Vysoká škola ekonomická v Praze Fakulta financí a účetnictví katedra bankovnictví a pojišťovnictví studijní obor: Finanční inženýřství

# Comparison of warrant pricing models

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#### Prohlášení

Prohlašuji, že diplomovou práci na téma "Comparison of warrant pricing models" jsem vypracovala samostatně a veškerou použitou literaturu a další prameny jsem řádně označila a uvedla v přiloženém seznamu.

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## Poděkování

Velké díky za pomoc a vedení při zpracování mé diplomové práce patří vedoucímu práce, prof. RNDr. Jiří Witzanymu, Ph.D., který ochotně věnoval svůj čas konzultacím a kvalitními připomínkami dovedl práci do své finální podoby.

#### Abstrakt

Témou diplomovej práce je oceňovanie warrantov. Budú predstavené štyri modely, základný Black-Scholes model pôvodne určený pre oceňovanie opcií, jeho alternatíva upravená o mieru rozdelenia, Galai-Schneller model a Ukhov model implementujúci volatilitu firmy. V praktickej časti bude ocenených štrnásť warrantov kótovaných na Hong Kong burze. Podnetom diskusie bude vhodný výber risk-free sadzby a metódy odhadu volatility. V praktickej časti práce sú prezentované výsledky ocenenia, pričom autorka uvádza 3 možnosti odhadu volatility, štandardnú metódu rovnakých váh, GARCH model a implikovanú volatilitu an základe tržných dát. Najlepšie výsledky mal Galai-Schneller model v kombinácii s implikovanou volatilitou. Black Scholes model určený pre oceňovavnie opcií mal výsledky najhoršie.

Kľúčové slová: warrant, Black, Scholes, Galai, Schneller, GARCH, implied volatility

#### Abstract

The theme of this master's thesis is warrant pricing methods. Four models will be presented, the benchmark Black-Scholes model originally developed for option pricing, the diluted version, the Galai-Schneller model and the Ukhov model implementing the firm volatility. Fourteen warrants quoted on the Hong Kong Stock Exchange are priced in the practical part. Appropriate risk-free rate and volatility measures are discussed. In the empirical part of the work results of the pricing models are presented. Author used three volatility estimation methods, the standard equal weighted, GARCH and implied volatility. The best results were given by the combination of the Galai-Schneller model and the implied volatility. The Black-Scholes model intended for option valuation performed the worst.

Keywords: warrant, Black, Scholes, Galai, Schneller, GARCH, implied volatility

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# **1** Introduction

The main topic of this thesis is the pricing of call warrants. A call warrant is similar to a call option, it gives the investor a right to buy company stock for the given strike price. Unlike options, the shares received for exercising a warrant are newly issued, and as such increases the total outstanding shares of the company. This has an effect on the shares price, also known as the dilution effect.

The result of this thesis is that the implied volatility calculated on the current option prices gave the best results for all the four used models. The Galai-Schneller outperformed the other three, whereas the Black Scholes model was the worst estimation method. The combination of the Galai-Schneller model and the implied volatility gave quite precise estimations of the real warrant prices.

The one thing all investors have or should have in common is the motivation to be rewarded for the undertaken risk. It is long ago since basic pricing techniques earned significant returns on financial markets and high computational capacities were only available to the privileged ones. The same goes for financial instruments, which have gone their own long way to the present form of not only basic derivatives, but also multiple new exotics.

Along with many other derivatives, warrants are yet another financial product ready to spin investors' heads around when they decide where to invest their money. They can be classified as non-standardized options being issued by either a financial institution or a nonfinancial corporation. Exercising the warrant happens directly between the investor and the issuer. Although warrants can be traded separately, they are usually attached to a bond issue making them more lucrative to a potential investor.

Within this work we will focus on pricing warrants with four different models, the benchmark Black-Scholes model, later enhanced to the diluted BS model, the Galai-Schneller model and finally the Ukhov model. The first two variations of the Black-Scholes model do not require much computational capacity and could be calculated on paper in an insignificant amount of time. On the other hand, the Galai-Schneller and Ukhov model require iteration algorithms to be used. This is because in both model the equations describe the value of the warrant as both endogenous and exogenous variable. On top of that, the Ukhov model prices warrants with the help of a system of two non-linear equations. These two would be quite timeconsuming to be solved by hand and are not a good subject for a practical exam at school.

This brings us to the choice of computing system. For reasons to be discussed in the practical part of this thesis RStudio was picked. It has proven to be a powerful tool allowing not only a very subjective approach, but also providing a wide range of libraries with predefined algorithms.

Having said all this, with great power comes great responsibility. Today you can simply google an option price calculator. With such powerful tools for pricing financial derivatives and lot of packages doing most of the tedious work for us, we must not forget the theory that supported all this in the first place. Pricing financial derivatives should not be driven with the sole vision of profit. The real reason is trying to understand how the prices move, what processes they are following and how could we best describe this with the language of mathematics.

It is the authors biggest wish to provide the reader with a work detailed enough to deeply understand the presented models. Basic knowledge of mathematical statistics, economy, financial markets and stochastic processes is nonetheless required, otherwise this would be a book and not a diploma thesis. The goal is to explain used models in detail, use them on a set of warrants, and come to logical conclusions based on the empirical results.

This thesis is divided into two major sections, the theoretical and the practical part. In the theoretical part the reader will be briefly introduced to the basic theory of stock prices, developing the Black-Scholes model and enhancing it into the diluted model. After that two advanced models will be explained and methods for estimating volatility discussed. In the practical part we will implement this theory, build models, price warrants and most importantly present the outcomes of the analysis. Different approaches to estimating volatility were used. The standard method for calculating volatility with equal weights, the GARCH model and implied volatility taken from option prices were used to present a broader analysis of the warrant prices. This helps to illustrate the differences between them and their ability to forecast.

### **1.1 Literature Review**

This paper will present four models for warrant valuation. Black and Scholes (1973) developed their famous option pricing model, which was not only used to price options, but also warrants. However, soon it was observed, that these two financial derivatives require different pricing methods. That is where Galai and Schneller (1978) decided to correct the model with respect to the dilution effect. It was widely used in the upcoming years, until one of the authors, Galai himself in his 1991 work with Crouhy pointed out that this is not the ideal approach to pricing warrants. In this work he also suggested an alternative to the former diluted Black and Scholes model, that will also be presented in this paper. That was because the stock price used in the used Black-Scholes model was not enough to captivate the underlying asset characteristics.

This was solved later by Ukhov (2003) in his study where an algorithm for pricing warrants was built. At last, he addressed the problem of the warrant's effect on the firm equity value with respect to the warrant price itself. As reader might already suspect, the price of the warrant depends on the price of the warrant. This neat solution provided by the system of two equations connects the hard-to-be-estimated firm volatility with the stock volatility, and finally, the warrant price with the firm value. Hence the result of the model is not only the price of the warrant itself, but also the estimated volatility of the firm after the warrant issuance. It has to be noted that the model is presented on a non-dividend stock.

One of the issues presented most frequently is the fact that the warrant prices also depends on the credit risk of the issuer. This was proposed by Chan and Pinder (2000). Abinzano and Navas (2007) support this claim in their work. They criticize Ukhov's (2004) approach to valuation without credit risk, extend his model and apply their theory in the Spanish market. They also suggest using the Hull, White (1995) model for vulnerable option pricing for warrants without a dilution effect.

# 2 Theoretical background

In the theoretical part of the thesis our reader will be guided through the basic facts about warrants, stock prices all the way through to the four pricing models. The first model to be used is the Black-Scholes model originally designed for pricing options, which is also a good benchmark for comparison with the other three models. The second model presented is the diluted version of the Black-Scholes model which was later enhanced into the Galai-Schneller model. Last but not least is the Ukhov model, which is more complicated and attempts to get the most information from the observable variables.

## 2.1 Warrant basics

In contrary to plain vanilla options, warrants are not standardized. The terms and conditions of a warrant are stated in a disclosure document, issued by the investee. This includes information about the underlying instrument, exercise price and style, expiry date, conversion ratio, issue size, delivery style and anything else that might be considered important.

What also makes warrants different from options is the fact that when issued, only a predetermined number of warrants are made available. As we know, this is not true for options, whose issue size can theoretically be unlimited. However, the main difference that brings trouble to pricing them is the exercise style. Contrary to options, when a warrant is exercised, the institution issues additional shares. These are then sold to the investor at the strike price of the contract. Since more shares are issued, the value of the outstanding shares changes, which is known as the dilution effect.

The **dilution effect** occurs when a company issues more shares, or secondary offerings. This results in a lower stock price per share, since the company value remains, but the number of shares is higher. Suddenly the same share accounts for a lower percentage of the company value, being less valuable.

For a call warrant the investor is buying the right to buy the underlying instrument for a certain price. On the other hand, a put warrant gives him the right to sell it for a certain price.

#### 2.1.1 What influences the price of a warrant?

First, we will roughly describe the effect of basic warrant attributes to the price of the warrant itself. See **Table 1** for the impact of different variables on warrant prices.

Variable	Change	Call Warrant	Put Warrant
Stock price	+	+	-
Dividends	+	-	+
Volatility	+	+	+
Interest Rate	+	+	-
Time to Expiry	-	-	-

Table 1: Warrant attributes

The **stock price** of the underlying instrument is one of the biggest factors in the price of a warrant. A higher stock price results in a higher price of the call warrant, since there is a bigger chance of the stock reaching the strike price of the warrant and investor exercising.

Regarding **dividends**, call warrants move in the opposite direction of the dividend movement. This is influenced by the ex-dividend date. The ex-dividend date is the cut-off date for the ownership of the next dividend. If you purchase the stock before this date, the next dividend belongs to you, otherwise it belongs to the seller. Therefore, a fall in the stock price equal to the dividend size is expected immediately after the ex-dividend date. This expected drop results in a lower call warrant price, and vice versa.

