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Faculty of Economics

Main specialisation: Economic analysis

Is the Population Growth Affected by the Growth of GDP or vice versa? The Case of Africa

Diploma Thesis

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Rok: 2020

I hereby declare on my word of honor that I have written the diploma thesis independently with using the listed literature.

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Prague, 22.05.2020

I would like to express my sincere gratitude to my supervisor, Ing. Pavel Potužák, Ph.D., for beneficial advice and comments provided while writing the diploma thesis, and to Prof. Dr. Şaban Nazlıoğlu for providing code and valuable feedback on my model of Bootstrapped Panel Granger Causality model.

Abstract

The diploma thesis aims to analyse whether GDP growth per capita affects population growth or vice versa. The analysis of the relationship between GDP growth per capita and population growth is done by two models. The Bootstrapped Panel Granger Causality model finds no causality when population growth is a dependent variable and a few cases of causality in Africa for GDP growth per capita as dependent variable. The Dynamic Panel Data model estimates GDP growth per capita as a significant explanatory variable in all cases. Population growth is a significant explanatory variable in all cases except for the situation when the model includes health control variables. Used data for this analysis meets an established conditions for using both models.

Key words: endogenous population growth, gdp growth per capita, bootstrapped panel-granger causality model, dynamic panel data model

JEL classification: J13, O11

Abstrakt

Cílem této diplomové práce je analyzovat, jestli růst HDP na obyvatele ovlivňuje populační růst nebo naopak. Analýza vztahu mezi růstem HDP na obyvatele a populační růstem je provedena dvěma modely. Bootstrapped Panel Granger Causality nenašel žádnou kauzalitu, kdy je populační růst závislou proměnnou, a našel několik případů kauzality v Africe, kdy růst HDP na obyvatele je závislou proměnnou. Dynamic Panel Data model odhaduje, že růst HDP na hlavu je signifikantní vysvětlující proměnná ve všech případech. Populační růst je signifikantní vysvětlující proměnná ve všech případech kromě situace, kdy model zahrnuje kontrolní proměnné pro zdraví. Použitá data pro tuto analýzu splňují stanovené podmínky pro použití obou modelů.

Klíčová slova: endogenní populační růst, růst HDP na obyvatele, bootstrapped panel-granger causality model, dynamic panel data model

JEL klasifikace: J13, O11

Contents

Introduction

Models of endogenous growth are studied in detail at a lot of universities. But only a few variables, which are used in these models, have an exogenous form and not many economists research their properties and relationships with other variables. I have decided to analyze the population growth and research how it is connected to the GDP growth per capita.

The theoretical part of the thesis focuses briefly on the history of endogenous theory of economic growth. Next section contains a description of the first theories and models of the population growth, such as Leibenstein (1955) and Becker (1960). Then I focus on the theory written by Becker and Lewis (1973), which uses Hicks-Slutsky equations to describe the relationship between quality and quantity of children. Another chapter describes the Solow – Swan model with endogenous population growth in detail. In this chapter, I demonstrate a difference in the exogenous form and endogenous form of the population growth in the model. Next, I describe Niehans (1963) extension of the Solow – Swan model. Nerlove – Raut (1997) theory follows up in the previous chapter. The Nerlove – Raut model uses a three-factor production function and planar analysis to describe behavior of the population growth and the GDP growth. As the last model I describe the one written by Barro and Becker (1998) which uses microeconomic analysis in the equilibrium growth framework. The further part contains data description used in the empirical part.

The empirical part consists of two sections. The first is the Bootstrapped Panel Granger Causality model, which is used to estimate Granger causality between the population growth and the GDP growth per capita in each country. The second section describes the Dynamic Panel Data model which estimates the effect of the population growth on the GDP growth per capita and vice versa with different groups of control variables. The reason for using these two models is described in the relevant chapters.

The thesis aims to answer a question whether the population growth influences the GDP growth per capita or vice versa. This is investigated using the above-mentioned models.

1. Literature review

1.1. History of endogenous growth theory

The modern theory of economic growth was based mainly on ideas of economists, such as Adam Smith (1776), David Ricardo (1817), Joseph Schumpeter (1934), Frank Ramsey (1928), and others. Their ideas as equilibrium dynamics, diminishing marginal returns, competitive behavior, and many others were taken as a foundation stone in growth models.

From the chronological point of view, the first growth theory was built by Frank Ramsey (1928) in his article 'A mathematical theory of saving'. Although this article was written in 1928, this model was used mainly in the 1960s. It could be said that his theory of intertemporal separable utility function is used in many theories, in the same way as Cobb-Douglas production function.

Another growth theory, which tried to use Keynesian analysis, was made by Harrod (1939) and Domar (1946). Their theory is based on production function without the rate of technology growth and low elasticity of substitution between inputs. The Harrod-Domar model was replaced by the Solow-Swan (1956) growth theory, the first neoclassical growth model.

Solow-Swan model led to two main predictions. First is conditional convergence, which assumes that every country has a long-term equilibrium point. If the economy has a lower starting point, it tends to have some positive per capita GDP growth. Larger distance between the starting point and the equilibrium point leads to a higher rate of the GDP growth per capita. The second prediction is based on the rate of technology growth. In the absence of any technological improvement, and if the economy is in the steady-state, growth per capita has to be zero.

Next model was described by Cass (1965) and Koopmans (1963). Authors take the Ramsey analysis and make the model which is able to preserve conditional convergence with better transitional dynamics. Equilibrium of the Cass-Koopmans model is supported by a decentralized, competitive framework. Production factors (labor and capital) are paid their marginal products, total income exhaust the total product, and there is no economic profit under the assumption of constant returns to scale. This is consistent with the Pareto optimum rule.[1]

Another contribution to the theory of economic growth was made by Arrow (1962) and Sheshinski (1967). Their idea uses the concept of learning-by-doing, which starts to spill over discoveries into the whole economy. In their view, technology is nonrival. Romer (1986) used the spillover idea in his article. He shows a competitive framework, which is generating the equilibrium rate of technological progress. This approach, unfortunately, violates Pareto optimal rule.

Aghion and Howitt (1992) and Grossman and Helpman(1991) continue on Romer's R&D theories. Their contribution to the R&D theories is the addition of other variables, which are important for long-term economic growth, such as taxation, government actions, infrastructure services, property rights and their protection and financial markets. Another assumption is that if the economy cannot run out of ideas, the growth rate can remain positive. Acemoglu (2002) used this approach to determine whether technological progress augment labor or capital.

Improvement in data availability and quality in the 21st century helps academic field to research endogenous growth theory deeper. These data are coming not only from statistical offices but from private companies as well, which gives another viewpoint for economic research. A lot of research is focused on topics like human capital (for example Goldin and Katz (2007), Furman and Macgavie (2007)), and the role of innovation for economic growth. Research of Lentz and Mortensen (2008), Akcigit and Kerr (2018), and Acemoglu (2018) focus on the problem of reallocation of resources due to the innovation process.

Another interesting topic these days is research focused on the growth slowdown hypothesis. Decker (2016) shows a slowdown in productivity performance according to U.S. nonfarm business sector data. This was proven by Akcigit and Ates (2018), who seek the problem in the decreased diffusion of information. Bloom (2017) tries to find a relationship between the number of research engaged and the productivity of these research.

[1] Pareto optimal rule or Pareto efficiency is a situation where no individual can be better off unless someone else is worse off.

1.2. History of the economic approach to population growth

The first theory of the population growth and its mechanics was proposed by Malthus in the 18th century. His theory was later rejected, mainly because he does not count with technological progress, which eliminates his predictions as described in Galor and Weil (2000). Yet, his theory can be considered as fundamental for most of the subsequent theories. The topic about population growth was not academically researched properly until the 1950s. At that time, academic field started to ask questions how the endogenous mechanism of economic growth and income affects the population growth rate. Leibenstein (1955) based his theory on the allocation of investments in the industry. His assumption for his theory is that decreasing the mortality rate is easier than decreasing the birth rate. Under this assumption, he expects rapid population growth, which could cause problems for underdeveloped countries. By his hypothesis, investing in urban industrial-commercial areas leads to the decline of the birth rate, instead of investing into agriculture, which can cause a very small decrease or even increase in the birth rate. Leibenstein (1955) constructs cost-benefit analysis in which he takes these investments into account together with the social cost and effects of urbanization on the fertility rate. He claims that positive population growth leads to lower capital-labor ratio, therefore to the lower potential output per capita. Thus the adverse effect of population growth tends to reduce the rate of reinvestment.

In another article written by Becker (1960), fertility is taken as an endogenous variable that can affect the economic system. He constructed a theoretical framework for the relationship between family income and fertility. The child is considered as a good, which varies over time from consumption durable good to production durable good depending on the child's age. With the help of microeconomic analysis, he shows the decision-making process of parents. Preferences of parents of having the child, together with the demanded quality of children and the ability to produce children, are considered in the utility function. For the cost function, Becker (1960) implies that we can count it as the present value of the expected outlays plus the imputed value of the parent’s service minus the present value of expected money return plus the imputed value of the child’s services. In principle, the cost function is easier to calculate. If the costs are positive, the child is a consumption good, and utility has to be higher than costs. If the costs are negative, the child is a production good, and utility has to be larger than zero. Becker (1960), in the empirical test of his theoretical framework, shows that income has a large impact on the quantity and quality of children and these properties are related. He tests how the desired number of children is related to income. His analysis of crude cross-sectional data gives a negative relationship, but this data does not consider contraceptive knowledge. With this knowledge taken as a constant, the relationship is positive, which he considers more consistent with the secular decline in child mortality.

Duesenberry (1960) in his comment criticizes the Becker approach as inadequate, mainly because he just considers cash expendables and not a non-cash cost in the cost function. He argues that time spent with children is given mainly by social convections in social classes, which are connected to income. This means that the number of children, which parents want to have, will be influenced mainly by the tendency to advance the standard of living for children together with advance of the parent's welfare. The next conclusion of Duesenberry (1960) criticism is that money and time spent on education will vary with the social class of parents. This argument was supported by Willis (1973) and De Tray (1973). Another point made by Duesenberry (1960) claims that the Becker approach is not considering the elasticity of the substitution in the family utility function between parent's income and the level of living for children.

1.3. Becker – Lewis model

Becker and Lewis (1973) decided to react to this criticism. They made a new analysis which includes the shadow price of children with respect to their number. Under the assumption that parents are in control and have care about children's number and welfare, they introduce nonlinearities and nonconvexities into budget constraints and modification for utility function in constraint to the traditional theory of consumer choice.

