{i,j}\}$

By knowing that $p_u + p_m + p_d = 1$ and by setting $\eta_{j,k} = M_{i,j} - x_{i+1,k}$ and $\Delta x_{i+1} = V_i\sqrt{3}$, Brigo and Mercurio conclude:

$p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}$
$p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2}$
$p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}$

In the case of Hull-White trinominal tree, the time differences between t_i and t_{i+1} do not have to be of the same length.

The mean and the variance of the process with respect to the filtration F_s:

$E\{r(t) \mid F_s\} = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$
$\text{Var}\{r(t) \mid F_s\} = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}]$

We have $x(t)$ that equals $r(t) - \alpha(t)$, and thus

$M_{i,j} = E\{x(t_{i+1}) \mid x(t_i) = x_{i,j}\} = x_{i,j}e^{-a\Delta t_i}$
$V_i^2 = \text{Var}\{x(t_{i+1}) \mid x(t_i) = x_{i,j}\} = \frac{\sigma^2}{2a}[1 - e^{-2a(\Delta t_i)}]$

The next step is to describe the displacement that we need to model the interest rate r. It will be denoted α_i and the present value of an instrument that pays 1 at node (i, j) and has no payoff if this node is not reached is denoted $Q_{i,j}$.

$Q_{i+1,j} = \sum_h Q_{i,h} q(h, j) exp(-(\alpha_i + h\Delta x_i)\Delta t_i)$

Finally, Brigo and Mercurio conclude the value of $r_{i,j}$ to be equal to $x_{i,j} + \alpha_i$ and the process of $r_{i,j}$ has the following progress:

with probability $p_u : r_{i+1,k+1} = x_{i+1,k+1} + \alpha_{i+1}$
with probability $p_m : r_{i+1,k} = x_{i+1,k} + \alpha_{i+1}$
with probability $p_d : r_{i+1,k-1} = x_{i+1,k-1} + \alpha_{i+1}$

6.3 Review of the results

The swaption parameters are the same, so it is possible to compare the results of the two models. The results are summarized in Table 6.1.

Table 6.1: The price of the European swaption

Strike price	0.055	0.045	0.030	0.010	0.005	0.001
LIBOR market model	0.0283	0.0541	0.1120	1.3557	2.4498	4.4791
Hull-White model	0.00	0.00	0.1181	1.5385	3.069	4.5999

These results are a product of a Monte Carlo simulation and thus differ in every trial. Table 6.1 shows the average of 5 results.

We can see that the two models price the swaption differently. In the Hull-White model, the swaption price becomes 0 when the price is 0.045 or higher. However, if the strike price declines, Hull-White produces higher prices than the LIBOR market model.

The smaller the strike price, the smaller becomes the difference in pricing. This is what we can see with the strike price 0.001 and lower, the difference in results of the two models is small. However, when the strike price is higher than 0.01, the difference becomes more observable. The prices are also presented graphically on Figure 7.1.

Chapter 7

Pricing a Bermudan swaption

Swaptions pricing depends on the type of swaption. In general, there exists a formula for European type of derivative. However, it is not possible to price derivatives with early exercise option, such as American or Bermudan type of derivative like this, and in this case, numerical approximations are selected. Swaptions have been priced with Black's formula for a long time and LIBOR market model is compatible with this formula.

Bermudan swaptions are swaptions with early exercise option. The valuation of such instruments is tricky and methods have been developed specifically for the purpose of valuing the Bermudan swaptions. We will describe the approach of Longstaff and Schwartz and the proposed Monte Carlo regression.

The first note is that they suggest using bond prices and their volatilities. The approach can be described with the example of a Bermudan swaption with an underlying swap with the maturity of 10 years. There will be 19 exercise dates, if the swap resets twice a year.

The zero coupon bond prices are put in a vector:

$$P(\cdot, 0.5y), P(\cdot, 1y), ..., P(\cdot, 10y)$$

The vector of 20 dimensions gets smaller in dimensions after every exercise date passes.

The corresponding interest rate, which is compounded continously:

$$r(t) \approx -2lnP(t, t + 0.5y)$$

The approximated dynamics:

$$dP(t, T_i) = -2lnP(t, t + 0.5y)P(t, T_i)dt + \sigma_{P_i}(t)P(t, T_i)dW_i(t) \tag{7.1}$$

where $T_i = 0.5i$ and $i = 1, ..., 20$.