**Volatility** change has a positive correlation with both call and put warrants, since investors always demand a higher reward for a higher risk.

The **interest rate** puts a number on the trade-off between either holding a call warrant and earning the IR on the rest of the money or holding the stock itself. Holding stocks is cheaper with a lower IR. Therefore, we also observe a positive correlation with call warrant prices.

Finally, with declining time to expiry, both the call and the put warrant prices are lower.

Generally, the long-term warrants with significant time to expiry have a great time value. This is because time allows the underlying stock price to change and the warrant to be in-the-money. With the approaching time to expiry their value is reaching zero, provided that the stock price did not appreciate (call).

#### 2.1.2 Stock prices

As mentioned earlier, stock prices are the main factor of warrant prices. To better understand the following pricing models, it is crucial to present basic theory about stock prices to the reader. This will help to explain why warrants cannot be priced deterministically and what that means. Ever since the start of the first exchanges, pricing stocks has been the keyword of many scholarly articles, but also speculations and downright illogical theories.

Luckily for us, pricing models have come a long way to their current form. The efficient market hypothesis has given way to broad discussions about the real availability of information on the market. If we believe the market to be in the strong efficient form, there is no point using technical or fundamental analysis nor would insider information be of any help. This is where advanced mathematical tools have come into place.

Nowadays, stock prices are modeled with stochastic differential equations with continuous time, which means that time periods can be divided into an infinite number of intervals. We assume that stock prices follow the stochastic process that is also the Markov process, having an attribute in equation (1).

$$P(x(t_n) \le x_n \mid x(t_{n-1}), \dots, x(t_1)) = P(x(t_n) \le x_n \mid x(t_{n-1}))$$
(1)

This states that the probability of the empirical value x at time  $t_n$  being smaller than or equal to prediction x at time  $t_{n-1}$  using all available historical information is the same, as if we didn't have any historical information. The Markov process has no memory and the random values at time  $t_n$  are strictly defined by their most recent values at time  $t_{n-1}$ . This contradicts any attempts of a technical analysis to predict the movement of stock prices based on historical data. One kind of Markov process is the Wiener process, also known as the Brownian motion. The Wiener process has the following properties:

1.  $z_0 = 0$ 

2. *z* has independent increments for every t>0, where the increments  $z_{t-u} - z_u$ ,  $u \le 0$ , are independent of  $z_s$ , s < t.

3. z has Gaussian increments,  $z_{t+u} - z_t \sim N(0, u)$ 

4. z has continuous paths,  $z_t$  is continuous in t

The equation (2) of the generalized Wiener process had to be altered for two reasons; it allows negative stock prices and variability would be higher for higher stock prices.

$$dx = adt + bdz \tag{2}$$

We describe the stock prices with a general Itô process in equation (3).

$$dS = \mu(S, t)dt + \sigma(S, t)dz \tag{3}$$

The unknown function  $\mu(S, t)$  is treated in equation (4).

$$\frac{E(dS_t)}{S_t} = \frac{E(S_{t+dt} - S_t)}{S_t} = \mu \, dt$$
<sup>(4)</sup>

Let us assume that the return on our investment should neither be dependent on the current stock price, nor on the currency we are investing in. We only want to be rewarded for the investment horizon. Therefore, the expected change in the price of the stock should only be equal to the percentage return times the change in time, and equivalently for volatility, which brings us to equation (5).

$$\mu(S_t, t) = \mu S_t, \qquad \sigma(S_t, t) = \sigma S_t \tag{5}$$

For stock prices we finally get the geometric Brownian motion equation (6).

$$dS_t = \mu S_t dt + \sigma S_t dz \tag{6}$$

Stock prices are expected to follow GBM, where Z is a Wiener process. We assume lognormal distribution of the stock prices.

## 2.2 Warrant pricing models

In this section the theory of building warrant pricing models will be explained. Starting from simple assumptions about the basic decision process when exercising a warrant, we will continue with the benchmark Black-Scholes model. The diluted version of the Black Scholes model was first presented by Galai and Schneller (1978) and is the second model used in this paper. This is a reasonable adjustment because the original Black and Scholes model was developed for option pricing, where no dilution is occurring.

The Galai-Schneller model with the adjusted underlying asset value used in the Black-Scholes formula of the model will be the third approach. Here we encounter the implicit equation for the warrant price for the first time. The second time we encounter it in Ukhov's (2004) model, which attempts to explain the warrant price with the use of observable variables. A more complex format of the model tries to captivate the firm value and volatility in order to better estimate the warrant value.

#### 2.2.1 Black Scholes model

Once the economic world accepted that stock prices might follow the GBM, it was a question of time when attempts to find an explicit formula to value options would be successful. When a "handy" size of portfolio was picked and Itô lemma applied on it, the famous Black-Scholes equation (7) was derived.

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC = 0$$
<sup>(7)</sup>

We cannot solve equation (7) yet, since we do not have any boundary conditions and it is a partial differential equation. However, when we add the payoff function (8) of the European call option, equation (7) provides a solution.

$$C(S,T) = \max(S - K, 0) \tag{8}$$

The Black Scholes model holds under these strong assumptions:

- Underlying asset price is lognormally distributed, has constant drift and volatility, and follows the geometric Brownian motion process  $dS_t = \mu S_t dt + \sigma S_t dz$ .
- No dividends on stock
- Constant risk-free rate for all maturities, and unrestricted lending and borrowing
- No taxes or transaction costs
- Perfectly divisible assets
- No arbitrage opportunity
- Continuous trading

The basic Black and Scholes equation (9) describes the value of the European call option as equal to the difference between the spot price of the underlying asset multiplied by the cumulative distribution function of the standard normal distribution with respect to  $d_1$  (10), and the discounted strike price multiplied by the same with respect to  $d_2$  (11).

$$C(S,t) = S\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$
(9)

- *S* underlying asset spot price
- *K* strike price
- *t* time to maturity
- *r* risk free rate
- $\sigma$  volatility of the underlying stock

$$d_{1} = \frac{\ln\left(\frac{S_{0}}{K}\right) + \left(r + \frac{\sigma^{2}}{2}\right)t}{\sigma\sqrt{t}}$$

$$d_{2} = d_{1} - \sigma\sqrt{t}$$
(10)
(11)

#### 2.2.2 Diluted Black Scholes model

Galai and Schneller (1978) enhanced the standard Black-Scholes model to its diluted version. Let us take, for example an equity firm of value at time t equal to X. Galai and Schneller assume the firm has an investment policy independent from its financing decisions, and it is also not affected by changes in the capital structure of the firm. That is why the cash received from warrant issuance would be distributed to the shareholder as dividends. In case that the warrants are exercised, the company would receive an income of KM, where K is the strike price of warrants and M is the number of issued warrants. The value of the firm rises. The condition of warrant exercise is in equation (12).

$$\widetilde{W} = \frac{E_t + MK}{N+M} = \frac{S_t + MK/N}{1 + M/N} > K$$
<sup>(12)</sup>

- $E_t$  equity value of the firm at time t
- *N* outstanding number of shares (before the warrant is exercised)
- K warrant strike price
- *M* number of warrants
- $\widetilde{W}$  warrant value after conversion
- S stock price at time t

The warrant will be exercised if the stock price after exercising a warrant is higher than the warrant strike price. M/N is the ratio of warrant issued to the number of outstanding shares of the company.

Furthermore, Galai and Schneller (1978) compare the price of a company with and without warrants. The price of a firm without warrants at time  $t_0$  is equal to the sum of outstanding shares

times the current stock price  $E = NS_0$ . Two thing can happen at time  $t_1$ ; either the stock price is lower than the strike price, or it is higher. If the stock price is lower, both the warrant and a potential call option would be worthless.

On the other hand, if the stock price is higher than the warrant (option) strike price, it will be exercised. We can further adjust equation (12) into (13).

$$\frac{S_t + MK/N}{1 + M/N} - K = \frac{S_t - K}{1 + \frac{M}{N}} = \widetilde{W}$$
<sup>(13)</sup>

Equation (13) expresses the warrant value after conversion  $\widetilde{W}$ . It would be equal to the change in the price of the company divided by the percentage growth in the number of outstanding stocks. After the warrant exercise, the value of the company is equal to (14)

$$E = M\widetilde{W} + M(\widetilde{W} - K) \tag{14}$$

The price of a corresponding option on a company without warrant would be (15)

$$C = S_t - K \tag{15}$$

From here we can see that the value of a call warrant can be expressed with the value of a call option (16). We take equation (13) and combine it with equation (15) for the price of a call option.

$$\widetilde{W} = \frac{S_t - K}{1 + \frac{M}{N}} = \frac{C}{1 + \frac{M}{N}}$$
(16)

The value of a call warrant can be computed as a ratio of the corresponding call option value. And finally, Galai and Schneller (1978) present the idea of calculating a warrant price as a diluted price of an option (17).

$$\frac{1}{1 + M/N} (S\Phi(d_1) - Ke^{-rt}\Phi(d_2))$$
(17)

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It simply dilutes the European call option price by the factor of the outstanding shares divided by the final sum of the shares after the exercise of the warrant. Note that the deviation from the standard Black-Scholes model is very low when the relative number of issued warrants to the total number of shares is low.

#### 2.2.3 Galai-Schneller model

Due to the widely misused diluted Black and Scholes model Galai and Schneller (1991) had to step up and redefine their previous pricing model for warrants. The main thought behind this model is considering the change in the value of the firm as a consequence of the warrant issuance. It is incorrect to decide whether to exercise a warrant based on the difference between the current stock price and the strike price of the warrant. This is because at the moment we decide to exercise the warrant, the dilution comes into effect and the stock price immediately drops. Again, this is not the case for options.