Assume a pair of parents who are individual decision-makers. They consume a single composite consumption good (c), their utility function is also determined by the number of children (n) and their quality (b) of each one of them. This quality can have different characteristics for parents. The approach of Becker and Lewis is pragmatic, thus they consider quality as something that can be measured, for example, expenses on education, health, sport, etc. Quality is therefore a single composite good spent on children. For simplicity n is considered as a continuous variable, each child is identical, and parents do not prefer any child and treat them equally, so variable *b* is the same for every child. Thus, the parent's utility function is

$$u^*(c, b, n), where\ u^*{}_i > 0, \qquad i = 1{,}2{,}3. \tag{1}$$

To keep this example as simple as possible, let's assume that parents anticipate identical children, who will be born at the beginning of the parent's decision period. In other words, parents do not have any unexpected child. Parent's income *I* is spend on *c* for themselves and *bn* on their children. Parent's budget constraint is

$$c + bn \le I \tag{2}$$

The condition of nonlinearity is given by the term *bn*. Parents feasible bundles (*c*,*b*,*n*) are not convex. But this is not eliminating the possibility for utility function to be monotonicly increasing and quasiconcave. As shown in Nerlove, Razin and Sadka (1987, Ch. 5), traditional theory holds with linear budget constraint, but in most countries, economies are developing differently. When the family is deciding about having kids, it could be at different times when the economy is in decline or has a higher growth. This growth has a significant relationship with the income of parents. That's why Becker and Lewis (1973) consider a nonlinearity condition. The Becker-Lewis model makes possible to have small fertility of parents even if the child is a normal consumption good. The authors also consider the following consumer optimization problem

$$max\ u^*(c, b, n), \ \ such\ that\ c + bn \le I \tag{3}$$

From the term *bn* quality is the price of quantity and vice versa. In this case, Becker and Lewis (1973) argue that they cannot use condition for normality[2], because they are not asking if certain good is a normal good. Therefore, they created the hypothetical problem, where

[2] Condition for normality means that good is a normal good. Thus, increase in income leads to increase in demand as well.

they search for a sign of N with respect to I. The optimum (c, b, n) depends on the level of I, namely the elasticity of the variable to I ($C(I)$, $B(I)$, $N(I)$). Let's reformulate the problem on:

$$\begin{gathered} max\, u^*(c,b,n), \;\; such\ that\ c + p_b b + \ p_n n \ \leq I + M, \\ where\ \ p_b > \ 0, p_n \ > \ 0 \end{gathered} \tag{4}$$

Terms p_b and p_n are prices of quality and quantity of children and M can be interpreted as lump-sum transfers. By applying a mathematical approach, we receive an optimal bundle of c, b, and n:

$$\begin{gathered} \bar{C}(p_b, p_n, I + M), \qquad \bar{B}(p_b, p_n, I + M), \\ \overline{N}\,(p_b, p_n, I + M) \end{gathered} \tag{5}$$

where latter functions are conventional Marshallian demand functions and exhibit normality, so:

$$\bar{C}, \bar{B}, \bar{N} > 0 \tag{6}$$

It is straightforward to see a relationship between (c,b,n) and ($\bar{C}, \bar{B}, \bar{N}$) when we compare equation (3) and equation (4). If we evaluate $p_b = N(I), p_n = B(I)\ and\ M = N(I)B(I)$, then bundle ($c$,$b$,$n$) is equal to ($\bar{C}, \bar{B}, \bar{N}$):

$$\begin{gathered} \bar{C}\big(N(I), B(I), I + N(I)B(I)\big) = C(I), \\ \bar{B}\big(N(I), B(I), I + N(I)B(I)\big) = B(I) \\ \bar{N}\big(N(I), B(I), I + N(I)B(I)\big) = N(I) \end{gathered} \tag{7}$$

For our purpose we will differentiate just last two expressions with respect to I, where lower indexes are partial derivation:

$$
\begin{aligned}
&(\overline{B_2} + N\overline{B_3} - 1)\frac{dB}{dI} + (\overline{B_1} + B\overline{B_3})\frac{dN}{dI} = -\overline{B_3} \\
&(\overline{N_1} + B\overline{N_3} - 1)\frac{dN}{dI} + (\overline{N_2} + N\overline{N_3})\frac{dB}{dI} = -\overline{N_3}
\end{aligned}
\tag{8}
$$

Next, we apply Hicks-Slutsky equations to the hypothetic problem that is created from Eq. (4). From the above equation, we can see that $\overline{B_1} + B\overline{B_3}$ is the substitution effect of the "price" of the quality of children on the quantity of children demanded. Expression $\overline{B_2} + N\overline{B_3}$ is the substitution effect of the "price" of the quantity of children on the quality of children demanded. The same applies for $\overline{N_1} + B\overline{N_3}$ and $\overline{N_2} + N\overline{N_3}$. When we denote:

$$
\begin{aligned}
\overline{S_{bb}} &= \overline{B_1} + B\overline{B_3} \\
\overline{S_{bn}} &= \overline{B_2} + N\overline{B_3} \\
\overline{S_{nb}} &= \overline{N_1} + B\overline{N_3} \\
\overline{S_{nn}} &= \overline{N_2} + N\overline{N_3}
\end{aligned}
\tag{9}
$$

By symmetry we got $\overline{S_{bn}} = \overline{S_{nb}}$ which we can substitute and solve *dN/dI*:

$$
\frac{dN}{dI} = \frac{\overline{N_3}(1 - \overline{S_{nb}}) + \overline{B_3\, S_{nn}}}{(1 - \overline{S_{nb}})^2 - \overline{S_{bb} S_{nn}}} \tag{10}
$$

By using elasticity terms, the equation above becomes:

$$
\eta_{nI} = k\,\frac{\bar{\eta}_{nI}(1 - \overline{\varepsilon_{nb}}) + \bar{\eta}_{bI}\overline{\varepsilon_{nn}}}{(1 - \overline{\varepsilon_{nb}})^2 - \overline{\varepsilon_{bb}\varepsilon_{nn}}} \tag{11}
$$

In a similar way,

$$\eta_{bI} = k\frac{\bar{\eta}_{bI}(1-\overline{\varepsilon_{nb}}) + \bar{\eta}_{nI}\overline{\varepsilon_{bb}}}{(1-\overline{\varepsilon_{nb}})^2 - \overline{\varepsilon_{bb}\varepsilon_{nn}}}, \tag{12}$$

Where the terms are:

$$\eta_{nI} = \frac{dN}{dI}\frac{I}{N},$$

income elasticity of number of children with respect to parents income *N(I)*,

$$\bar{\eta}_{nI} = \overline{N_3}\frac{\bar{I}+\overline{NB}}{\bar{N}},$$

income elasticity of number of children in the steady state $\bar{N}()$, (considered positive),

$$\bar{\eta}_{bI} = \overline{B_3}\frac{\bar{I}+\overline{NB}}{\bar{B}},$$

income elasticity of quality of children in the steady state $\bar{B}()$, (considered positive),

(13)

$$k = \frac{I}{\bar{I}+\overline{NB}} < 1,$$

$$\bar{\varepsilon}_{nn} \equiv \frac{\bar{S}_{nn}p_n}{\bar{N}} = \frac{\bar{S}_{nn}\bar{B}}{\bar{N}},$$

own – price elasticity of number of children in the steady state $\bar{N}()$,

$$\bar{\varepsilon}_{bb} \equiv \frac{\bar{S}_{bb}p_b}{\bar{B}} = \frac{\bar{S}_{bb}\bar{N}}{\bar{B}},$$

own – price elasticity of quality of children in the steady state $\bar{B}()$,

$$\bar{\varepsilon}_{nb} \equiv \frac{\bar{S}_{nb}p_b}{\bar{N}} = \bar{S}_{nb},$$

cross – substitution elasticity.

From equation (11) and equation (12) we can clearly assume if cross – substitution elasticity is equal to one (this means than quality and quantity of children have unitary elasticity of substitution), then both income elasticities are positive due to negativity of the own - price elasticity, and quality meets a normality condition. Thus, the increase in income leads to an increase in the quantity and quality of children.

If cross – substitution elasticity is larger than one and with the assumption that total expenditure is increasing proportionally with the increase in income, *N(I)* or *B(I)* has to increase. In this example suppose an increase of *B(I)*, so $\bar{\eta}_{bI} > 0$). The numerator of equation (2) is negative hence denominator is negative too, and the income elasticity of *N(I)* is positive. Under these assumptions, we can say that the quality and quantity of children are again positive with an increase in income.

In the last example that is cross – substitution elasticity is smaller than one, there are two possibilities. The first possibility is when the denominator of equation (11) or equation (12) is positive. This can happen if own - substitution elasticities are relatively low. So if the income elasticity of quality is quite higher than the income elasticity of quantity, the child quantity falls with income and an increase in quality. The second possibility has a denominator smaller than zero. Then ceteris paribus own – substitution elasticities are relatively high, the quality of children is decreasing and the quantity of children is increasing with the increase in income.

Becker and Lewis (1973) discuss these findings. For the first example, they consider the pure substitution effect of the increase in the child cost. If we increase shadow price of quality relative to shadow price of quantity and shadow price of consumption, for example better contraceptive methods are developed exogenously, parents will have less children. This decrease leads to higher quality (cross – substitution elasticity is < 1). The Becker-Lewis theoretical model has the same conclusion as De Tray's (1973) article, which empirically studied an increase in mother's education. This increase has a positive effect on the quality of children and a negative effect on quantity of children. As the second example, they now consider pure substitution effects of equal percentage increases of quality and quantity. Then the income-compensated elasticity with respect to changes in quality of children and quantity of children tends to have higher numerical value than quality-compensated elasticity. Again this corresponds to De Tray's (1973) empirical research on the increase of wage for women that has a greater influence on the number of children than on quality of children. In conclusion, Becker-Lewis (1973) model was able to react to criticism of the previous model written by Becker (1960) and give a solution that is in line with empirical research.

As an extension for Becker – Lewis model we can consider work from Nishimura and Zhang (1995). They used the overlapping-generations model, where children transfer part of their income to the parents. This assumption adds another variable into the budget constraint of parents:

$$c + bn \leq I + an \qquad (14)$$

where a is a part of the income of children transferred to parents. To see how much parents will be consuming themselves, and how they determine the quality and quantity of children, we have to use a one-loop Nash equilibrium theorem. In this overlapping case, there could be two equilibria as a transfer from child to parents ("gift" equilibrium) in all periods and a transfer from parents to a child ("bequest" equilibrium) in all periods. Other equilibria could be a transfer in different periods between parents and children. But only two steady-state equilibria exist. First is when parents have zero savings, second when parents have positive savings. That gives an agent a problem of multiple equilibria solutions, which can lead to irrational decisions. Another complication is the assumption of agents, which takes the action of other agents as given. Raut (1996) suggests that this could lead to the maximum consumption of parents in the first period with small savings to get maximum transfer from children's income in the next period. That's why he used a sequential game framework and subgame perfection. This approach allows to take into account the influence of agent, who behaves out of the equilibrium. The author puts another complication for using overlapping-generation model. Parents can have no children if they have access to the social security system, which can replace income from children. Another solution was proposed by Azariadis and Drazen (1993). They suggest using a nonsequential bargaining framework.