The Brownian motion W_i is correlated, the vector W is twenty-dimensional:

$$dZ_i dZ_j = exp(-k \mid i - j \mid)dt$$

Finally, the algorithm basis function at time T_i,

$$1, P(\cdot, T_i), ..., P(\cdot, T_{20}), \frac{1 - P(T_i, T_{20})}{\sum_{j=i+1}^{20} 0.5P(T_i, T_j)}, [\frac{1 - P(T_i, T_{20})}{\sum_{j=i+1}^{20} 0.5P(T_i, T_j)}]^2,$$

$$[\frac{1 - P(T_i, T_{20})}{\sum_{j=i+1}^{20} 0.5P(T_i, T_j)}]^3$$

with the first, second and third power of the swap rate $S_{i,20}$ as the last three terms in the expression.

The swaption parameters are:

The starting date of the swaption: 11/04/2018

The exercise date of the swaption: 11/04/2023

The maturity of the swaption: 11/04/2028

The strike price of the instrument: 0.045

There has been 40 000 Monte Carlo simulations performed.

Table 7.1: The price of the Bermudan swaption

Strike price	0.055	0.045	0.030	0.010	0.005	0.001
LIBOR market model	0.1267	0.2116	0.4805	2.3315	3.9148	6.0293
Hull-White model	0.0004	0.0067	0.1815	2.7097	4.6105	6.6798

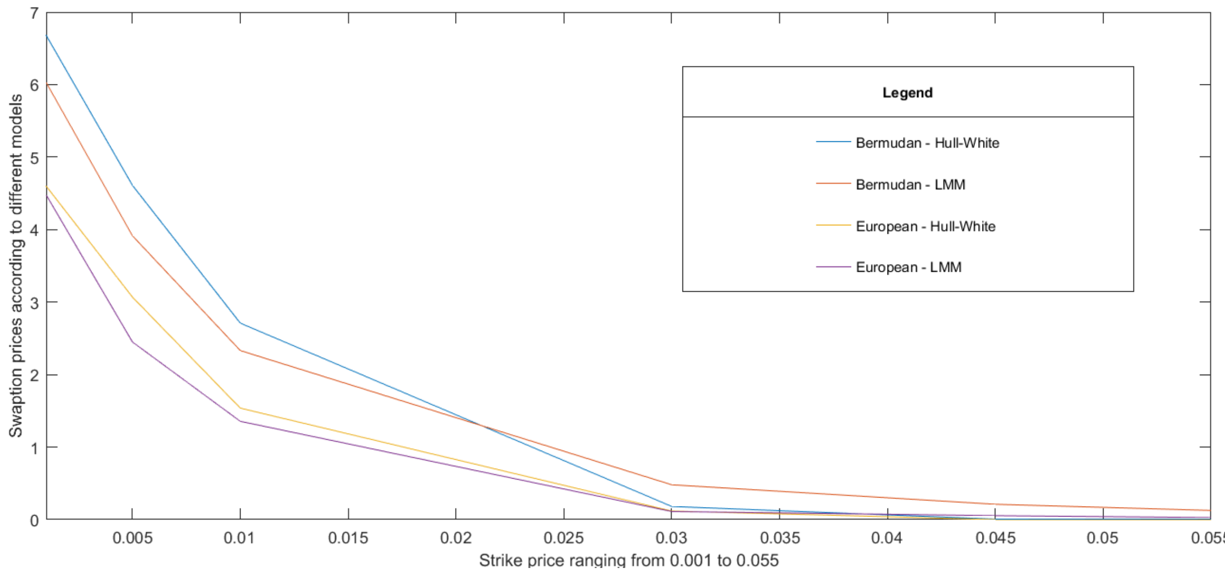

Figure 7.1: Results of pricing European and Bermudan swaptions in Hull-White and LIBOR market model. Source: Matlab

Again, we can see that LIBOR market model and the Hull-White model predict slightly different prices, which is caused by the different methodology of the models.

A final summary can be seen in Figure 7.1. We can see that the Bermudan type of swaption with otherwise same features is valued above the European type of contract by both models. This theoretical assumption has been confirmed. The Bermudan swaption gives the holder an advantage of more exercise dates and this possibility is valued by the market by the difference in the prices of the two.