The condition of exercising a warrant stays the same as in the older Galai-Schneller model (12). After an issue of warrants is fully exercised, the stock price is equal to the sum of the equity and the strike price times the number of exercised warrants, over the total shares outstanding after the exercise. It logically follows that we would only attempt to exercise the warrant if the exercise price is lower than the share price after exercising. Note that this means that the investor might not exercise the warrant even if the exercise price is lower than the current share price. This is because exercising the warrant would lead to lower stock prices, which would push the stock price under the exercise price.

Hence the earnings from holding a warrant are the maximum from zero and the difference between the share price after exercising and the warrant strike price. We can see the condition appearing in (18) describing the payout to be suspiciously similar to the condition of the European call option payout.

$$W_t = Max\left(\frac{E_t + MK}{N + M} - K, 0\right)$$
(18)

•  $W_t$ - value of warrant at time t

Now we rearrange the equation to exclude the predetermined original number of shares and the number of sold warrants from the payout function. We suddenly have the dilution factor in front of the payout condition (19). This strongly resembles the diluted Black-Scholes model, with the only difference being in the price of the asset inside the brackets.

$$W_{t} = Max \left(\frac{E_{t} + MK - NK - MK}{N + M}, 0\right)$$

$$= Max \left(\frac{1}{N + M}(E_{t} - NK), 0\right)$$

$$= Max \left(\frac{N}{N + M}\left(\frac{E_{t}}{N} - K\right), 0\right)$$

$$= \frac{N}{N + M} Max \left(\frac{E_{t}}{N} - K, 0\right)$$
(19)

Equation (19) shows that the payoff of a warrant can be derived from the equity value. The warrant price at time *t* can be determined as the value of a call option. Parameters of the call option would be  $\frac{E_t}{N}$  as the price of the underlying asset, *t* as time to maturity, *K* as the exercise price.

The next question is, how do we calculate the equity value? Although it is usually calculated as the stock price times the number of shares, we also have to include the warrants claim. This leads us to Equation (20), where the equity value of a firm is equal to the stock price S times number of shares N plus the number of warrants M times the warrant price W.

$$E_0 = NS_0 + MW_0 \tag{20}$$

When we put (20) into equation (19) we get equation (21) giving us the underlying asset value to be used for the call option formula. However, we can see that the warrant price is both endogenous and exogenous. This causes an iteration problem, which can be solved by iteration algorithms, such as the Newton's method.

$$W_0 = \frac{N}{N+M} E\left[Max\left(\frac{NS_0 + MW_0}{N} - K, 0\right)\right] * e^{-rt}$$
(21)

13

The price of the European warrant is calculated as shown in (22) where the value Call(...) is the equation (23) of the Black-Scholes equation with an altered asset value.

$$W_{0} = \frac{N}{N+M} Call \begin{cases} UAV = S_{0} + \frac{M}{N}W_{0} \\ Time \ to \ maturity = t \\ Exercise \ Price = K \\ Risk - free \ rate = r \\ Volatility = \sigma \end{cases}$$
(22)

$$W(S,0) = (S_0 + \frac{M}{N}W_0)\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$
<sup>(23)</sup>

Compared to the diluted Black-Scholes model, this model requires iterations to be solved due to the implicit equation for the price of a warrant. Iteration methods require an initial approximation of the price. Some models are sensitive to the initial guess and might not give any results if the guess is not done well. The best input for the iteration algorithm would be the price of a warrant calculated according to the Black-Scholes model.

#### 2.2.4 Ukhov model

The last model to be presented in this paper is the Ukhov  $(2004)^1$  model. The warrant price is inherited from Galai and Schneller (1991) (24), (25). This equation describes the value of a warrant under the Black-Scholes assumptions.

$$W_0 = \frac{1}{M+N} \left[ E\Phi(d_w) - e^{-rt} N K \Phi(d_w - \sigma \sqrt{t}) \right]$$
(24)

<sup>&</sup>lt;sup>1</sup> UKHOV, A.D., 2004, "Warrant pricing using observable variables", Journal of Financial Research, vol. 27, no. 3, 329 - 339

$$d_w = \frac{\ln(E/NK) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$
(25)

A few complications arise within this equation. The first of them is the company value *E*, that Galai-Schneller model replaced with the value of the firm after the warrants are exercised. We assume a log-normal distribution of the company value. Once again, the value of the warrant depends on the value of the firm, which depends on the value of the warrant. We can already see that solving this model will require iterations.

The second problem is the  $\sigma$  and its estimation. Usually it is estimated as the stock logarithmic return volatility, but we must emphasize that this is only an estimation. The *true* sigma should be the standard deviation of the company value process, not of the stock prices. Observing the volatility of a firm is rather difficult, since to estimate  $\sigma$  we first need to know the value of firm *V*. To do that, let us first define  $\Omega_S$  (26).

$$\Omega_S = \frac{\Delta_S E}{S} \tag{26}$$

- $\Omega_S$  elasticity of the stock price
- Δ<sub>S</sub> stock hedge ratio measuring the price change of one share of stock when the firm value E changes by one unit
- E firm value
- S stock price

The elasticity of stock prices is expressed with respect to the firm value weighted by the stock hedge ratio. In equation (73)  $\sigma_S$  is related to  $\Omega_S$ . The volatility of the stock price can be expressed as the sum of the elasticity of the stock price multiplied by the firm volatility.

$$\sigma_{\rm S} = \Omega_{\rm S} \sigma \tag{27}$$

Now that the volatility of the firm is described using the stock volatility, we can put (26) into (27) and we get a formula for the stock volatility using the firm volatility, stock hedge ratio, stock price and the firm value (28)

$$\sigma_S = \frac{\Delta_S E}{S} \ \sigma \tag{28}$$

Let us express firm volatility from (28). The standard deviation of a firm can be estimated as the stock volatility divided by the firm value times the hedge ratio over the stock price (229).

$$\sigma = \frac{\sigma_S}{E \,\Delta_S / S} \tag{29}$$

Before being able to calculate  $\sigma$ , the hedge ratio of the stock  $\Delta_S$  equation should be derived. For this the hedge ratio of a warrant,  $\Delta_w$  is defined. The same as for  $\Delta_S$ ,  $\Delta_w$  is the hedge ratio of a warrant that measures the change in the price of a warrant when the value of the firm changes by one unit. To be able to calculate it, we need to use equation (24) and express the warrant delta as the partial derivation of the warrant value with respect to the firm value. As we know, derivations express the change in a function when the change of the input of the function is infinitesimal. A simple derivation can only be used if we are deriving a function of one variable. Since the value of warrant in (24) is expressed as a function of many variables, we need to derivate it partially. This allows us to calculate the change in the price of a warrant with respect to the firm value if all other factors that could influence the warrant price are held constant. We can write (30):

$$\Delta_{w} = \frac{\partial W_{0}}{\partial E} = \frac{\partial \left[\frac{1}{M+N} \left[ E\Phi(d_{w}) - e^{-rt} N K \Phi(d_{w} - \sigma \sqrt{t}) \right] \right]}{\partial E}$$
(30)

Luckily for us, expression (30) is not hard to derivate with respect to *E* as the second part inside the brackets will be equal to zero since it contains no *E*, so we are only left with the first part. Equation (31) expresses the delta of the warrant in terms of the outstanding number of shares, the total warrants issued and the distribution function of  $d_w$ . Keep in mind that  $d_w$  is a function of *E*, *N*, *K*, *r*, *t*,  $\sigma$ .

$$\Delta_W = \frac{1}{N+M} \Phi(d_W) \tag{31}$$

Now  $\Delta_W$  will be brought together with  $\Delta_S$ . What happens if the price value changes by one unit? The price of the stock changes by  $\Delta_S$  and the price of the warrant changes by  $\Delta_W$ . The weighted average of the stock and warrant deltas is equal to the firm delta  $\Delta_V = 1$  (33).

$$1 = N\Delta_S + M\Delta_W \tag{33}$$

The weights are the outstanding number of shares and the size of warrant issue, respectively.

$$\Delta_S = \frac{N + M - M\Phi(d_W)}{N(N+M)} \tag{34}$$

Now that we have  $\Delta_S$  expressed (34), the Ukhov model can be built. A system on two non-linear equations (35) needs to be solved to get the equilibrium values ( $E^*, \sigma^*$ ).

$$\begin{bmatrix} 0 &= E - MW_0 - SN \\ 0 &= \sigma(E\Delta_S/S) - \sigma_S \end{bmatrix}$$
(35)

Once resolved, the warrant price W is equal to (36).

$$W = \frac{E^* - SN}{M} \tag{36}$$

Overall, this model is building on the theory of Black-Scholes, Galai-Schneller and others. It enhances the formerly mentioned models by including the firm volatility and firm value into the calculations. Resolving the system of equations by hand is not really an option, therefore a pricing algorithm had to be developed.

#### 2.2.4.1 Newton's method

Most of the time mathematicians sit above scripts is caused by attempts to find some *x*. What is this *x*?

That depends. The more things there are that the x depends on, the longer it takes to find it. If we have a simple equation x = 1 + 2, it takes less than a few seconds. If it is more complicated, let's say  $x^3 - 0.5x^2 + 5 = 0$  it might take a minute. How long would mathematicians sit there if given the Ukhov model to solve? Such a type of equation does not necessarily have an explicit solution that can be calculated precisely. Sometimes we have to be satisfied with an approximate solution. These equations are not new and have been dealt with ever since falling apples defined the laws of physics.

One of the methods most used in modern algorithms is the Newton's method of approximation. This one is also used later on as a part of RStudio library to estimate warrant prices with the Ukhov model. For a simple introduction of the algorithm let us have a univariate function f(x);  $x \in R$  and an initial estimate  $x_0$ .

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{37}$$

#### • $f'(x_0)$ – value of the first derivative of f at point $x_0$

Then under a certain set of assumptions, the most important being the differentiability of the function in  $x_0$ , the approximation  $x_1$  of the root of the function f(x) given by (37) is a better one than  $x_0$ , A preset level of precision is reached by repeating the process generally described as (38).