1.4. Solow-Swan model with endogenous population growth

In this chapter, I will describe the Solow-Swan model witfh endogenous population growth. In the first part, I will derive a standard Solow-Swan model with exogenous growth. The second part will focus on the extension made by Niehans (1963). Next, I will add the three factor production function, which is inspired by an article made by Lee (1986) and developed by Nerlove and Raut (1997). The Solow – Swan model is a structural model. This structure can be made by the two-sector economy - households, firms, and the market.

Households have inputs and assets of the economy at their disposal. They choose how to split their income on consumption and savings. Households are independent. Thus, each household makes a decision how much time to spend on work and how many children they want to have. Firms hire capital and labor from households and produce output. Firms property are owned by households. Existence of the market allows firms to sell their output to households or to other firms. Households sell their inputs to the firms on the market too. The price is established by the interplay of demand and supply.

As an input, we consider capital, labor, and the level of technology. Capital is represented by all durable physical inputs, such as machines, buildings, and so on. These inputs were created in the past by the same production function. Capital is rival good, so one good cannot be used by two companies. The second input is labor. We consider labor as every action made by the human body that produces output. From the macro perspective, labor consists of time spent on work. Labor is considered as a rival input. The third input, technology or knowledge, gives workers an instruction how to make output goods from inputs. Technology is, in contrast to other inputs, a non-rival input, thus two firms can use the same technology at one time.

We consider at the moment a standard Solow-Swan model with exogenous population growth in discrete time, a closed economy with no government interventions or foreign market. Production has a constant return to scale with only two inputs:

$$Y_t = F(K_t, L_t) \tag{15}$$

where Y_t = output, K_t = capital stock, L_t = labor which we assume is the same as population. By dividing these variables by L_t we get:

$$y_t = f(k_t) \tag{16}$$

where $f(k) = F(k,1)$. Another assumption of this model is equality of savings, S_t, and gross investments, I_t. s is the saving rate and a constant fraction of output:

$$I_t = S_t = sY_t \tag{17}$$

Every rational household chooses a saving rate by a cost-benefit analysis of consumption today versus consumption tomorrow. For that, we need to consider preference parameters, wealth, and interest rates. For now, the saving rate will be an exogenous constant variable.

Capital stock changes with a gross investment, minus a constant rate of depreciation (every time period, some of the capital wears out, thus it cannot be used in production), δ, of the current capital stock. Capital in this model is homogenous, thus every unit of capital is the same:

$$K_{t+1} = (1-\delta)K_t + I_t = sF(K_t, L_t) + (1-\delta)K_t \tag{18}$$

Labor grows at the constant exogenous rate $\bar{n}$. Workers have the same skill, and every worker offers one unit of labor at the time as described in Nerlove and Raut (1997):

$$L_{t+1} = (1+\bar{n})L_t \tag{19}$$

We normalize initial population $L_0 = 1$. Combining equation (18) and equation (19) in per capita terms, we receive:

$$\begin{gathered}(1+\bar{n})k_{t+1} = sf(k_t) + (1-\delta)k_t \\ k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+\bar{n}} = g(k_t), k_0 \; given\end{gathered} \tag{20}$$

Solow-Swan model dynamics are described by a path of k_t, capital depreciates at a rate of depreciation, the population grows constantly, and investment is made by the proportion of output. Hence stationary solutions are:

$$k^* = g(k^*) \tag{21}$$

and the shape of the function g determines the local stability of these solutions. We need to have conditions which give us the nonnegative globally stable steady-state solution:

$$\begin{gathered} g'(0) > 1, \\ g'(\mathrm{k}) < 1, \text{for some } \mathrm{k} > 0, \end{gathered} \tag{22}$$

and g is concave. Production function only satisfies $g(k)$ conditions if:

$$\begin{gathered} f(0) = 0, \\ f'(0) > \frac{\delta + \bar{n}}{s}, \\ f'(\mathrm{k}) < \frac{\delta + \bar{n}}{s}, \text{for some } k > 0, \end{gathered} \tag{23}$$

and f is concave. Condition of concavity of the function f gives us necessarily unique solution when equation (21) holds:

$$k^* > 0, \text{for at that point } |g'(\boldsymbol{k}^*)| < 1 \tag{24}$$

Under these conditions, we can see clearly that $k^* = 0$ is an unstable solution.

The next section is described in Nerlove and Raut (1997). We substitute exogenous population growth in equation (19) for endogenous form of population growth. This form depends on the exogenous saving rate. For simplicity suppose that the growth rate of population depends on the level of per capita consumption:

$$\frac{L_{t+1}}{L_t} = 1 + n[(1-s)f(k_t)] = h(k_t) \tag{25}$$

And $n(c_m) = 0$ for some level of per capita consumption, $c_m = f(k_m)$. Let's substitute denominator of equation (18):

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{h(k_t)} = g(k_t), k_0 \ given \tag{26}$$

The capital-labor ratio still determines the dynamics of the economy with a system that is univariate and more complex with endogenous population growth formula.

Unfortunately, our previous conditions on s and δ and concavity of f no longer give us stationary points or explicitly the local stability or instability of such equilibria. However, we can compare the location and properties of a nontrivial steady state with steady states from the Solow-Swan model with exogenous population growth. Assume $\bar{k}^*$ as a stationary point in equation (20), thus:

$$\bar{k}^* = \frac{sf(\bar{k}^*) + (1-\delta)\bar{k}^*}{1+\bar{n}}$$

Or (27)

$$\frac{\bar{n}+\delta}{s}\bar{k}^* = f(\bar{k}^*)$$

Conditions applied previously on production function are satisfied, because $\bar{k}^* = 0$ is a stationary point and $f(k)$ intersects a straight line with a slope defined by $(\frac{\bar{n}+\delta}{s})$ in $k^* > 0$ as well. So with endogenous population $n(k) = n[(1-s)f(k)]$, from equation (26), we have:

$$\left[\frac{n(k^*)+\delta}{s}\right]k^* = f(k^*) \tag{28}$$

where k^* corresponds to the stationary point. Equation (27) shows us a linear solution of k, which relates to the Solow-Swan model with exogenous population growth. However, Equation (28) gives us a nonlinear function, which for simplicity I define as:

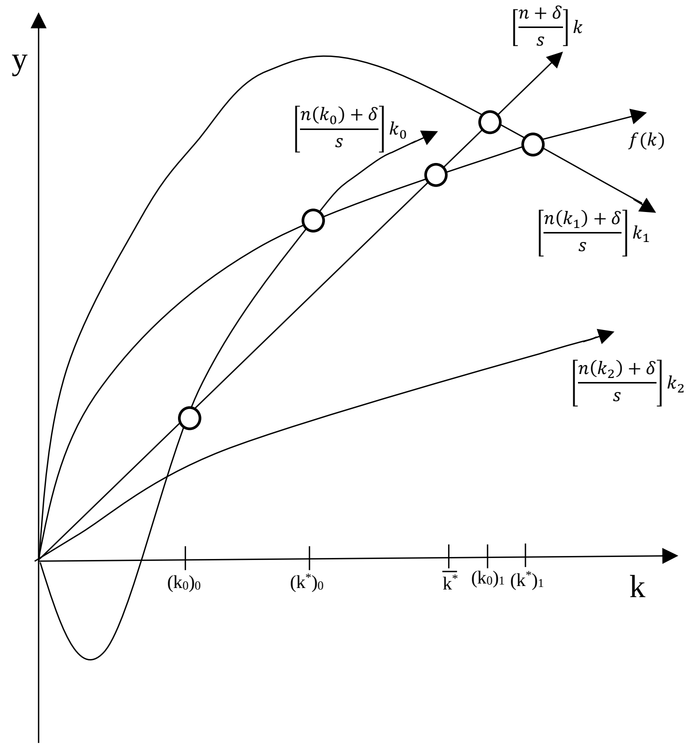
$$\left[\frac{n(k)+\delta}{s}\right]k = \rho(k) \tag{29}$$

As before, a stationary point is set by $\rho(k^*) = f(k^*)$, where properties are given by function $n(k)$. If $n(k)k = 0$ as $k = 0$, and thus y and $(1 - s)y = 0$, $\rho(0) = 0$. Consider the assumption that $n(k)$ is increasing in k and it is positive for values greater than some small value and $n(k) > \bar{n}$ for some $k > 0$. Then

$$\frac{n(k)+\delta}{s} + \frac{n'(k)}{s}k = \rho'(k) \tag{30}$$

which has to be greater than $(\bar{n} + \delta)/s$ since $n' > 0$, thus only one unique solution exists where $n(k_0) = n'$. Function $\rho(k_0)$crosses the line of production function at one point in Fig. 1. If $k_0 < \bar{k}^*$ then capital-labor ratio, k^*, in the Solow-Swan model with exogenous population growth is less than $\bar{k}^*$ in the Solow-Swan model with endogenous population growth, else $k^* > \bar{k}^*$. The shape of $n(k_1)$ gives values which increase by smaller values of k, and these values have to turn down and recross the line $\bar{n}$ with $n' < 0$. That creates another equilibrium point where the capital-labor ratio is greater than $\bar{k}^*$. Another solution is $n(k_2)$ which is falling when values of capital-labor ratio are very low, thus never reach the level $\bar{n}$. With this condition, there is no nontrivial stationary point or there have to be large values of the capital-labor ratio which can give us an equilibrium point. Other described solutions are shown in Fig 1.

Figure 1: Cases of the shape of ρ(k)



Source: Nerlove and Raut (1997), modified

From this perspective, it is clear that only endogenizing population growth does not give us a solution for the shape *n(k)*, and it does not explain the dynamics of the model. One solution to this problem could be using a utility-maximizing model to show the nature of function *n(k)*.

Suppose a nontrivial steady-state solution was found for the Solow-Swan model with endogenous population growth. To describe the dynamics of the model we need to differentiate *g* with respect to *k* in equation (26) and utilize equation (28). We get

$$g'(k^*) = \frac{sf'(k^*) - (1-\delta) + k^* n'(k^*)}{1 + n(k^*)}. \qquad (31)$$

This expression is greater than -1 if

$$f'(k^*) > \frac{-[1+n(k^*)]-(1-\delta)+k^*n'(k^*)}{s}. \tag{32}$$

That can be fulfilled only if $n'(k^*)$ has not got a very large positive value. For subsequent local stability analysis, I assume that $n(k)$ is a value that satisfies the condition above. Our focus is whether $g'(k^*) \gtrless 1$, thus

$$f'(k^*) \gtrless \frac{n(k^*) + \delta + k^* n'(k^*)}{s} = \rho(k^*). \tag{33}$$

From Fig. 1, we can assume if $\rho(k)$ crosses $f(k)$ from below, this stationary point is stable. If $\rho(k^*)$ cross $f(k)$ from above, this stationary point is unstable. As we can see from equation (33), $\rho(k^*)$ is mainly the transformation of $n(k)$, thus there arises a possibility of quite nonmonotonic behavior in $(1 - s)y$ and therefore in k. Usual Solow-Swan model with exogenous population growth gives us only one equilibrium point, but with endogenous population growth, we have multiple equilibria, where some of them are unstable. Even if the endogenous population growth has the same growth rate as exogenous population growth we will get an unstable equilibrium. This arises from the condition in equation (33) with expression in equation (31). One solution to this problem is the utility-maximizing model of endogenous fertility.