The yellow and the blue line represent the results for the Hull-White model. We can see that in the case of European, as well as Bermudan swaption, it results of the computations are in general higher than in the case of LIBOR market model (the red and the purple line), with the only exception being the Bermudan swaption with strike price 0.03 and higher.

Conclusion

We have presented the evolution of interest rate models from short rate models to market models, with a focus on market models, especially LIBOR market model, as well as several notions that are related to calibration of the model.

We have performed a calibration of LIBOR market model. It has the ability to fit today's term structure of the interest rates and thus has gained popularity in practice.

Interest rate derivatives play a major role among derivatives. They can be used for speculation, risk hedging or as an opportunity for arbitrage.

In the thesis, we have priced a European and a Bermudan swaption using one of the short rate models and a market model. The Hull-White model can be seen as less convenable for pricinge an interest rate derivative such as a swaption than a market model.

We can conclude that Hull-White model and LIBOR market model produce slightly different results. This can be assigned to the fact that these two models present a very different approach to interest rate modeling and the theory behind the two models is very different. At this point, we are not able to tell which results are more reliable and it requires further research.

In theory, LIBOR market model offers a more sophisticated approach and therefore should offer a more reliable results in comparison to Hull-White model.

We have worked with the basic form of LIBOR market model. Some of the extensions of LIBOR market model that have been suggested since its appearing and are presented in the section 3.5 of this thesis. They could offer a further research on this topic.

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List of abbreviations

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- ATM = At the money;
- $CC(A)$ = Cascade Calibration (Algorithm);
- HW = Hull-White model;
- IRS = Interest Rate Swap (either payer or receiver);
- ITM = In the money;
- LFM = Lognormal forward-Libor model (Libor market model, BGM model);
- LSM = Lognormal forward-swap model (swap market model);
- MC = Monte Carlo;
- OTM = Out of the money;
- $PVBP$ = Present Value per Basis Point (or annuity);
- $RCCA$ = Rectangular Cascade Calibration Algorithm;
- TSV = Term Structure of Volatilities;
- SDE = Stochastic differential equation;
- I_n: the $n \times n$ identity matrix;
- $B(t), B_t$: Money market account at time t, bank account at time t ;
- $D(t, T)$: Stochastic discount factor at time t for the maturity T;
- $P(t, T)$: Bond price at time t for the maturity T;
- $r(t), r_t$: Instantaneous spot interest rate at time t;
- $R(t, T)$: Continuously compounded spot rate at time t for the maturity T;

- $L(t, T)$: Simply compounded (LIBOR) spot rate at time t for the maturity T;
- $f(t, T)$: Instantaneous forward rate at time t for the maturity T;
- $F(t; T, S)$: Simply compounded forward (LIBOR) rate at time t for the expiry. maturity pair T, S;
- $F^f(t; T, S)$: Foreign simply compounded forward (LIBOR) rate at time t for the expiry -maturity pair T, S;
- $T_1, T_2, \ldots, T_{i-1}, T_i, \ldots$: An increasing sequence of maturities;
- τ_i: The year fraction between T_{i-1} and T_i;
- $Fi(t) : F(t; Ti.1, Ti)$;
- $S(t; Ti, Tj), Si, j(t)$: Forward swap rate at time t for a swap with first reset date T_i and payment dates $T_{i+1}, \ldots, T_j$;
- $Ci, j(t)$: Present value of a basis point (PVBP) associated to the forward.swap rate $S_{i,j}(t)$, i.e. $\sum_{k=i+1}^{j} \tau_k P(t, T_k)$;
- Q_0: Physical/Objective/Real.World measure;
- Q: Risk-neutral measure, equivalent martingale measure, risk-adjusted measure;
- Q_d: Spot LIBOR measure, measure associated with the discretely rebalanced bank-account numeraire;
- W_t, Z_t: Brownian motions under the Risk Neutral measure;
- 1_A, $1\{A\}$: Indicator function of the set A;
- E: Expectation under the risk-neutral measure;
- E^Q: Expectation under the probability measure Q;
- $Corr^i(X, Y)$: correlation between X and Y under the T_i forward adjusted measure Q_i;
- $Var^i(X)$: Variance of X under the T_i forward adjusted measure Q_i; i can be omitted if clear from the context or under the risk-neutral measure;
- $Cov^i(X)$: covariance matrix of the random vector X under the T_i forward adjusted measure Q^i;
- $Std^i(X)$: standard deviation (acting componentwise) of the random vector X under the T_i forward adjusted measure Q_i;