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$
(38)

**Image 1** shows the graphic representation of Newton's approximation.



*Image 1:* Newton's method

First, we set a random initial point  $x_0$  and do an orthogonal intersect with f, thus reaching point  $f(x_0)$ . Next we take a tangent at  $f(x_0)$ , which is a derivation f' with respect to  $x_0$  and lay through a line that intersects the x axis. We repeat the steps until we get close enough, meaning our guess at time t differs from the guess at time t-l by an insignificant amount a (39).

$$|x_t - x_{t-1}| < a; \ t, a \in R \tag{39}$$

Note that for certain kinds of function Newton's algorithm oscilates or diverges and other iterration methods have to be used.

The multivariate version is similar to the univariable one, with a little difference. Since now we have multiple variables that our approximation depends on, a matrix is needed. The iteration equation takes form (40) where *F* is a function  $F: \mathbb{R}^m \to \mathbb{R}^m, m \in \mathbb{N}$ .

$$x_n = x_{n-1} - J_F(x_{n-1})^{-1}F(x_{n-1})$$
(40)

 $J_F(x_n)$  is a *m* x *m* Jacobian matrix. Instead of calculating the inverse of the matrix to left-multiply  $F(x_n)$  let us rearrange the equation into (41)

$$J_F(x_{n-1})(x_n - x_{n-1}) = -F(x_{n-1})$$
(41)

#### 2.2.5 Note on American warrants

As it was out of scope of this work to include a thorough evaluation of American warrants along with the already presented models for European warrants, only a suggestion for future research will be mentioned. All the theory above applies to European warrants. A problem occurs with the American version. As with all other American financial derivatives, an American style warrant can be exercised anytime during its lifetime. The author suggests using the relationship between the European call warrant and European call option to derive the American call warrant price from the price of an American call option.

We will assume that the decision-making mechanism when exercising an American call option and American call warrant is the same (as for the underlying asset value and the current stock value). The only difference is in the definition of the underlying asset value. Therefore, an American warrant can be priced as (42). We can simply use any existing model for American options, e.g. binomial trees, at the price  $S_0 + \frac{M}{N}W_0$  and adjust the result to the dilution ratio.

$$W_{A} = \frac{N}{N+M} Call_{A} \begin{cases} UAV = S_{0} + \frac{M}{N} W_{0} \\ Time \ to \ maturity = t \\ Exercise \ Price = K \\ Risk - free \ rate = r \\ Volatility = \sigma \end{cases}$$
(42)

## 2.3 Estimating variables

Using a complex model to estimate financial derivatives prices does not necessarily result in their correct valuation. It is the unknown input variables which play the main role in the reliability of our calculations. Let us say we use the Black-Scholes model to estimate the warrant price. We need seven variables on input, from which only 5 are certain (time to maturity, number of shares, number of warrants, strike price, stock price). The other two, the risk-free rate and volatility, have to be estimated. First, we will discuss which reference rates can be used to estimate the risk-free rate. After that, techniques for volatility estimation will be presented.

#### 2.3.1 Risk free rate

As a well-known concept, the Markowitz portfolio implies that every investor is looking for the best reward he can get, based on his risk preferences. On one side of this coin we have great returns from investments to modern technologies, such as cryptocurrencies, on the other very low returns on state bonds. What the articles about happy Bitcoin investors tend to leave out is the major risk complementing every above-average return.

Although empirically a risk-free asset does not exist, as nothing is absolutely certain, the financial world works with the term risk-free rate. This is the rate of the possibly lowest risk bearing assets an investor can buy, mainly used to diversify a portfolio or as an obligatory part of a financial institution's balances for regulatory purposes. We could naturally conclude that what is good enough as a risk-free asset for financial market regulators, is also good enough for our estimations. However, no such strong premises should be left unquestioned.

Note that government issued bonds apply for a good risk-free rate proxy only for the local subjects. If an investor buys a bond of a foreign government (with respect to the used currency), he also faces exchange-rate risk. The risk-free rate can be calculated from Treasury bonds (long term US), or Treasury bills (short term US). The bootstrapping method is applied to derive the required term of the rate based on available bills and bonds.

Another possible estimation is the interbank offered rates, such as LIBOR, PRIBOR or EURIBOR. Despite being widely spread as risk-free rate proxies, since 2017 discussions have occurred about the LIBOR's strong impact on financial markets, and the need to push away from it. The main risk-free approximation should transition from LIBOR to a near-risk-free rate before 2021, following the main idea that a robust rate that is not easily manipulated, better reflects the real rates of the market and does not present such a risk to the financial stability. Following this example, ESTER (European Risk-Free Rate) will replace EURIBOR as of 2021 and will be published before October 2019. When compared with the US equivalent SOFR, SONIA is the rate on unsecured transactions, whereas SOFR is on the secured ones.

Since the warrants used in this study are traded on the Hong Kong exchange as well as the underlying assets, a different rate had to be used. The Hong Kong Association of Banks<sup>2</sup> publishes daily data on the HKD (Hong Kong dollar) interest settlement rates. These are the estimated offer rates for which banks can deposit money in the interbank market in Hong Kong.

They are quoted as overnight, 1week, 2weeks, 1month, 2months, 3months, 6months and 12months. These rates are calculated as the average quote from the contributing banks excluding the three highest and lowest bids. The contributing banks are a list of 12 to 20 banks appointed by the Hong Kong Association of Banks with respect to their activeness on the current business day. It is calculated with precision to 5 decimal places and is not published during the Hong Kong public holidays.

To better understand the role of HKAB<sup>3</sup> in the Hong Kong bank system, we have to first introduce the HKMA, or the Hong Kong Monetary Authority, the equivalent of the Czech National Banks. HKMA is the regulatory authority that supervises the local bank system, manages reserves and looks after the stability of Hong Kong's financial system. It grants banking licenses and as such is also responsible for the sector's risk management. Coming back to HKAB, even if a bank was granted a license from HKMA, it cannot operate fully without being a member of the HKAB. Banks are represented by a chosen senior executive representative at the HKAB meetings.

The role of HKAB is to help banks consult law reforms, as well as new legislations and changes in regulations. It also invests into the further education of banks, provides a channel of communication for the members, promotes best practices and create rules for the banking business.

#### 2.3.2 Volatility

Known under many names; volatility, dispersion or the standard deviation is a measure of variance of the values within a dataset. For example, in a group of 10-year old children, the variance of the height would be relatively small compared to a group of people aged between 5-20 years. Volatility helps us to measure how much the data in the dataset can differ from the mean value. As mentioned earlier, this is an important measure with a significant impact on the price of warrants. Higher volatility means a higher risk of a bad (or good) outcome, and investors require being rewarded for this risk. Markowitz portfolio theory shows that each investor has

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http://www.hkab.org.hk/DisplayInterestSettlementRatesAction.do?Submit=Search&lang=en&year=2019&mo nth=1&day=2

his own risk preference, which he pairs with the expected return. A risk-seeking investor could ask for an 8% return for investing into a startup whereas a risk-avoiding one would be satisfied with 2% return for an investment into a stable company.

Volatility is one of the biggest question marks when pricing any financial derivate. It can either be estimated from historical data or calculated from current market prices of relevant financial derivatives. Examples of historical data methods are the following:

- standard method with equal weights
- autoregressive conditional heteroscedasticity model, or (ARCH (m))
- general autoregressive conditional heteroscedasticity model (GARCH (p,q))
- exponentially weighted moving average (EWMA)

In order to estimate parameters of the more advanced models' different approaches are available, i.e. the most likelihood method. In this work we will focus on historical data estimation and the GARCH (p,q) model.

#### **2.3.2.1** Standard method with equal weights

The simplest method is to estimate volatility from historical data, as suggested in Hull<sup>4</sup>. One of the available methods to calculate realized volatility from historical data is volatility based on daily log returns (43).

$$sd = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$
(43)

- *n* number of observations
- $u_i = \log\left(\frac{s_i}{s_{i-1}}\right)$ , i = 1, 2, ..., n logarithm of the daily returns
- $\overline{u}$  mean value of u

For easier calculation, usually the modification of (43) is used, see (44).

<sup>&</sup>lt;sup>4</sup> HULL, J., 2012, *Options, futures, and other derivatives*. Boston: Prentice-Hall, ISBN 978-0-13-216494-8.

$$sd = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)} \left( \sum_{i=1}^{n} u_i \right)^2}$$
(44)

The calculated volatility is daily if calculated from daily data. We can change the volatility time horizon by multiplying it with  $\sqrt{t}$ . The main issue of this model is that it values older observations the same as the most recent ones. Therefore, it might happen that current volatility is either under- or overestimated, depending on the length of the horizon it is calculated on and present depressions or growths. In order to have a robust model, it is always better to have a longer time horizon. This, however, depends on the form of data we want to evaluate. It is not reasonable to calculate insurance reserves with volatility estimated over the last two weeks. At the same time estimating volatility on a few years' horizon for intra-day trading is unreasonable and will not have any information value. In this thesis we work with a two-year range of daily stock prices.

#### 2.3.2.2 GARCH (p,q) process

Another option for volatility estimation based on historical data is the GARCH or the general autoregressive conditional heteroscedasticity process. GARCH estimates the error at time t based on the size of the previous errors. For explanatory purposes we will present the GARCH (1,1) process, see equation (45).

$$\sigma^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{45}$$

- $V_L$  long-run variance rate
- $\gamma$  weight of  $V_L$
- $\alpha$  weight of  $u_{n-1}^2$
- $\beta$  weight of  $\sigma_{n-1}^2$
- $\sigma_{n-1}^2$  instantaneous variance rate at time n-1

For GARCH (1,1) the following equations hold:

$$\alpha + \beta + \gamma = 1 \tag{46}$$

$$y_t = \mu_t + u_t \tag{47}$$

$$u_t = \sigma_t \varepsilon_t \tag{48}$$

- $\mu_t$  can be an ARMA equation or a constant...
- $\varepsilon_t$  errors are independent and identically distributed random variables

In GARCH (p,q) *p* is the number of lags used to calculate  $\sigma_n^2$  and *q* is the number of lags for  $u^2$ . GARCH models are mean reverting, which means that in long run volatility is being pulled back to the long-term average value  $V_L$ .