1.4.1. Niehans extension

Niehans (1963) made a model with an endogenous population and savings. The main idea of this paper is to find a connection between modern growth theory and the Ricardian tradition of treating labor as an endogenous factor and comparing it with the Malthusian theory. Niehan's model supposed two classes. “Proletariat” is not creating savings and “capitalist” class are those who divide their income into consumption and saving part with the assumption that no offspring are formed beyond reproduction. Niehans (1963) define production function as:

$$X = L^{\alpha} K^{\beta} \ (1 > \alpha > 0;\ 1 > \beta > 0) \tag{34}$$

In the Niehans model population increases by the proportion of the difference between actual wage and some level of minimum wage, w_m, which people are willing to accept. The actual wage is the same as the marginal product of labor. The equation of population growth is defined as:

$$\frac{dL}{dt} * \frac{1}{L} = \frac{\dot{L}}{L} = p\left(\frac{\partial X}{\partial L} - w_m\right) \tag{35}$$

Niehan's model supposed that *p*, which he defines as marginal propensity to proliferate, is always positive based on Malthusian theory. Similarly, capital accumulation was defined as a difference between the marginal return of capital and some minimum return r_m, which could be defined as a special case of the natural interest rate in the steady state:

$$\frac{dK}{dt} * \frac{1}{K} = \frac{\dot{K}}{K} = s\left(\frac{\partial X}{\partial K} - r_m\right) \tag{36}$$

In this equation, *s* is a marginal propensity to save out of profits. Equations (34), (35), (36) are determining this model. For another analysis Niehans shows variation between *log(L)* and *log(K)*, where *w* and *r* are constant:

$$\left.\frac{d(logK)}{d(logL)}\right|_{w\,=\,constant} = \frac{1-\alpha}{\beta} \tag{37}$$

$$\left.\frac{d(logK)}{d(logL)}\right|_{r\,=\,constant} = \frac{\alpha}{1-\beta} \tag{38}$$

In the next chapters the author is discussing different types of returns to scale, and how it will inflict the equilibrium between actual wage and rate of return of capital, which is not important for this thesis. What is interesting is part III., where Niehans (1963) shows one class model with infinite growth and constant returns to scale, so $\alpha + \beta = 1$. By setting assumptions with production function from equation (15):

$$r_t = f'(k_t) > 0, \tag{39}$$

$$w_t = f(k_t) - k_t f'(k_t) > 0 \tag{40}$$

We can substitute from equation (17) and equation (19):

$$I_t = s(r_t)Y_t, s' > 0, s(0) = 0 \tag{41}$$

and

$$L_{t+1} = [1 + n(w_t)]L_t, n' > 0 \tag{42}$$

With condition if $n(w_t) < 0$ for w_t less than minimum wage.

This model is a mere modification of the Solow-Swan model, where the growth of an economy is determined by the dynamics of the capital-labor ratio and population:

$$\frac{L_{t+1}}{L_t} = h(k_t) = 1 + n[f(k_t) - k_t f'(k_t)] = 1 + n(k_t) \tag{43}$$

Substitution of equations (39) and (41) in equation (18) yields

$$k_{t+1} = \frac{(1-\delta)k_t + s[f'(k_t]f(k_t)}{1 + n(k_t)} = g(k_t) \tag{44}$$

Now we have savings, which depend on capital-labor ratio via the marginal product of capital. Using same notation $s(k) = s(f'(k))$ all conditions above should be repeated with modification:

$$\rho(k) = \frac{n(k) + \delta}{s(k)} \tag{45}$$

The problem of this model is, that it can be showed that a nontrivial stationary point may not be determined.

Corchón (2016) extends this model by adding Malthusian ideas on the labor supply. In his conclusions, the model yields several steady state values of per capita income. An increase in total factor productivity is compensated by a decrease in the capita-labor ratio in the stable steady state.

Stamova and Stamov (2013) include a delay process of recruitment in the labor force and impulse-response effects on the capital-labor ratio. They found out that when population growth is not constant, past per capita income is bounded, and the small variation of the initial capital-labor ratio does not changed fundamentally the economic growth process.

1.5. The Nerlove - Raut model

Both previous models with endogenous population growth have a more complex structure of dynamic behavior, due to independence on the concavity of the production function and exogenous parameters. Nerlove-Raut (1997) model is built on the three-factor production function and it is inspired by the article of Lee(1986) on Malthus and Boserup theory.

Lee (1986) in his essay describes a difference between Malthusian and Boserup's theory of the relationship between population and technology. He uses a blank graph with population growth on the horizontal axis and technology on the vertical axis. He assumes fixed natural resources. The Malthusian theory describes points where the welfare of people based on the combination of technology and population is increasing, decreasing, or stable. Lee shows how shapes and locations should look like for the Malthusian theory. On the other hand, the Boserup approach is quite different. Boserup suggests that the population is dense relative to technology, and this will determine whether technology progress occurs. Where population is sparse, technology growth will be in decline.

As a result, Lee (1986) concludes that synthesis between Malthusian and Boserup theory is possible. Using a phase diagram he can show intuitive solutions that can be easily modified. He admits that his result is made by two controversial assumptions: diminishing returns are set in both labor and technology when they increase, while resources are stable; there also exist costs for maintaining the current level of technology. In his opinion, there is a need for maintaining physical and human capital so the technology would not be forgotten. The Lee analysis combines Malthusian and Boserup theory into one diagram. Malthusian theory sets other conditions, which limit the Boserup phase space. Thus, technology progress will occur only for a limited portion of the Boserup phase space, and moreover it creates an equilibrium point. Lee (1986) includes other conditions such as preventive checks, too-strong institutions, and exogenous mortality. These conditions set a lower level of technology, under the assumption of the high density of population.

In the Nerlove-Raut model, labor receives marginal product of labor, and the rest of "surplus" is given to capitalists who save all of it. The third production factor can be for example, stock of knowledge, natural resources, environmental quality, etc., but for further analysis, it will be just factor Z, which varies over time. Because of the usage of this third factor, a constant return to scale is not assumed. As discussed in Nerlove (1993) univariate dynamics cannot be used, but we have to apply the planar analysis.[3] In contrast with Malthus's theory, who probably thinks about Z as constant, Z will change over time, in response to levels or changes in the capital or labor. Boserup claims a reversible process in response to population pressure as described above.

To describe the Nerlove-Raut model we replace production function (15) by:

$$Y_t = F(K_t, L_t, Z_t) \tag{46}$$

with the assumption of constant returns to scale for all three factors. For the per capita terms, it will be:

[3] Planar analysis is a decomposition of a complex structure into flat planes. Box with 6 sides can be described from 6 different views. Each view is a flat shape (plane).

$$y_t = f(k_t, z_t) = F\left(\frac{K_t}{L_t}, 1, \frac{Z_t}{L_t}\right) \tag{47}$$

$F_L = f - kf_k - zf_z$ is the marginal product of labor. Labor is paid its marginal product:

$$w_t = f(k_t, z_t) - k_t f_k - z_t f_z. \tag{48}$$

If we assume that laborers save nothing, which is consistent with the Niehans model, and the growth of population is determined by w_t we have:

$$\begin{aligned} \frac{L_{t+1}}{L_t} &= 1 + n[f(k_t, z_t) - k_t f_k(k_t, z_t) - z_t f_z(k_t, z_t)] \\ &= 1 + n(k_t, z_t). \end{aligned} \tag{49}$$

If savings are made by the entire surplus of capitalists and owners of the Z factor and if savings can be used only to augment the capital stock, Eq. (17) is replaced by:

$$\begin{aligned} I_t &= K_t F_{K_t} + Z_t F_{Z_t} \\ &= Y_t - N_t w_t \\ &= N_t(y_t - w_t) \end{aligned} \tag{50}$$

Then

$$K_{t+1} = (1 - \delta)K_t + N_t(y_t - w_t) \tag{51}$$

Again, we combine equation (49) and equation (51) in per capita terms. We receive:

$$1 + n(k_t, z_t)k_{t+1} = (1 - \delta)k_t + (y_t - w_t) \tag{52}$$

$$k_{t+1} = \frac{(1-\delta)k_t + (y_t - w_t)}{1 + n(k_t, z_t)} = g(k_t, z_t)$$

Function $g(k_t, z_t)$ depends only on k_t and z_t because $y_t = f(k_t, z_t)$, and w_t is the function of (k_t, z_t). Nerlove-Raut (1997) assume that the evolution of Z is

$$Z_{t+1} = H(K_t, N_t, Z_t) \tag{53}$$

where H is a homogenous of degree one. Similarly, we combine equation (53) with equation (50). This leads to the motion of factor Z equation in per capita terms:

$$\begin{gathered} 1 + n(k_t, z_t) z_{t+1} = \psi(k_t, z_t) \\ z_{t+1} = \frac{\psi(k_t, z_t)}{1 + n(k_t, z_t)} = h(k_t, z_t) \end{gathered} \tag{54}$$

where $\psi(k_t, z_t) = H(k_t, 1, z_t)$. We can describe the system of equation (52) and equation (54) as a planar system in k_t and z_t. Let:

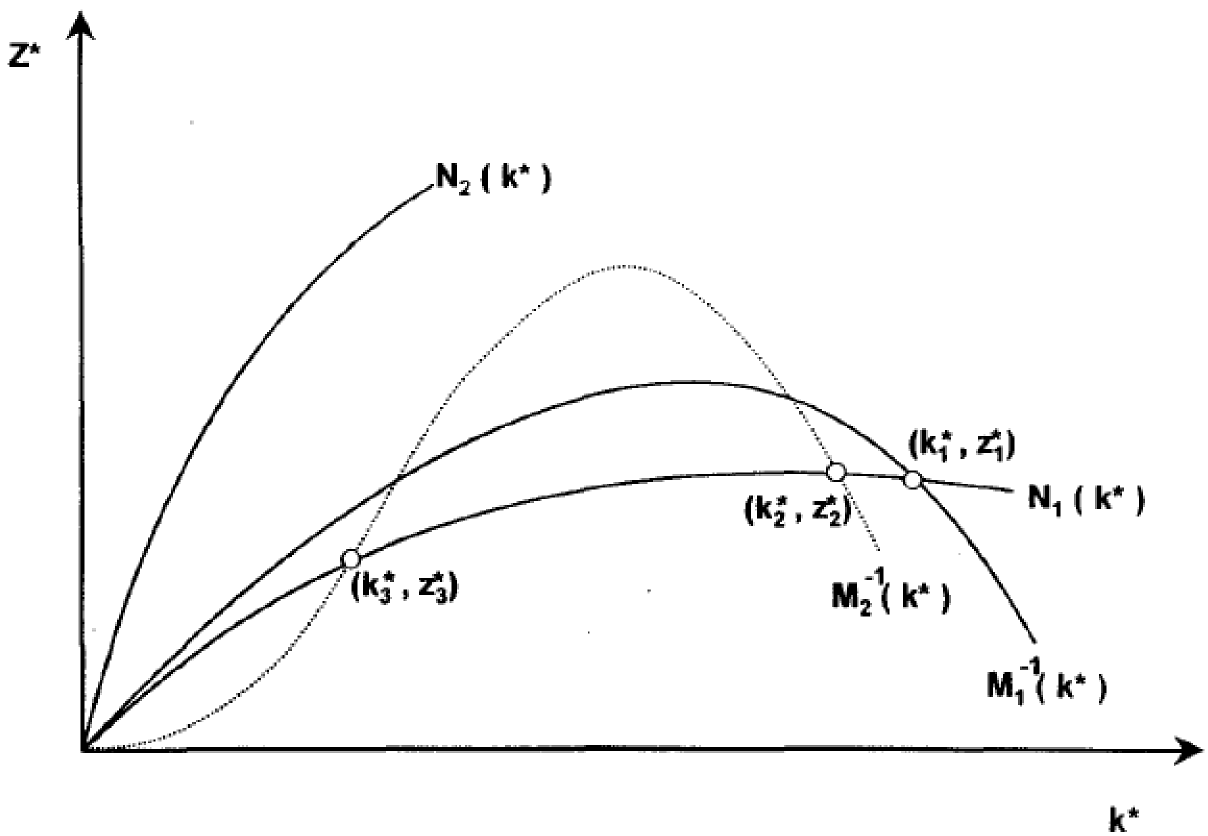
$$k^* = M(z^*), z^* = N(k^*), \tag{55}$$

M() and/or N() can have several branches with one or more discontinuities. By plotting this function into the graph we can get the stationary point, where these two functions cross. Derivatives of these functions can be obtained at any point of continuity of any branch due to the implicit function theorem. Thus:

$$\begin{gathered} M' = \frac{dk^*}{dz^*} = \frac{\varphi_z}{1 - \varphi_k} = \frac{(1 + n^*) g_z + n_z k^*}{1 - [(1 + n^*) g_k + n_k k^*]} \;, \\ N' = \frac{dz^*}{dk^*} = \frac{\psi_k}{1 - \psi_z} = \frac{(1 + n^*) h_k + n_k k^*}{1 - [(1 + n^*) h_z + n_z k^*]}. \end{gathered} \tag{56}$$

If we consider general circumstances, we can determine $k^* = 0 = z^*$ as a stationary point, thus one branch of M() and one branch of N() have to start at point [0;0]. From an economic point of view, we consider only the first quadrant of the graphical solution because any negative value has no economic sense.

Figure 2: Cases of the shape of N() and M()



Source: Nerlove and Raut (1997)

In Fig. 2 Nerlove and Raut (1997) plotted some examples of curves, which always start at point [0;0], but branches can start from other points. From the graph, we can see that branch M_1^{-1} was considered as increasing and then decreasing. Since N_1 has a lower slope than M_1^{-1} we have one stationary point (k_1^*, z_1^*). In another example when N_2 has a greater slope than M_1^{-1} there is no stationary point possible. When M_2^{-1} is not considered as a strictly concave function, we have two stationary points, which both have their own different local stability on function N_1. If we assume N as a not strictly increasing curve, we have the possibility of many nontrivial equilibria. With this method of different branches, we can see the dynamic properties of the Nerlove - Raut model. However, with this level of abstraction, it is not possible to determine the nature and existence of stationary points. For this purpose, let the per capita surplus available for investment be:

$$s_t = y_t - w_t = k_t f_k(k_t, z_t) + z_t f_z(k_t, z_t) \tag{57}$$

and indicate functions or values calculated at a nontrivial stationary point (k^*, z^*) by affixing an asterisk.

1.5.1. Nerlove Raut model example

We choose the Cobb-Douglas production function:

$$y_t = k_t^{\sigma} z_t^{\mu}, 0 < \sigma, \mu;\ \sigma + \mu < 1, \tag{58}$$

Linear approximation of $n(w_t)$ gives us:

$$1 + n(k_t, z_t) = \frac{L_{t+1}}{L_t} = \frac{w_t}{w_m}, w_m > 0, \tag{59}$$

This yields equation (52):

$$k_{t+1} = \frac{(1-\delta)k_t - s_t}{w_t / w_m}, \tag{60}$$

where w_m = minimum wage, $s_t = (\sigma + \mu) y_t$ and $w_t = y_t - s_t = (1 - \sigma - \mu) y_t$. Thus function M^{-1} showed above becomes

$$M^{-1}(k^*) = z^* = \left\{ \frac{(1-\delta)(k^*)^{1-\sigma}}{\left[\frac{1-(\sigma+\mu)}{w_m}\right]k^* - (\sigma+\mu)} \right\}^{1/\mu} \tag{61}$$

With the assumption of linear approximation in logs terms in $\psi(k_t, z_t)$:

$$z_{t+1} = k_t^{\alpha} z_t^{\beta}, 0 < \beta < 1 \tag{62}$$

but with the possibility of α to be negative.[4] Function N showed above becomes

$$N(k^*) = (k^*)^{\frac{\alpha}{(1-\beta)}} \tag{63}$$

Conditions set above can be violated, based on how we define our factor *z* for example in the case of environmental quality or safety. Per capita of *z* factor may deteriorate at higher capital-labor ratios. If all conditions above are held, we get $0 < \alpha < (1- \beta)$ under condition $(\alpha+ \beta)<1$, so $N(z^*)$ is a concave function, since $0< \beta<1$ has been set. Shape of $M^{-1}(k^*)$ in equation (61) is set by parameters and a relationship which they have between each other. For example, if μ is an even number, $M^{-1}(k^*)$ increases from 0 to positive infinity as k^* starts from 0 and continues in the path of $k_m = w_m \frac{\sigma+\mu}{[1-(\sigma+\mu)]}$.[5] If k^* has large values, $M^{-1}(k^*)$ is a decreasing function of k^* where

$$\frac{dz^*}{dk^*} = \frac{-z^*}{\mu} \frac{\frac{\sigma*(1-\sigma-\mu)}{w_m}k^* + (\sigma+\mu)(1-\sigma)}{k^*\left[\frac{(1-\sigma-\mu)}{w_m}k^* - (\sigma+\mu)\right]} < 0 \;\; for\, k^* > k_m. \tag{64}$$

---

[4] Alpha is coefficient for z_{t+1}. *Z* can be a factor which negatively influences production function (for example environment pollution).
[5] K_m is a stock of capital when wage is equal to minimum wage

From this equation and assumptions of $k^* > k_m;\ \alpha, \beta > 0;\ \alpha + \beta < 1;\ \sigma, \mu > 0;\ \sigma + \mu < 1$, we have a unique nontrivial stationary point, which is a stable equilibrium of this problem. A big role in this problem is the value of the minimum wage, which defines the location of this equilibrium. Growth of population is then determined by values k^* and z^* from equation (59):

$$\frac{L_{t+1}}{L_t} = \frac{[1-(\sigma-\mu)](k^*)^{\sigma}(z^*)^{\mu}}{w_m}$$

$$\frac{L_{t+1}}{L_t} = \frac{[1-(\sigma-\mu)](k^*)^{\sigma+(\frac{\alpha\mu}{1-\beta})}}{w_m} \tag{65}$$

If

$$w_m = [1-(\sigma-\mu)](k^*)^{\sigma+(\frac{\alpha\mu}{1-\beta})}, \tag{66}$$

then the population is stationary. If w_m exceeds value given by in equation (66), the population starts to decline (possibly even into zero). If w_m is lower than value in equation (66) on right hand side, population starts to increase. Thus, a small value of w_m leads to a higher rate of population growth at the stationary point and the lower consumption per capita.

For this example of the Nerlove-Raut model we write elasticities of functions of k_{t+1} in equation (60) and z_{t+1} in equation (62) with respect to their arguments k_t and z_t as $\xi_k, \xi_z, \eta_k, \eta_z$

$$\xi_k = \frac{w_m}{1-(\sigma+\mu)}\left[\frac{(1-\delta)(1-\sigma)}{y^*}\right],$$

$$\xi_z = \frac{w_m}{1-(\sigma+\mu)}\left[\frac{-\mu(1-\sigma)}{y^*}\right], \tag{67}$$

$$\eta_k = \alpha$$

$$\eta_z = \beta$$

Following the approach in Nerlove (1993) we receive:

$$\text{tr J} = \frac{\frac{w_m}{y^*}(1-\delta)(1-\sigma)}{1-(\sigma+\mu)} - \beta$$

$$\det J = \frac{\frac{w_m}{y^*}(1-\delta)}{1-(\sigma+\mu)}\big(\alpha\mu+\beta(1-\sigma)\big). \tag{68}$$

$$\det J = \left[\frac{\big(\alpha\mu+\beta(1-\sigma)\big)}{1-\sigma}\right] tr\ J + \beta\left[\frac{\big(\alpha\mu+\beta(1-\sigma)\big)}{1-\sigma}\right]$$
$$= A\ tr\ J + V$$

Equation (40) determines a straight line in the tr J-det and J plane with slope

$$\text{A} = \frac{\alpha\mu}{1-\sigma} + \beta \tag{69}$$

and intercept

$$\text{B} = \frac{\alpha\beta\mu}{1-\sigma} + \beta^2 \tag{70}$$

If our assumptions are possible: $0 < \sigma < 1; 0 < \mu < 1; \sigma + \mu < 1; 0 < \beta < 1$ with indefinite sign in parameter α.

We consider two possibilities for α. If $\alpha > 0$, the capital-labor ratio increases, and the effect on the stock of Z is positive. A and B have a positive sign. It follows the issue of stability or instability, which is determined by magnitude in tr J in Eq. (68). Only a small ratio of minimum wage to the per capita output has stable equilibrium because term (1 - δ) is likely to be close to one. If $\alpha < 0$, A and B are ambiguous. The Nerlove – Raut model displays the importance of the minimum wage ratio on the capital-labor ratio or per capita output. If this ratio is large, the existing equilibrium might be unstable.