- $\mathcal{N}(\mu, V)$: Multivariate normal distribution with mean vector μ and covariance matrix V ; Its density at x is at times denoted by $\mathcal{N}(\mu, V)(x)$.
- $Bl(K, F, v)$: The core of Black's formula:
- $PFS(t, \mathcal{T}, \tau, N, R)$: Price at time t of a payer forward-start interest rate swap at times $\mathcal{T} = [T_1, \ldots, Tn]$ with first reset date T_1 and payment dates $T_2, \ldots, T_n$ at the fixed rate R; As usual τ_i is the year fraction between T_{i-1} and T_i and can be omitted, and N is the nominal amount and can be omitted;
- $RFS(t, \mathcal{T}, \tau, N, R)$: Same as above but for a receiver swap;
- $PS(t, T, \mathcal{T}, \tau, N, R)$: Price of a payer swaption maturing at time T, which gives its holder the right to enter at time T an interest rate swap with first reset date T_1 and payment dates $T_2, \ldots, T_n$ (with $T_1 \geq T$) at the fixed strike-rate R; As usual τ_i is the year fraction between T_{i-1} and T_i and can be omitted, and N is the nominal amount and can be omitted;
- $RS(t, T, \mathcal{T}, \tau, N, R)$: Same as above but for a receiver swaption;
- $ES(t, T, \mathcal{T}, \tau, N, R, \omega)$: Same as above but for a general European swaption; ω is $+1$ for a payer and ω -1 for a receiver, and can be omitted.
- $FSCpl(T_j, T_{k-1}, T_k, \tau_k, \delta)$: Price at time 0 of a call option, with maturity T_k, on the LIBOR rate $F_k(T_k.1)$, with $T_{k-1} > T_j$, where the strike price is set as a proportion δ of either the spot or forward LIBOR rate at time T_j.
- $LSO(t, T_{i-1}, T_i, \tau_i, N, K, \omega, \psi)$: Price at time t of the spread option on two-currency LIBOR rates
- $LP(t, T_{i-1}, T_i, \tau_i, N, K, \omega)$: Price at time t of the option on the product of the two LIBOR rates $L(T_{i-1}, T_i)$ and $L^f(T_{i-1}, T_i)$, whose payoff at time T_i, in domestic currency,
- Instantaneous (absolute) volatility of a process Y is $\eta(t)$ in $\mathrm{d}Y_t = (\ldots)\mathrm{d}t + \eta(t)\mathrm{d}W_t$.
- Instantaneous level-proportional (or proportional or percentage or relative or return) volatility of a process Y is $\sigma(t)$ in $\mathrm{d}Y_t = (\ldots)\mathrm{d}t + \sigma(t)Y_t\mathrm{d}W_t$.
- Level-proportional (or proportional or percentage or relative) drift (or drift rate) of a process Y is
  $mu(t)$ in $\mathrm{d}Y_t = \sigma(t)Y_t\mathrm{d}t + (\ldots)\mathrm{d}W_t$.

item $DC(Y)$: $1 \times n$ vector diffusion coefficient of a diffusion process Y driven by the vector (correlated) Brownian motion Z with $\mathrm{d}Z = C\mathrm{d}W$, with C a $n \times n$ matrix, $C"C' = \rho$, the $n \times n$ instantaneous correlation matrix, and W vector n-dimensional standard Brownian motion. In other terms, if $\mathrm{d}Y_t = (\ldots)\mathrm{d}t + v_t C\mathrm{d}W_t = (\ldots)\mathrm{d}t + v_t\mathrm{d}Z_t$. then $DC(Y) = v_t$.

- $F_n(t;T,S)$: Simply compounded forward (LIBOR) rate at time t for the expiry. maturity pair T,S in the nominal economy;
- $F_r(t;T,S)$: Simply compounded forward (LIBOR) rate at time t for the expiry. maturity pair T,S in the real economy;
- *Tenor*: the length of the underlying swap, $T_\beta - T_\alpha$