#### 2.3.2.3 Implied volatility

Another approximation of volatility used in this thesis is the implied volatility. Historical volatility is based on the past market data. On the other hand, implied volatility is the volatility suggested by the current option prices. If we know the value of the call option, we can calculate the volatility that this value implies. We can take *S* as the stock price of the underlying asset, *K* as the strike price of the option, *r* as the risk-free rate, *T* as the time to maturity and *c* as the current price of the European call option. Then the only unknown variable in the Black-Scholes equation (9) is the volatility  $\sigma$ . Since sigma cannot be expressed explicitly from the equation, an iteration algorithm has to be used. The result of this is the implied volatility of the firm on the market. We can use it to price our models and compare the results. The volatility obtained from the current market data should bring us the best results. This is because it also contains the volatility risk premium of the stock. The volatility risk premium is the premium investors require for the unexpected market volatility.

## 2.4 Evaluation methods

The most important part of any analysis is the results. All the work done would be useless without a proper model evaluation and the following conclusions. Evaluation methods are always limited by the used models. In our case we will only be able to use simple estimators of errors due to having only a small number of observations. Advanced statistic criteria would be useful in different cases. To remind the reader of the available options, we mention the mean squared error given by equation (49)

$$MSE = \frac{1}{n} \sum_{i=0}^{n} (y_i - \hat{y}_i)^2$$
(49)

The mean squared error is a quite simple, yet popular estimation of the difference between the real and the estimated values. The second power helps to eliminate possible bias due to negative and positive error cancelling each other out.

The mean absolute deviation measure is similar to MSE but it does not use squares of the difference. It is given by equation (50)

$$MAD = \frac{1}{n} \sum_{i=0}^{n} |y_i - \hat{y}_i|$$
(50)

Another option is the MAPE, or mean absolute percentage error (51)

$$MAPE = \frac{100}{n} \sum_{i=0}^{n} \frac{|y_i - \hat{y}_i|}{|y_i|}$$
(51)

With the help of MAPE we can better express the ability of the prediction model to forecast prices. The results are in percentage. Combining all three of these estimates should give us a better insight into the explanatory power of our models.

# **3** Model implementation

In the second part of this thesis, warrants traded on the Hong Kong exchange will be priced. First, we will investigate how the data was prepared before it was ready to be used in the presented models. Next, the construction of our models within RStudio will be discussed. RStudio was chosen because the software was widely used by author during her studies, the documentation is very user-friendly and since its use is wide-spread, a large number of computing packages are available. This was very beneficial when dealing with the system of two non-linear equations within the Ukhov model.

Compared to other economically, financially and mathematically oriented software on the market, RStudio should not be forgotten. In contrast to Eviews, it provides a modern user interface and is more customizable. For some people this could also be a reason not to use it, since at least a basic knowledge of programming and algorithmic thinking is required in order to use it properly. This rewards the ones who dare, and in the end simplifies the work. After multiple revisions of my code in RStudio, I was able to price a new set of warrants with 4 different models (with already downloaded stock prices and warrant information) in less than 10 minutes. All it takes is a little knowledge of for-cycles and a few sleepless nights. Matlab would also be a feasible option if one has a license. Another option would be a good free variant of Matlab on the market, GNU Octave with similar syntax and an open-source code.

## **3.1 Preparation**

Each warrant is priced at the issue date. This means that for each warrant we will need to calculate the underlying stock volatility, use a risk-free rate of the appropriate maturity, find out the outstanding shares at the time of issuance and the number of issued warrants, maturity of the warrant, strike price and the spot price of the stock.

Fourteen warrants were picked from the Hong Kong exchange for our model evaluation. They were not hand-picked to provide ideal results. Only the following conditions had to be filled in order to be included on the list – warrants were listed in November or December 2019; they were also traded on this day and their maturity was either half year or one year. Although the Hong Kong exchange is a world-wide institution, warrants are not very liquid here and none of them was traded continuously within even just one month. That is also one of the reasons why author decided to spot-value multiple warrants rather than to price a single warrant in a wider time frame.

To simplify dealing with different entitlement ratio of warrants within models, one more assumption was made. From now on we assume that having a warrant with the entitlement ratio  $e_r$ , warrant price w and number of issued warrants M is the same as having a warrant with an entitlement ratio 1, warrant price  $w^*e_r$ , and number of issued warrants  $M/e_r$ . For example, we would have the issue of 10,000 warrants at price w = 0.2 per warrant with an entitlement ratio  $e_r = 10$ , meaning we can buy one stock at strike price K for ten such warrants. We assume this would be the same as an issue of 1000 warrants at price w = 2 per warrant with an entitlement ratio  $e_r = 1$ . In both cases the price of having a right to buy one stock is 2. Following this assumption, we multiplied the price of warrant by its entitlement ratio and divided the number of issued warrants by the same.

Our 14 warrants had together 8 underlying assets, all of them quoted in HKD on the Hong Kong Stock exchange. Of course, all the data was available on our regular resources of stock data. An interesting fact is that HK exchange naming convention is a little different from the one we are used to. Instead of using a three- or more letter code, they name their stock as 0016.HK, 2345.HK and so on. As an advanced economy Hong Kong's strong banking sector is well developed and therefore no language barriers were encountered.

#### 3.2 Models

#### 3.2.1 Black-Scholes implementation

The benchmark Black-Scholes pricing model has a simple implementation in RStudio. With all required variables already prepared, all we need to do is to write the equation of the European call option formula and fill in the correct variables. The cumulative normal distribution function of the standard normal distribution is an implemented function of RStudio, removing the need for statistic tables in our hands.

#### 3.2.2 Diluted Black-Scholes implementation

The diluted version of the BS model was not much harder to write. The former results were multiplied with the dilution factor.

#### **3.2.3** Galai-Schneller implementation

This model was a little harder to implement. There was one thing we had to keep in mind; the warrant price was expressed using the warrant price itself. This means we could not get an explicit equation for a warrant price but needed to solve the equation with iterations.

The algorithm runs as follows:

- 1. A comparing value to begin the iteration with was chosen (Black Scholes model results)
- 2. The warrant price was assigned equal to this comparing value
- 3. The warrant price based on the current stock price, strike price of the warrant, time to maturity, volatility, risk-free rate, number of issued warrants, number of shares and the underlying asset value expressed as  $S_0 + \frac{M}{N}W_0$  was calculated.
- 4. The calculated warrant price assigned to the comparing value
- 5. If the warrant value and comparing value are the same with precision on 4 decimal digits, stop. If not, repeat from step 2.

This way we can be sure that the algorithm stops at some point (due to the decimal rounding) and that our result will be quite precise. The code can be seen in Appendix 1.

Let us think about why the algorithm comes to a solution. This is for two reasons, one being the fact that the European call price can be always calculated since it is given with an explicit equation. The second reason is that the comparing value is converging to the warrant price. There are multiple approaches to the iteration problem but given the relatively small amount of data it was enough to write it rather than use some of the more complex options.

#### **3.2.4** Ukhov model implementation

The Ukhov model has proven to be the biggest challenge of this work, mostly because of its practical implementation. The code is not long, however the logic behind it required some time

to be developed. We had to use a special package "nleqslv" designed to solve systems of nonlinear equations. Three different functions were written. The first one captivated the whole system of two equations as described above. The second and the third ones were called from within the first one, and calculated the non-linear parts, such as the BS model variation and the hedge ratio of the stock. No iteration code was needed from the author's side this time, as the "nleqslv" package provided a handy function to resolve the proposed system of equations. After this process came to an end, the warrant price was calculated simply using the formula W =

$$\frac{E^*-SN}{M}.$$

One problem that had to be resolved in order for this model to be useful was caused by the outstanding number of shares and issued warrants. The computation of this model requires Jacobian matrices. There are a few pre-conditions for the matrix to provide a solution. Even though it is possible to allow singular Jacobian, other ways of bypassing this issue exist. Why is a singular Jacobian a problem? Because it is a sign of a diverging solution, similar to the conditions of convergence for the limit  $\lim_{n\to\infty} \left(\frac{1}{a}\right)^n$ ,  $a \in R$ . This limit converges if and only if  $a \in (-\infty; -1) \cup (1; \infty)$ . When  $a = \pm 1$  the limit oscillates and if  $a \in (-1; 1)$  the value of the limit grows above all bounds. This means a singular Jacobian would not provide a solution.

There are multiple ways how to deal with this. One of them is to change the initial estimation you input into the equation. The same as the comparing value into the Galai-Schneller model, this model also required an initial estimation of a vector. The other way that was applied is to make sure the values are within the same range. We had strike prices under 100, but at the same time an outstanding number of shares in billions. Rounding might lead to loss of data, which leads to almost singular matrices. Since it is the dilution ratio and not the values themselves that matter, we could divide the outstanding shares and issued warrants by 100,000,000 to get results. All codes can be viewed in Appendix A.

## 3.3 Results

#### **3.3.1 Empirical volatility**

The next part of preparations was calculating the stock volatilities based on the historical data. We used two-year data from 01.01.2016 until the day before the warrant listing date. A small problem occurred because of missing values, which was dealt with by replacing the missing values by the mean of the stock in the period. Log returns of the stock prices were calculated. We will take a closer look at the log stock returns of the stocks in relation to their sigma. This is meant to demonstrate the relationship between high jumps in stock returns to the volatility of the returns.

To present the practical meaning of volatility, a series of graphs of stock returns can be found in Appendix B. Here it is clearly visible that a stock with higher volatility has bigger jumps in the returns. It is also noticeable that the stock returns look like random variables, or even white noise without any at first-sight visible underlying process or deterministic part.