1.6. The Becker – Barro model

Becker and Barro (1988) created a model, which finds a relationship between fertility, some bequest in the form of human capital, parental altruism, and economic growth. Parents are deciding about the number of children and capital (human and physical), which will be bequeathed to each child. The quality factor in the Becker-Lewis presented in chapter 1.3. is considered as the bequest of the human capital of a parent to a child. The parent's utility function is set up by satisfaction from their consumption and from having children. Thus, the parent choices are motivated by a trade-off between altruism towards their children and their utility function. The difference between this model and the Becker-Lewis model is the term of quality. The Becker – Barro model has given to quality an explicit interpretation as a bequest of human and/or physical capital. Another addition of Becker and Barro (1998) is what proportion of children's utility function influence the parent's utility function. Parents decide how much trade-off will happen between their consumption and children's needs. This model does not try to explain why children should "repay" parent's altruism or what the parent's motivation is to have children when they are old. Becker and Barro (1998) assume that it is beneficial for parents to have a surviving child in their old age and child has the tendency to have "reverse altruisms", therefore they care about the welfare of their parents.

A combination of microeconomic models of fertility and human and physical capital leads to certain general equilibrium considerations. Parent's utility function is made of the number of children n, their consumption c, children utility u, and utility of children in the next period u_{t+1}:

$$u_t = u(c_t, n_t, u_{t+1}) \tag{71}$$

If there is no difference between children and parents, thus:

$$u_{t+1} = u(c_{t+1}, n_{t+1}, u_{t+2}) \tag{72}$$

and so on. Under the assumption of one-parent families, each child receives the same bequest from her parent. Bequest, b_t has the form of physical capital, which is added to the child endowment. Thus, the parent maximizes his utility function with respect to the budget constraint:

$$c_t + (b_t + a)n_t \leq I_t + b_{t-1} \tag{73}$$

where a denotes additional exogenous costs(or benefits) of having a child and I_t is the parent's income. Because it is difficult to solve the problem for a general solution, Barro and Becker (1998) decided to use the additively separable utility:

$$u_t = v(c_t, n_t) + \beta(n_t)n_t\hat{u}_{t+1} \tag{74}$$

Where $\beta(n_t)$ stands for a degree of altruism per child and $\hat{u}_{t+1}$ is a the parents estimate of children utility. It is assumed that this degree will be decreasing with n, so $\beta' \leq 0$. Under the assumption of perfect foresight, we can replace the term $\hat{u}_{t+1}$ for u_{t+1}. If we relax this assumption, then $\hat{u}_{t+1}$ is the maximum taken over c_{t+1}, b_{t+1}, n_{t+1} given by I_t and b_t and expectation of parents about the endowment of a child $\hat{I}_{t+1}$. It does not need to be equal to u_{t+1}.

1.6.1. Nonrecursive formulation

Let the function $f_t(b_t, \hat{I}_{t+1}) = \hat{u}_{t+1}$ be parent's expectations of a child's future utility. Thus parents maximization problem can be rewritten as:

$$\begin{gathered} \max_{c,n,b}\{v_t(c_t, n_t) + \beta(n_t)n_t f_t(b_t, \hat{I}_{t+1})\} \\ such\ that\ c_t + (b_t + a)n_t \leq I_t + b_{t-1}, \end{gathered} \tag{75}$$

Where I_t, $\hat{I}_{t+1}$ and b_{t-1} are given. If we omit the condition of integer for the number of children and assume the existence of an interior solution, the first-order conditions are

$$\begin{gathered} v_1 = \lambda \\ v_2 + \beta(n_t)[1 - \varepsilon_\beta]f(b_t, \hat{I}_{t+1}) = \lambda[b_t + a] \end{gathered} \tag{76}$$

$$\beta(n_t)n_t f'\left(b_t, \hat{I}_{t+1}\right) = \lambda n_t,$$

where $\varepsilon_\beta = -n\beta'/\beta$ is the elasticity of the degree of altruism with respect to the number of children. Term λ is the marginal utility of the parent's consumption in optimum. If $\beta' = 0$ then elasticity is zero and altruism is constant. If elasticity is less than one, having another child has a positive effect on the direct marginal utility. Canceling n_t, the last equation gives us the marginal utility of increasing bequest, and it is equal to the marginal utility of consumption of parents.

In the end for the nonrecursive formulation, an increase in exogenous endowment of previous parent's bequest may not increase the number of children. An increase in cost reduces fertility because the parent has to substitute his consumption and bequest to their children. If altruism is a constant number, increasing altruism does not need to increase the number of children. It will just increase the parent's utility. An important note is that bequest is only possible by altruism. If there is no altruism, there will be no bequest. Even if altruism is very low, parents still will have more children than when parents have no altruism.

1.6.2. Recursive formulation

The previous examples with a nonrecursive formulation of the problem do not need to give us a growth path of population or bequest. Perfect foresight and rational expectations necessarily coincide in a deterministic world. When we change the expectations of parents, equations will always change about the change of expectations. Thus, if children have the same utility function as their parent and with the assumption about separability, recursive utility function becomes dynastic utility function of the parent in period zero:

$$\sum_{t=0}^{\infty}\left[\prod_{\tau=0}^{t}\gamma(n_{t-1})\right]v(c_t\,, n_t) \tag{77}$$

$$where\ \gamma(n_t) = \beta(n_t)n_t\ and\ \gamma(n_{-I}) \equiv 1$$

Constraints condition requires equality. For the need of convergence of the infinite sum, we set *v(.)* as bounded and:

$$0 < \beta(n_t)n_t < 1 \tag{78}$$

Another assumption we need to make is about endowments. Children have the same exogenous endowment. It is independent on the parent's bequest. The same assumption was applied on parents too. Thus the previous equation becomes:

$$\max_{(c_t, n_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[\prod_{\tau=0}^{t} \gamma(n_t) \right] v(c_t\,, n_t),$$
$$such\ that\ c_t + (b_{t+1} + a)n_t = I + b_t, \qquad t \geq 0 \tag{79}$$

The difference between budget constraint in equation (79) is that inequality was replaced by equality. It is never an optimal solution for parents to leave anything to children if the number of children, their utility, and consumption are all desirable.

The solution to equation (79) can be computed by dynamic programming techniques. For all $t \geq 0$, let utility function from bequest ($u^*(b_t)$) be as a value function of future generation t at a given level of the bequest. Then the Bellman equation of equation (79) becomes:

$$u^*(b_t) = \max_{(c, n, b_{t+1}} \{v(c_t, n_t) + \beta(n_t)n_t u^*(b_{t+1})\},$$
$$such\ that\ c_t + (b_{t+1} + a)n_t = I + b_t, \tag{80}$$

Making a solution from this task will cause one problem. Under the assumption of concavity of the utility function $v(c_t, n_t)$ and $\beta(n_t)n_t$, value function $u^*(b_t)$ is not concave. Barro and Becker (1988) deal with this problem by adding another assumption about the parents part of the additively separable function. Parents own utility function from consumption and children just depends on the parent's consumption, thus:

$$v_n = 0 \tag{81}$$

Barro and Becker (1988) found out that the analysis is too difficult to be carried explicitly. They added two more assumptions:

a. Degree of altruism toward children has a constant elasticity with respect to the number of children:

$$\beta(n_t) = \beta_0 n_t^{-\beta_1} \tag{82}$$

b. Constant elasticity of utility from own consumption

$$v(c_t) = c_t^{\sigma}, 0 < \sigma < 1. \tag{83}$$

To use the micro model on the equilibrium growth framework, Barro and Becker (1989) address more extensions and describe why to use it:

c. Replace *I* with w_t, variable wage rate of the adult
d. Replace b_t by $(1+r_t)k_{t-1}$, where r_t is the rate of return to physical capital and k_t is physical capital per capita bequeathed to the individual child.

This will change parents budget constraint to:

$$c_t + (k_t + a)n_t = w_t + (1 + r_t)k_{t-1} \tag{84}$$

Parents still want to maximize dynastic utility function, which is:

$$u_0(c_0, n_0, c_1, n_1, \ldots\ldots) = \sum_{t=0}^{\infty} \left[\prod_{\tau=0}^{t} \gamma(n_{\tau-1}) \right] c_t^{\sigma}, \tag{85}$$

with respect to the budget constraint in equation (84). This creates maximization problem:

$$\max_{(c_t,n_t)_{t=0}^{\infty}} u_0(c_0, n_0, c_1, n_1, \ldots\ldots) = \sum_{t=0}^{\infty} \left[\prod_{\tau=0}^{t} \gamma(n_{\tau-1}) \right] c_t^{\sigma}, \tag{86}$$

$$such\ that\ c_t + (k_t + a)n_t = w_t + (1 + r_t)k_{t-1}$$

Exogenous and endogenous parameters, which parents have, are:

Exogenous	Endogenous
w_t = adult wage	c_t= own consumption
k_{t-1} = bequest from their parents	n_t = number of children
r_t = rate of return to capital	k_t = physical capital bequest per child

Cost of a raising a child (a), altruism ($-\beta_1$), elasticity (σ) and the coefficient (β_0) are parameters of the parent's problem as well.

Another assumption of this model is that parents can foresee their wage and rate of returns in the future. These assumptions are under investigation of current research, how realistic these assumptions are. But for the development of the next part of the model, these assumptions are crucial. To get a positive number of children we need to set a condition:

$$\sigma < 1 - \beta_1 \tag{87}$$

We can define

$$N_i = \prod_{t=0}^{i-1} n_t \tag{88}$$

as the number of descendants in the ith generation of parents. Then total consumption becomes $C_i = N_i c_i$. Therefore utility in period zero, which is a more detailed description of dynastic utility function in equation (85), is:

$$
\begin{aligned}
u_0 &= c_0^{\sigma} + \beta_0 n_0 n_0^{-\beta_1} c_1^{\sigma} + \beta_0^2 (n_0 n_1) * (n_0 n_1)^{-\beta_1} c_2^{\sigma} + \cdots \\
&= c_0^{\sigma} + \beta_0 N_1^{1-\beta} c_1^{\sigma} + \beta_0^2 N_2^{1-\beta} c_2^{\sigma} + \cdots \qquad (89) \\
&= c_0^{\sigma} + \beta_0 N_1^{1-\beta-\sigma} (N_1 c_1)^{\sigma} + \cdots
\end{aligned}
$$

Next, we need to set a condition for parents, so they have a motivation to produce children at all. Differentiating utility in period zero with respect to the number of children gives us:

$$
\frac{\partial u_0}{\partial N_i} = (1 - \beta_1 - \sigma) \frac{\beta_0^i N_0^{1-\beta_1-\sigma} (N_i c_i)^{\sigma}}{N_i} > 0 \qquad (90)
$$