The GARCH model was also implemented. GARCH(1,1) was used based on the model results and respective criterions. The "rugarch" package had to be installed. The *ugarchspec* function of this package allows you to specify different process variants, i.e. sGARCH, eGARCH, iGARCH, define the error distribution model, ARMA(m,n) process and so on. We used the regular GARCH process to model the stock volatilities.

Implied volatilities were received from the price of an option of the same strike price as the warrant (or the closest). As these are already calculated in the data provided by the Hong Kong exchange, no additional work was required. **Table 2** shows volatilities of the respective warrants at the pricing date.

		Equal		Implied
Warrant	Stock .HK	weight	GARCH	Volatility
17405	0016	0.2635	0.2739	0.2538
14255	0016	0.2631	0.2532	0.2677
13654	0016	0.2626	0.2351	0.2826
13787	0700	0.3977	0.3800	0.3144
17092	0939	0.2708	0.2286	0.2899
16243	1928	0.3995	0.3694	0.3281
15869	1928	0.3993	0.3662	0.3151
14943	2018	0.5440	0.5663	0.5221
12835	2018	0.6599	0.6109	0.5089
13680	2318	0.4318	0.3965	0.2966
12583	2318	0.4318	0.3965	0.2620
12588	2318	0.4316	0.3657	0.2472
15029	2600	0.6008	0.4031	0.5225
14474	2628	0.3280	0.3141	0.3462

Table 2: Estimated volatilities

No pattern such as a constant undervaluation or overvaluation with a specific method is visible. Implied volatility should give the best results, as it is reversely calculated from the current option prices, therefore includes also the volatility premium. We have a few warrants that are quoted on the same stock, i.e. warrants 17405, 14255 and 13654. These have different volatilities. This is because they are priced on a different date. We can see the difference is not that big for the equal weight method, which is caused by the fact that all observations have the same weight. That would mean that adding additional observation would not change the mean value so much. On the other hand the GARCH model has bigger differences, which would correspond to the latest volatilities having the highest weight. Implied volatility, being calculated from the current market data and depending on the underlying financial derivative price, can give different results even within one day. That is because the implied volatility of two options quoted for the same stock at the same date would be different if their strike price, or time to maturity is different.

To get a better idea about the relationship between warrants, stocks, firms and industries we can refer to **Table 3.** The Sun Hung Kai Properties had the lowest equal weighted and GARCH volatility in the sample. The lowest implied volatility was calculated on an option of Ping An Insurance Company of China (0.247), but volatility calculated a week earlier was higher (0.297)

on a different option. Highest volatility belonged to AAC Technologies Holdings Inc., where all three volatility estimations were higher than in any other case.

Stock .HK	Name	Industry
0016	Sun Hung Kai Properties Ltd.	Real estate
0700	Tencent Holdings Ltd.	Conglomerate
0939	HSBC Holdings	Finance
1928	Sands China	Hospitality
2018	AAC Technologies Holdings Inc.	Technology
2318	Ping An Insurance Company of China	Finance
2600	Aluminium Corporation of China Ltd.	Metals
2628	China Life Insurance Company Ltd.	Finance

Table 3: Underlying stock companies

In Excel we plotted four distributions – one empirical and three theoretical normal distribution  $N(\mu, \sigma)$ , where  $\mu, \sigma$  are calculated based on the equally weighted standard method, GARCH and the implied volatility. Here we see an obvious bias of the real log stock returns from the normal distribution plotted with the historical volatility. This is where implied volatility performed better in some cases, and GARCH in other.



Graph 1: Distribution of 0016.HK stock log return on warrant 17405 volatilities

**Graph 1** shows the stock return distribution of the 0016.HK stock with respect to the 17405 warrant volatilities. The three volatility estimators were similar to each other and it is visible that the stock log return has lower volatility, and might also follow a different probability distribution.



Graph 2: Distribution of 0939.HK stock log returns on warrant 17092 volatilities

On **Graph 2** we can see an improvement compared to **Graph 6**. Here GARCH with the lowest volatility predicted the stock data the best.

Finally, **Graph 3** shows the distribution of the prices of stock 1928.HK of Sand China, a technological company. Here the implied volatility was the lowest; therefore, it could best fit what seems to be a distribution of a much lower standard deviation. This also resulted in the bad prediction of the respective warrant prices for warrants 16243 and 15869. In all three graphs, we see undeniable fat tails of the real log returns on stocks. This is a result of more frequent extreme values on the market, as well as a higher frequency of close-to-mean values. On Fama and French Forum<sup>5</sup> the authors agree that the returns are rather symmetrical around the mean but compared to normal distribution, they have fat tails. Fat tails would result in more frequent occurrences of extreme variables on both sides. An alternative fat tail variant would be a low degree t-distribution or symmetric non-normal ones.

<sup>&</sup>lt;sup>5</sup> <u>https://famafrench.dimensional.com/questions-answers/qa-are-stock-returns-normally-distributed.aspx</u>



Graph 3: Distribution of 1928.HK stock log returns on warrant 15869 volatilities

**Table 4** shows the basic data about modelled warrants. All of them were call European style warrants traded in Hong Kong dollars, either half-year or year to maturity, with no dividends payed in the last year. Warrants with higher stock prices have a higher entitlement ratio. Warrants that are very close to reaching the spot stock price are more expensive, such as warrant 14255 or 14943. On the other hand, warrants that have big difference between the strike and the spot price are very cheap to buy, i.e. 13787 or 12583.

Warrant	UL	Date	r	Μ	Ν	W	K	ER	S	t
17405	0016	12/31/2019	2.43	2.00E+07	2.90E+09	0.560	126.80	10	119.30	0.5
14255	0016	12/10/2019	2.35	2.00E+07	2.90E+09	0.620	118.80	10	113.30	0.5
13654	0016	11/29/2019	2.35	2.80E+07	2.90E+09	0.400	128.88	10	113.90	0.5
13787	0700	12/4/2019	2.35	1.00E+08	9.55E+09	0.178	360.88	100	328.60	0.5
17092	0939	12/31/2019	2.43	5.00E+07	2.40E+11	0.246	7.68	1	6.73	0.5
16243	1928	12/27/2019	2.43	7.00E+07	8.09E+09	0.182	45.93	10	41.40	0.5
15869	1928	12/20/2019	2.43	7.00E+07	8.09E+09	0.123	45.88	10	39.60	0.5
14943	2018	12/18/2019	2.41	4.00E+07	1.21E+09	0.800	72.85	10	67.60	0.5
12835	2018	11/8/2019	2.26	3.00E+07	1.21E+09	0.410	61.88	10	51.60	0.5
13680	2318	12/3/2019	2.35	1.55E+07	7.45E+09	0.510	95.00	10	90.00	0.5
12583	2318	12/2/2019	2.35	3.00E+07	7.45E+09	0.081	118.88	5	89.55	0.5
12588	2318	12/2/2019	2.35	1.85E+07	7.45E+09	0.246	99.99	10	89.55	0.5
15029	2600	12/18/2019	2.41	4.00E+07	3.94E+09	0.255	2.95	1	2.58	0.5
14474	2628	12/12/2019	2.40	4.00E+07	7.44E+09	0.265	25.08	5	20.85	1

Table 4: Warrant attributes

- UL underlying stock code
- R risk-free rate to date
- T time to maturity to date
- M number of issued warrants
- N outstanding number of shares
- W warrant value to date
- K strike price
- ER entitlement ratio
- S underlying stock price to date

Note that warrants 16243 and 15869, both belonging to the same underlying asset 1928, have almost the same strike price, and also the calculated volatilities are very similar. Therefore, most of the difference in their prices would be caused by the movement of the stock price in the week they are quoted apart. Another good example of the impact of the market variables are warrants 12583 and 12588 quoted in the same day for a different strike price and different entitlement ratio. The former is three-times lower in market price (0.081) for two reasons. First, half of the price is reduced due to the entitlement ratio difference. Second, the strike price is much higher for the first of the two warrants. Now let us take a look at 13680 and 12588, again both quoted on 2318.HK stock. They are listed one day apart, with only the implied volatility being a little different. However, the price of 13680 is twice as much (0.51) than that of the warrant 12588 (0.246). This is caused by the difference in the strike price of the warrants, 95 and 99.99, respectively.

		Equal		
Warrant	Actual	weight	GARCH	Implied
17405	5.600	6.4924	6.8418	6.1694
14255	6.200	6.6359	6.3237	6.7831
13654	4.000	3.8272	3.0582	4.4074
13787	17.800	25.8357	24.2154	18.2776
17092	0.246	0.2198	0.1528	0.2518
16243	1.820	3.1585	2.8128	2.3421
15869	1.230	2.4898	2.1386	1.6129
14943	8.000	8.7585	9.1902	8.3363
12835	4.100	6.2409	5.5396	4.1027
13680	5.100	9.3336	8.4359	5.8992
12583	0.405	3.2681	2.5990	0.6263
12588	2.460	7.4817	5.8357	2.9946
15029	0.255	0.3128	0.1714	0.2562
14474	1.325	1.4514	1.3438	1.5936

Table 5 shows the Black-Scholes model results by the respective volatility estimation method.

Table 5: Black-Scholes model

Most of the times we observe the models over valuating the real warrant prices. It is clear at first

sight that implied volatility estimations gave much better results than the other two methods.

<b>Table 6</b> shows the distributed Black-Scholes model.
---

		Equal		
Warrant	Actual	weight	GARCH	Implied
17405	5.600	6.4879	6.8371	6.1652
14255	6.200	6.6313	6.3193	6.7784
13654	4.000	3.8235	3.0553	4.4032
13787	17.800	25.8330	24.2129	18.2757
17092	0.246	0.2198	0.1527	0.2517
16243	1.820	3.1558	2.8103	2.3401
15869	1.230	2.4876	2.1368	1.6115
14943	8.000	8.7296	9.1599	8.3088
12835	4.100	6.2254	5.5258	4.0926
13680	5.100	9.3317	8.4342	5.8980
12583	0.405	3.2655	2.5969	0.6258
12588	2.460	7.4798	5.8343	2.9939
15029	0.255	0.3096	0.1697	0.2536
14474	1.325	1.4498	1.3424	1.5918

Table 6: Diluted Black-Scholes model

The expected slight decline of the predicted price due to the dilution effect caused better results in most of the cases. However, the dilution factor for these companies is very small and did not bring too much additional explanatory power.

		Equal		
Warrant	Actual	weight	GARCH	Implied
17405	5.600	6.2163	6.5552	5.9031
14255	6.200	6.3742	6.0711	6.5171
13654	4.000	3.5987	2.8612	4.1565
13787	17.800	24.7560	23.1843	17.4305
17092	0.246	0.2064	0.1422	0.2371
16243	1.820	3.0204	2.6854	2.2299
15869	1.230	2.3627	2.0239	1.5178
14943	8.000	8.4370	8.8565	8.0268
12835	4.100	5.9613	5.2821	3.8931
13680	5.100	9.0067	8.1339	5.6690
12583	0.405	3.0242	2.3902	0.5503
12588	2.460	7.1612	5.5663	2.8226
15029	0.255	0.2986	0.1621	0.2439
14474	1.325	1.3723	1.2686	1.5094

In **Table 7** we can see the results of estimating warrant prices with the new Galai-Schneller model.

Table 7: Galai-Schneller model

The Galai-Schneller model performed even better than the diluted Black Scholes model. The estimations are not so overvalued as before, and mostly for the implied volatility we can observe quite precise predictions.

**Table 8** shows the Ukhov model. The Ukhov model seems to do quite well, but further comparison will have to be done before evaluating if it performed better than the Galai-Schneller model. For a few warrants, i.e. 13787, 14943, 13680 or 12583 there is a huge difference between pricing with equal weight and GARCH, or with implied volatility. In these cases, we observe the former two to result in bad estimations of the real warrant prices, whereas the implied volatility almost copies the real price. For thirteen warrants we can see that the Black Scholes model overvalued the real market price, even with the implied volatility. This could be caused by the fact that the implied volatility is calculated on option prices, which are higher than the respective warrant prices.

		Equal		
Warrant	Actual	weight	GARCH	Implied
17405	5.600	6.4922	6.8416	6.1692
14255	6.200	6.6358	6.3235	6.7830
13654	4.000	3.8266	3.0576	4.4069
13787	17.800	25.8356	24.2153	18.2775
17092	0.246	0.2198	0.1528	0.2518
16243	1.820	3.1584	2.8126	2.3420
15869	1.230	2.4895	2.1384	1.6126
14943	8.000	4.3304	4.3280	8.3374
12835	4.100	6.2400	5.5386	4.1018
13680	5.100	9.3336	8.4359	5.8992
12583	0.405	3.2674	2.5980	0.6260
12588	2.460	7.4816	5.8356	2.9945
15029	0.255	0.3126	0.1713	0.2562
14474	1.325	1.4512	1.3437	1.5934

Table 8: Ukhov model





**Graph 4** presents the results of pricing warrant 17405 quoted on the 0016.HK stock. Here we see the lowest dispersion in the actual results with the results predicted with the implied volatility. Best results were received from the Galai-Schneller model.



Graph 5: Pricing of warrant 14255

**Graph 5** shows the results of pricing warrant 14255 quoted also on the 0016.HK stock. Here the results of the Galai-Schneller model are visibly better than the rest. Interesting fact is that in this case implied volatility predicted the price the worst, whereas the GARCH estimation predicted it the best.



Graph 6: Pricing of warrant 13654

On **Graph 6** we can see the predicted prices of the 13654-warrant quoted on 0016.HK. None of the models' results were distinctive from the other. Equal weight gave the best results, whereas the GARCH model the worst ones.



Graph 7: Pricing of warrant 13787

On **Graph 7** the results of pricing the warrant 13787 on the underlying stock 0700.HK are presented. Here the implied volatility clearly outperformed the other two methods for all models. Equal weight estimations gave the worst results.



Graph 8: Pricing of warrant 17092

Pricing of warrant 17092 quoted on 0939.HK is presented on **Graph 8**. The best results are achieved with implied volatility. Regarding models, none of them was visibly better than the rest.



Graph 9: Pricing of warrant 16243

Graph 9 shows the results of pricing warrant 16243 on the stock 1928. Here we see that none of the combination of models with volatility estimations gave as good results as they did before. Nonetheless, the implied volatility is still the best. Galai-Schneller model seems to have the lowest deviation from the actual market price from the choice of used models.



Graph 10: Pricing of warrant 15869

Results of pricing the warrat 15869 also on stock 1928.HK can be found on Graph 10. Implied volatility and Galai-Schneller model seem to be the best estimators. This is the third time we have seen equal weight with the worst results, GARCH on the second place and implied volatility as the best proxy.



Graph 11: Pricing of warrant 14943

Graph 11 shows the results of pricing warrant 14943 on the underlying asset 2018.HK. Interesting on this warrant is the fact the Ukhov gave quite bad estimations for both equal weight and GARCH volatilities.



Graph 12: Pricing of warrant 12835

The warrant 12835 results depicted by Graph 12 show that the implied volatility is giving very accurate valuations of the actual market price.



Graph 13: Pricing of warrant 13680

Both **Graph 13, Graph 14** and **Graph 15** shows common pattern. Equally weighted volatility performs the worst, GARCH is the second best and the implied volatility, although still overestimating the real price, performs the best. In the case of the 12583 warrant, the implied volatility prediction is much better than the rest, which overpriced the warrant more than five times.



Graph 14: Pricing of warrant 12583



Graph 15: Pricing of warrant 12588



Graph 16: Pricing of warrant 15029

**Graph 16** shows warrant 15029. Interesting fact here is that the GARCH model estimation resulted in undervaluation of the real price. This might be caused by a low current market volatility of the asset, not counting the risk premium that is included in the implied volatility. Finally, **Graph 17** shows the last warrant, 14474. Deviations from the real stock price were not significant for GARCH volatility, also all the four models have shown similar prediciton power.



Graph 17: Pricing of warrant 14474

**Graph 18** shows the plotted mean squared errors of the models. **Graph 19** shows the mean absolute deviations and **Graph 20** the mean absolute percentage error. The respective values of the MSE, MAD and MAPE evaluators can be found in **Table 9**, **Table 10** and **Table 11**.



Graph 18: Model MSE

	Equal		Implied
Model	weight	GARCH	volatility
<b>Black-Scholes</b>	8.9632	8.9470	0.1883
Diluted BS	8.9470	5.4333	0.1854
Galai-Schneller	7.1118	4.1103	0.0827
Ukhov	9.8830	6.3080	0.1882
T-11. 0. MOE			•

Table 9: MSE

The mean squared error estimator shows that implied volatility performed the best. As we have seen above, it gave much better results in almost all the cases. Historical volatility calculated on the daily log stock with equal weights returns gave the worst results. If we were to compare the models' prediction power, the Galai-Schneller model gave the best results regardless of the used volatility method. On the other hand, the Black Scholes model gave the worst results. This is expectable, since this would be the price of an option with the same factors as the warrants, but option should be more expensive since there is no dilution effect occurring at the exercise date. The diluted Black Scholes model was second best, leaving Ukhov at the third place.



Graph 19: Model MAD

	Equal		Implied
Model	weight	GARCH	volatility
Black-Scholes	1.9545	1.9496	0.3652
Diluted BS	1.9496	1.5924	0.3619
Galai-Schneller	1.7240	1.4060	0.2399
Ukhov	2.1623	1.7739	0.3650

Table 10: MAD

The mean absolute deviation estimator shows smaller differences between the prediction power of the three volatility measures. Again, the Galai-Schneller model gave the best results for all volatilities.



Graph 20: Model MAPE

	Equal		Implied
Model	weight	GARCH	volatility
Black-Scholes	96.22%	95.99%	15.12%
Diluted BS	95.99%	76.96%	15.05%
Galai-Schneller	87.36%	70.59%	10.82%
Ukhov	98.80%	79.24%	15.11%
T.1.1. 11. MADE			

*Table 11:* MAPE

The mean absolute percentage error shows the percentage with which our estimates differ from the market values. Here equally weighted standard volatility performed the worst. Implied volatility has undeniably the best results for all three measures. When it comes to models, Galai-Schneller has the best results again. Black and Scholes and Ukhov gave the worst results at this measure.

Considering the above, the Galai-Schneller model outperformed the other three. Regardless of volatility estimates used, it had the lowest mean average percentage error, mean absolute deviation and mean absolute error. Implied volatility had the best results for all models. Further enhancement of the models could be done, as suggested by Chan and Pinder (2000) and Abinzano and Navas (2007). They suggest that the credit risk of the issuer should be taken into consideration. Exhaustive research on a broad number of warrants comparing different models has yet to be done, since the evaluation of credit risk must be done on an individual basis for each issuer.

# **4** Conclusion

The broad choice of warrants presented in this work helped to uncover the problem of using stock volatility for the four introduced pricing models. The observed volatility on log stock returns could be modeled with normal distribution but would not provide as good results as using other distributions with fatter tails.

Being the biggest factor in the warrant price that we can influence, it is important to compare different methods of volatility estimations. We included both historical and implied volatility, using the standard method with equal weights and the GARCH method and the implied volatility based on the option call price of the related underlying stock. Even though the historical methods managed to provide an adequate solution in some cases, implied volatility outperformed both of them. Regarding the model prediction power, the outcome is that the Galai-Schneller model outperformed the other ones. Broader research for other markets would give a better comparison on the prediction power of the four used models.

Warrant price estimated with respect to the implied volatility by Black Scholes model should be equal to the price of an option with the same attributes. This would be higher than the real market warrant price, due to the dilution factor and the market pricing. So even if we take the implied volatility reversely computed from the current option prices, the Black Scholes model would overestimate the real warrant prices.