From the above condition in equation (90) its obvious, that only when condition holds, parents make children. Setting Lagrangian function from maximization problem in Equation (86) yields

$$
u_0^* \equiv \sum_{i=0}^{\infty} \left\{ c_i^{\sigma} \beta_0^i N_i^{1-\beta_1} - \lambda_i [c_i + (k_i + a) n_i - w_i - (1 + r_i) k_{i-1}] \right\} \qquad (91)
$$

We can obtain first-order conditions by differentiating the Lagrangian with respect to endogenous variables and set them equal to zero. Taking ratios of these equation yields three intertemporal “arbitrage” conditions:

$$
\frac{\lambda_{j+1}}{\lambda_j} = \frac{n_j}{1 + r_{j+1}} \qquad (92)
$$

$$= \frac{V_{j+1}}{V_j} \frac{c_j}{c_{j+1}}$$

$$= \frac{n_j (k_j + a) \sum_{i=j+2}^{\infty} V_i}{n_{j+1}(k_{j+1} + a) \sum_{i=j+1}^{\infty} V_i}$$

$$where\ V_j = \beta_0^j N_j^{1-\beta_1}\, c_j^{\sigma}$$

Since

$$\frac{V_{j+1}}{V_j} = \beta_0 n_j^{1-\beta_1} \left(\frac{c_{j+1}}{c_j}\right)^{\sigma} \tag{93}$$

we can rewrite terms in "discounted" values between ratios of consumption and total costs (bequest + cost to raise children):

$$\frac{\lambda_{j+1}}{\lambda_j} = \frac{n_j}{1 + r_{j+1}} = \beta_0 n_j^{1-\beta_1} \left(\frac{c_{j+1}}{c_j}\right)^{\sigma-1} \tag{94}$$

This ratio of birthrate to returns to physical capital is an intertemporal link in the trade-off between the cost to raise a child in different generations and per capita consumption of parents in each generation. The conclusion of the Barro – Becker model are:

1. Consumption per capita can be higher only if the cost to raise a child rises. It is not dependent on the degree of altruism, fertility, or interest rate

2. Change in the interest rate leads to a change in fertility (n_j). Higher interest rate gives a higher fertility rate. Parents can consider children as a form of savings and increase in the interest rate leads to higher motivation to a higher fertility. Fertility is affected by the degree of pure altruism too.
3. Change in the initial capital does not influence future consumption when the cost of raising a child is stable. Higher initial capital just raises initial consumption and increases fertility proportionately.
4. Imposed tax on children in the *j*th generation and an increase in initial wealth have an influence on higher consumption and reduce fertility. The rate of growth population remains unchanged, only the level of population will change.
5. Condition of the constancy of exogenous parameters *w*, *r,* and *a* together with the specific form of the altruism function and utility function gives us a unique steady state of all endogenous variables across all generations. This steady state is stable and does not depend on the initial position.

This model is a foundation for the next articles in Becker and Barro (1989) and Becker et al. (1990). These articles are an extension in the mathematical form and economic explanation of this model. The 1990 model tries to explain the economy-wide fertility and per capita consumption. For this purpose, the authors set an assumption that rates of return on investments in human capital rise instead of decline, as the stock of human capital increases.

This model was extended by other authors. For example, Bosi, Boucekkine, and Seegmuller (2016) extended the model to apply a heterogeneous agent problem with different capital endowments. They found out that all individual endogenous parameters are equal in the next period, and it does not depend on sector specialization.

Pestieau and Ponthiere (2012) ask the question how the structure of early and late children in terms of period and goods affects fertility timing. Instead of one reproduction period, they created a model with two periods. In their findings, they have two possible answers of why parents have children at a later age. The first reason is the fall in cost connected to healthcare when parents decide to have children later. The second explanation is the increase in hourly productivity leads to an increase in opportunity costs of early children.

2. Data description

For my thesis, I created two datasets. The first dataset covers data only for the population growth and growth of GDP per capita. Data are taken from the World Bank database. They range from 1960 to 2018 and they include 30 African countries. This dataset is used in the first model. Unfortunately, there are no other data with the same range which can be used as control variables. For this purpose, I created the second dataset with a shorter range (1990 – 2017) with some control variables, which I divided into subcategories described below.

2.1. Index group

For the first group, I use two indexes. First is the Human Development Index (HDI), which is created by the United Nations. It consists of indicators for health, knowledge, and economic variables. The final results are counted for normalized indicators using a geometric mean. The score is scaled from 0 to 100 where the highest value is the best. This indicator was taken due to its high correlation with GDP growth as described in Hudakova (2018) for European countries. Data range from 1990 to 2017 and they are taken from the database of the United Nations.

The second index is the Free market index. This index is composed of indicators such as property rights, government integrity, judicial effectiveness, tax burden, business freedom, labor freedom, monetary freedom, trade freedom, investment freedom, and financial freedom. Countries are graded again on a scale from 0 to 100. The overall score is determined by averaging the values with given weights. Cebula and Clark (2013) estimated a significant impact of economic freedom on real GDP per capita in OECD countries. The first value was taken in 1995 and data are available at heritage.org.

2.2. Safety group

The safety group consists of binary variables. It depends whether the country was in the civil war (inside the country) or participated in the war against another nation (outside of the country). The next binary variable is based on the number of deaths in the war. If there were more than 1,000 deaths, the value of the binary variable is one. The meaning of this variable is to have some control if there is significance of real fights with casualties or countries agreed on a kind of non-attack agreement, which can affect the dependent variable. Data for wars were downloaded from two databases. First is the Peter Brecke dataset which consists of data from the year 1500 to 2000. The next data were used from the Uppsala Conflict Data program. These binary variables were chosen based on research by Tir and Diehl (1998) and Murdoch and Sandler (2002), which show significant effects on the population growth and GDP growth.

I added tourism arrivals on 1,000 people as another variable, which can be used as an indicator of the safety of the country. Ministries of Foreign affairs have the list of countries where they do not recommend travelling, or they can even ban the option to travel into some countries. This can be caused mainly by cultural or safety reasons and it could be the next indicator of the safety of the country. These data are taken from the World Bank database from 1995.

2.3. Economy Group

For the economy group, I use four variables. First of them is a development aid provided mainly by rich countries. This can have some effect as described in Collier (2007). He estimates that development aid does not help the countries, but most probably it has a negative effect on the development of the country. Most of the money is not focused on projects but ends up in the clerk or political pockets. Data for the development aid are downloaded from the World Bank database and divided by the population of the country. Data are available for most African countries from 1995.

Collier (2007) claims that total rents from natural resources can have a large effect not only on the stability of the country but for the development of a country in Africa as well. He argues that countries with a large number of natural resources do not focus on the development of the industry, but on the primary sector only. Thus, the economy of the African countries mainly depends on the prices of natural resources, which they extract. Data are at disposal in the World Bank database from 1995 for most of the countries.

Some economic models are setting a saving rate as one of the determinants of the GDP growth. For that reason, I use the gross domestic saving rate as percentage of the GDP as another variable from the World Bank database from the year 1995.

I include tourism arrivals into the country in the economy group too. For some countries in Africa, tourism is a large part of the income, mainly for island states or, for example, for Egypt and Tunisia.

2.4. Health group

For Health group variables I use those which should have an effect on dependent variables. One of them is the prevalence of HIV, as percentage of population from 15 to 49 of age. This variable can be considered as a problem mainly for the population growth. Data are at disposal in the World Bank and are taken from the year 2000.

Another problematic illness in Africa is malaria, which is another variable that can cause a significant effect on the dependent variables. I downloaded data for malaria deaths per 100,000 people.

The last variable is the prevalence of undernourishment as percentage of the population. Data are taken from the World Bank database since the year 2000. Insufficient nutrition can cause illness or death, which should have an effect on the GDP growth and population growth.

2.5. Handling with data

Most of the data are available for different periods and different countries. The rule of thumb which I applied was to use only periods and countries which have mostly all data points available. In all tables below, there is a number of countries (N) and time span of the data (T) used in the model. There were a few cases, in which the value was missing between periods, so I decided to replace these points by a middle value of the year before and after. If one value was missing only at the end or beginning of the dataset period, it was replaced by the mean of the previous two years or mean of two following years respectively.

3. Empirical Part

The aim of the thesis is to determine if the population growth influences the GDP growth per capita or vice versa. This problem was empirically studied in the past. Unfortunately, researchers focused mainly on two approaches by my knowledge. First, they studied causality between the population growth and the GDP growth per capita. Second, they used cross-section analysis with control variables. Examples of these approaches are in Table 1:

Table 1: Examples of empirical research

Author(s)	Time period	Countries	Methodology	Results
Thornton (2001)	1921- 1994	Argentina, Brazil, Chile, Colombia, Mexico, Peru, Venezuela	Cointegration analysis and Granger causality	No causality between population growth and GDP growth per capita
Furuoka, Munir (2009)	1961 – 2003	Thailand	Cointegration analysis and Granger causality	Population growth positively influences GDP growth per capita
Furuoka, Munir (2011)	1960 -2007	Singapore	Cointegration analysis and Granger causality	Positive both way causality
Furuoka (2013)	1960 - 2007	Indonesia	Cointegration analysis and Granger causality	Population growth positively influences GDP growth per capita
				14 countries: Population growth positively

Darrat and Al-Yousif (1999)	1948 - 1996	Twenty developing countries	Granger causality and Error Correction model	influences GDP growth per capita 2 countries: Population growth negatively influences GDP growth per capita 4 countries: Positive bidirectional effect
Singha & Jaman (2012)	1960 - 2010	India	Granger causality and Error Correction model	No causality
Garza – Rodriguez (2016)	1960 – 2014	Mexico	Vector Correction model	Positive effect in both ways
Chang, Chu, Deale, Gupta (2014)	1870 – 2013	21 countries	Bootstrapped Panel-Granger Causality test	5 countries: GDP growth per capita has an effect on population growth 4 countries: Effect of population

				growth on GDP growth per capita 2 countries: Bidirectional causality 11 countries: No causality
Afzal (2009)	1981 - 2015	Pakistan	OLS	The negative effect of population growth on GDP growth per capita
Kodoru, Tatavarthi (2016)	1985 - 2015	India	OLS	The positive effect of population growth on GDP growth per capita
Dao (2013)	1990 – 2008	45 African Countries	Cross-section regression analysis	The negative influence of population growth on GDP growth per capita

Note: Own construction

From the previous research, it is obvious that results mainly depend on methodology, countries, and time period used.

The conclusions whether the population growth influences the GDP growth or vice versa are different for countries and periods, therefore there is not a common conclusion. The closest research to the aim of this thesis is from Dao (2013). He estimates a negative influence of the population growth on the GDP growth per capita. To avoid replicating this research, I use different methodology and periods.