This being said, the outcome of our analysis is clear. Implied volatility and Galai-Schneller model represent a good method to describe the warrant pricing process of the market. Future research could be done including models with credit risk and the comparison with other markets. Even though there are more warrants issued every day, the available data is still not broad enough for warrant pricing to become as mainstream as valuating stock prices. This makes the topic of warrants much more interesting and good ground for future research.

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Image 1: Newton's method19
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# 6 Appendix A

## 6.1 Rstudio code

```
BlackScholes1 <- function(S,K,T,sigma,rf)
ł
    d1 <- ((\log(S/K) + (rf + (sigma^2)/2) * T) / (sigma * sqrt(T)))
    d2 \ll (d1 - sigma * sqrt(T))
    value <- (S * pnorm(d1) - exp(-rf^*T) * K * pnorm(d2))
    print(value)
   return(value)
}
#pricing warrants
 for (i in 1:12)
 {
    S <- as.numeric(warrants[i,25])
                                                                   # S = stock price
    K <- as.numeric(warrants[i,22])
                                                                   # K = strike price
    T <- 0.5
                                                                   # T = Expiration Date
                                                                   # sigma = volatility
    sigma <- sigmas[i]</pre>
                                                                    \# rf = risk free rate
    rf <- as.numeric(warrants[i,3])/100
    value = BlackScholes1(S,K,T,sigma,rf)
   results[i,3] = value
    M <- as.numeric(warrants[i,10])/as.numeric(warrants[i,23]) # M - number of warrants
    N <- as.numeric(warrants[i,26])
#diluted model
```

results[i,4] = (N/(M+N)) \* value

```
}
```

```
{
    UAV <- S + M/N * W0
    d1 <- ( (log(UAV/K) + (rf + (sigma^2)/2)* T) / (sigma * sqrt(T)) )
    d2 <- (d1 - sigma * sqrt(T)
    value <- (N/(N+M)) * (UAV * pnorm(d1) - exp(-rf*T) * K * pnorm(d2))
    return(value)
}</pre>
```

```
BS_iteration <- function(S,K,T,sigma,rf,M,N,W0,value)
{
    repeat
    {
        W0 = value
        value = BlackScholes(S,K,T,sigma,rf,M,N,W0,value)
        print(value)
        if (W0 == value)
        {
            return(value)
            break
        }
        }
        }
    }
```

```
}
```

}

}

```
#initializining table of results
```

```
names <- c("Warrant", "Actual", "Black-Scholes", "Diluted BS", "Darden", "Ukhov")
results <- data.frame(1:22, 1:22, 1:22, 1:22, 1:22)
colnames(results) <- names
for (i in 1:22)
{
    for (j in 1:6)
    {
        results[i,j] = 0
    }
    results[i,1] = warrants[i,1]
    results[i,2] = warrants[i,16]</pre>
```

}

```
#pricing warrants
```

warrants <- read\_excel("My\_data/New\_warrants/Warrant\_datafile.xlsx",col\_types =
c("numeric", "date", "numeric", "num

```
# valuating half year warrants
 T < -0.5
 for (i in 1:22)
 {
  S <- as.numeric(warrants[i,25])
                                                                    \# S = stock price
  K <- as.numeric(warrants[i,22])
                                                                    # K = strike price
  sigma <- sigmas[i]</pre>
                                                                    # sigma = volatility
                                                                    \# rf = risk free rate
  rf <- as.numeric(warrants[i,3])/100
  M <- as.numeric(warrants[i,10])/as.numeric(warrants[i,23])
                                                                    # M - number of warrants
  results[12,2] = results[i,2] * as.numeric(warrants[i,23])
  N <- as.numeric(warrants[i,26])
                                                                    # N - outstanding shares
  W0 <- 3
                                                                     # W0 - initial guess
  value <-3
  value = BS_iteration(S,K,T,sigma,rf,M,N,W0,value)
  results[i,5] = value
 }
# valuating one year warrants
 T <- 1
 indexed <- c(13, 14, 17, 9, 22)
 for (i in indexed)
 {
  S <- as.numeric(warrants[i,25])
                                                                     \# S = stock price
                                                                    # K = strike price
  K <- as.numeric(warrants[i,22])
  sigma <- sigmas[i]</pre>
                                                                    # sigma = volatility
  rf <- as.numeric(warrants[i,3])/100
                                                                    # rf = risk free rate
  M <- as.numeric(warrants[i,10])/as.numeric(warrants[i,23]) #ent. ratio
  N <- as.numeric(warrants[i,26])
                                                                    # N - outstanding shares
  W0 <- 3
                                                                    # W0 - initial guess
```

```
value <- 3
```

```
value = BS_iteration(S,K,T,sigma,rf,M,N,W0,value)
```

```
results[i,5] = value
 }
########
           UKHOV MODEL #######
# first we need the two equation system, x \le c(1:10)
# (V,S,T,K,sigma2,rf,1(ent.rat,k),N,M, sigma) = x here V, sigma2 obsolete...
# a - vector(V, sigma2)
 Ukhov <- function(a)
 {
    y \le numeric(2)
   #first equation of Ukhov system
   \# 0 = V - S*N - M * WarVal(V,T,K,sigma2,rf,1(ent.rat),N,M)
   y[1] = a[1] - x[2] *x[8] - x[9] * warVal(a)
   #second equation of Ukhov system
      0 = ((V * deltaS)/S) * sigma2 - sigma
   #
   y[2] = ((a[1] * deltaS(a)) / x[2]) * a[2] - x[10]
   #returning y
   У
 }
# warVal function calculating the diluted price of warrant based on Galai-Schneller
\# (V,S,T,K,sigma2,rf,1(ent.rat,k),N,M, sigma) =x
```

```
# (V,S,T,K,sigma2,rf,1(ent.rat,k),N,M, sigma) =x

warVal <- function(a)

{

result = 0

#mi = (log((k*V)/(N*K)) + (rf + 0.5 * sigma2^2) * T) / (sigma2 * sqrt(T))

mi = (log((x[7]*a[1])/(x[8]*x[4])) + (x[6] + 0.5 * a[2]^2) * x[3]) / (a[2] * sqrt(x[3]))

#result = ( 1/( M+N)) * ( (k*V * pnorm(mi)) - ((exp(-rf*T)) * N * K * pnorm(mi - sigma2 * sqrt(T)) ) )

result = ( 1/( x[9]+x[8])) * ( (x[7]*a[1] * pnorm(mi)) - ((exp(-x[6]*x[3])) * x[8] * x[4] * pnorm(mi - a[2] * sqrt(x[3])) ) )

result

}
```

```
# deltaS function
\# (V,S,T,K,sigma2,rf,1(ent.rat,k),N,M, sigma) =x
 deltaS <- function(a)
 {
   result = 0
   \#mi = (\log((k*V)/(N*K)) + (rf + 0.5 * sigma2^2) * T) / (sigma2 * sqrt(T))
   mi = (\log((x[7]*a[1])/(x[8]*x[4])) + (x[6] + 0.5 * a[2]^{2}) * x[3]) / (a[2] * sqrt(x[3]))
   #result = ((N + M - M * pnorm(mi)) / (N * (N + M)))
   result = ((x[8] + x[9] - x[9] * pnorm(mi)) / (x[8] * (x[8] + x[9])))
   result
 }
#pricing warrants with Ukhov model
 for (i in 1:12)
 {
   S <- as.numeric(warrants[i,25])
                                                 \# S = stock price
   K <- as.numeric(warrants[i,22])
                                                 # K = strike price
   T <- 0.5
                                                 # T = Expiration Date
   sigma <- sigmas[i]</pre>
                                                 # sigma = volatility
   rf <- as.numeric(warrants[i,3])/100
                                                 # rf = risk free rate
   M <- as.numeric(warrants[i,10])/(1000000*as.numeric(warrants[i,23]))
   N <- as.numeric(warrants[i,26])/1000000
                                                 # N - outstanding shares
   x <- c(100, S, T, K, 100, rf, 1, N, M, sigma)
   a <- c(1,1)
   value = nleqslv(a, Ukhov, control=list(btol=.01))
   results[i,6] = (value x[1]-S*N)/M
 }
****
########
            Estimating volatility
                                (example)
                                             #########
#volatility
stock1 <- read_excel("My_data/New_warrants/3333.HK.xlsx", col_types = c("date",
"numeric", "numeric", "numeric", "numeric", "numeric"))
```

```
### replacing missing values with mean ##
```

```
#mean stock1
counter = 0
sum = 0
for (i in 1:739)
{
    if (!is.na(stock1[i,5])) {
        counter = counter +1
        sum = sum + stock1[i,5]}
}
mean = sum/counter
for (i in 1:739)
{
    if (is.na(stock1[i,5])) { stock1[i,5] = mean}
}
```

```
names <- c("stock4", "stock5", "stock7",
"stock8","stock9","stock10","stock11","stock12","stock13","stock14","stock15","stock16","sto
ck17","stock18","stock19","stock20","stock21","stock22")
log_stock2 <- data.frame(1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:760, 1:7
```

```
for (i in 1:739)
{
    log_stock[i,1] = log(stock1[i+1,5]/stock1[i,5])
}
```

```
sigmas <- c(1:22)
# variance sd1
 mean <- mean(log_stock[,1])</pre>
 sum <- 0
 for (i in 1:739)
 {
   sum = sum + (log_stock[i,1] - mean)^2
 }
 sd1 = sqrt(sum/738)
 sigmas[1] = sd1 * sqrt(183)
## GARCH ##
ug_spec <- ugarchspec(variance.model = list(model = "sGARCH", garchOrder = c(1, 1)),
   distribution.model = "std")
   ugfit = ugarchfit(spec = ug_spec, data = log_stock2[500:759,1])
   ugfit
   forecast <- ugarchforecast(ugfit, n.ahead = 1)</pre>
```

# 7 Appendix B







Log stock return on underlying stocks.