My empirical analysis consists of two parts. First, I use the approach by Chang, Chu, Deale and Gupta (2014), and test causal relationships for African countries individually. Second, I use a Dynamic panel data model. As is shown in the previous research, each country can have different causality. By the first model, I want to determine how these two variables influence each other in a country-specific environment. The second model tries to show other possible effects, which can have an influence on both variables.

3.1. Bootstrapped Panel-Granger Causality model

Bootstrapped Panel-Granger Causality test (BPGC) is an approach proposed by Kónya (2006). The important feature of BPGC implies that all variables do not require to be tested for stationarity conditions. This method uses the unit root test and the cointegration test. Both tests are robust.

The first step of this approach is to determine a system of equations to establish zero restriction for causality according to the Wald principle. This is done by Seemingly Unrelated Regressions (SUR). Next, we need to generate a bootstrap critical value. In the Wald principle, there is a condition for the joint hypothesis. In this case, it is not a valid condition, since BPGC has a specific Wald test value and country-specific bootstrap critical value.

The system of equations is composed of two sets of the equations, specifically:

$$GDP_{1,t} = \alpha_{1,1} + \sum_{i=1}^{ly_1} \beta_{1,1,i} GDP_{1,t-i} + \sum_{i=1}^{lx_1} \delta_{1,1,i} POP_{1,t-i} + \varepsilon_{1,1,t} \tag{95}$$

$$GDP_{2,t} = \alpha_{1,2} + \sum_{i=1}^{ly_1} \beta_{1,2,i} GDP_{2,t-i} + \sum_{i=1}^{lx_1} \delta_{1,2,i} POP_{2,t-i} + \varepsilon_{1,2,t}$$

.

.

.

$$GDP_{N,t} = \alpha_{1,N} + \sum_{i=1}^{ly_1} \beta_{1,N,i} GDP_{N,t-i} + \sum_{i=1}^{lx_1} \delta_{1,N,i} POP_{N,t-i} + \varepsilon_{1,N,t}$$

and

$$POP_{1,t} = \alpha_{2,1} + \sum_{i=1}^{ly_2} \beta_{2,1,i} GDP_{1,t-i} + \sum_{i=1}^{lx_2} \delta_{2,1,i} POP_{1,t-i} + \varepsilon_{2,1,t}$$

$$POP_{2,t} = \alpha_{2,2} + \sum_{i=1}^{ly_2} \beta_{2,2,i} GDP_{2,t-i} + \sum_{i=1}^{lx_2} \delta_{2,2,i} POP_{2,t-i} + \varepsilon_{2,2,t}$$

(96)

.

.

.

$$POP_{N,t} = \alpha_{2,N} + \sum_{i=1}^{ly_2} \beta_{2,N,i} GDP_{N,t-i} + \sum_{i=1}^{lx_2} \delta_{2,N,i} POP_{N,t-i} + \varepsilon_{2,N,t}$$

where *GDP* refers to the GDP growth per capita, *POP* to the population growth, *N* to the number of countries in data (in this case N = 54), *t* to the time periods (in my case 1960 – 2018) and *l* is the lag length. Each equation has a different value for variables. Values can be cross-sectional correlated in error terms. Thus, for testing the Granger causality, there are four possible results:

1. Unidirectional causality from *POP* to *GDP*, thus not all $\delta_{1,i}$ are zero and all $\beta_{2,i}$ are zero

2. Unidirectional causality from *GDP* to *POP*, thus all $\delta_{1,i}$ are zero and not all $\beta_{2,i}$ are zero
3. Bidirectional causality between *GDP* and *POP* thus not all $\delta_{1,i}$ and not all $\beta_{2,i}$ are zero
4. No causality between *GDP* and *POP* thus all $\delta_{1,i}$ and $\beta_{2,i}$ are zero

The proposed system of equations is shaped by the lag length. Hence, the optimal lag length, which is important for the robustness test in results, needs to be set. As proposed in Konya (2006), maximal lags are allowed to differ across variables but they need to be the same across equations. In my regression system, I consider one to eight lags for each possible pair of ly_1, lx_1, ly_2, lx_2. For simplicity and easier computational process, which does not need a powerful computer, I suppose $ly_1 = lx_1$ and $1y_2 = lx_2$ The optimal lag length will be determined by the combination which minimizes Schwarz Bayesian Criterion (SBC) and Akaike Information Criterion (AIC) that are defined as:

$$AIC_k = ln|W| + \frac{2N^2q}{T}$$

and (97)

$$SC_k = ln|W| + \frac{N^2q}{T}\ln(T)$$

where *W* is the estimated covariance matrix, *N* is the number of equations, *q* is number of coefficients per equation, and *T* stands for the period length, all in system *k* = 1,2. The results of the optimal time lag are in Table 2.

Table 2: Optimal lag length result

LAG GDP	SBC	AIC	LAG POP	SBC	AIC
1	**89.43**	**86.20**	1	-199.49	-202.93
2	102.66	91.81	2	**-228.91**	-240.38
3	125.02	102.03	3	-226.31	**-250.40**
4	156.63	116.85	4	-202.16	-243.46

5	198.09	136.75	5	-155.80	-218.89
6	249.53	161.72	6	-98.34	-187.83
7	310.51	191.19	7	-37.79	-158.24
8	384.93	228.91	8	31.03	-124.99

Note: Bold number is the lowest number in the column. The optimal lag length is tested on the full sample of 30 African countries in time period 1960 – 2018.[6, 7] Own construction.

The optimal lag length for the GDP growth per capita as a dependent variable is one according to SBC and AIC criteria. Every other lag has a worse criterion. In other words, adding more lags (more explanatory variables) does not help to reach better predictions.

The optimal lag length for the population growth as a dependent variable according to SBC is two and according to AIC three. Unfortunately, the estimation, when the optimal lag is three for the population growth, gives us a big autocorrelation error, which increases bootstrap values. This can be caused by a deterministic trend. There are two possibilities how to deal with this problem. The first possibility is to use fewer lags on the level of values, but lag two does not give us reasonable results, only lag one, which is reported below. The second option is to use first-difference values.

3.2. Tests

Konya (2006) proposed to use BPGC if there is an assumption of some cross-sectional dependence between countries. If cross-sectional correlated errors are present, the system of SUR equations described above is more efficient than pooled OLS regression in case of determining causality. I use four tests for testing the cross-sectional dependence. First is the Lagrange multiplier (LM) test written by Breusch and Pagan (1980). For using the LM test, we set the estimation of the following model:

$$y_{it} = \alpha_i + \beta_i' x_{it} + u_{it} \; for \; i = 1,2 \dots \dots, N, t = 1,2 \dots \dots, T \qquad (98)$$

[6] Data from: World Bank, Population growth [online], accessed at 19-09-2019, available at: https://databank.worldbank.org/reports.aspx?source=2&series=SP.POP.GROW&country=#

[7] Data from: World Bank, Growth of GDP per capita [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/NY.GDP.PCAP.KD.ZG

where y_{it} is the GDP growth per capita, i is the cross-sectional dimension, t is the time dimension and x_{it} is [k;1] vector of the population growth, α_i and β_i stand for intercepts and slope coefficients respectively. This coefficient varies across countries.

The null hypothesis of the LM test has a form, H_0: $Cov(u_{it},u_{jt}) = 0$, for all t and $i \neq j$, alternative hypothesis: H_1: $Cov(u_{it},u_{jt}) \neq 0$, for all t and at least one pair of $i \neq j$. The equation for the LM test is:

$$LM = T \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\rho}_{ij}^2 \tag{99}$$

where $\hat{\rho}$ is the estimate of the pair-wise correlation of residuals from the pooled OLS equation (99) for each i. The LM statistics have asymptotic chi-square with $\frac{N(N-1)}{2}$ degrees of freedom. The LM test is suitable mainly for relatively small N and large T. For large T and large N, Pesaran (2004) proposed a scaled version of the LM test:

$$CD_{lm} = \left(\frac{1}{N(N-1)}\right)^{\frac{1}{2}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(T\hat{\rho}_{ij}^2 - 1\right) \tag{100}$$

The null hypothesis of this test is that the CD_{lm} test converges to the standard normal distribution. Peseran (2004) developed a more general cross-sectional dependence test (CD test) which has the form:

$$CD = \sqrt{\frac{2T}{N(N-1)}} \left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\rho}_{ij}\right) \tag{101}$$

The null hypothesis of the CD test has an asymptotic standard normal distribution. Pesaran (2004) claims that this test has the mean zero for fixed T and N. Next, CD test is robust for heterogenous dynamic models with breaks in the slope, coefficients and error variances, until unconditional means of y_{it} and x_{it} are time-invariant with symmetric distributions. This test was criticized by Pesaran (2008), where population sample has average pair-wise correlations zero in certain situations. The last test is Baltagi, Feng, Kao (2012) scaled LM test statistics for fixed effects of Pooled OLS. The equation of this test is:

$$LM_{BC} = \sqrt{\frac{2T}{N(N-1}}\left(\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\hat{\rho}_{ij}\right) \quad (102)$$

Under the null hypothesis, where T and N are large, LM_{bc} has a standard normal distribution. The results of this test are below in Table 3.

Another aspect of the BPGC is cross-country heterogeneity. This is tested by the Wald principal, thus the null hypothesis claims that slope coefficient is homogeneous against the alternative hypothesis that propose the slope coefficient is heterogeneous. To be able to apply the Wald principal test value, the data have to have a shape of small N and large T. The next assumptions are that explanatory variables are strictly exogenous and error covariances are homoscedastic. Pesaran, Yamaguta (2008) recommended a standardized Swamy test to test the slope coefficients. To get the test statistic with a probability value, we need to compute the Swamy test:

$$\tilde{S} = \sum_{i=1}^{N}\left(\tilde{\tilde{\beta}}_i - \tilde{\beta}_{WFE}\right)' \frac{x_i' M_\tau x_i}{\tilde{\sigma}_i^2}\left(\tilde{\tilde{\beta}}_i - \tilde{\beta}_{WFE}\right) \quad (103)$$

where $\tilde{\tilde{\beta}}$ is the estimator from pooled OLS, $\tilde{\beta}_{WFE}$ is estimator from the weighted fixed effect pooled estimation of the regression model in equation (98). M_τ symbolizes the identity of matrix and $\tilde{\sigma}_i^2$ is estimation of σ_i^2. Then standardized dispersion statistic is:

$$\tilde{\Delta} = \sqrt{N}\left(\frac{N^{-1}\tilde{S} - k}{\sqrt{2K}}\right) \quad (104)$$

This test has asymptotic normal distribution only if N and T are large, $\sqrt{N}/T$ is a large number and error terms have a normal distribution. For small sample datasets, Pesaran and

Yamaguta (2008) created an adjusted test. Under normally distributed errors, the bias-adjusted equation is:

$$\tilde{\Delta}_{adj} = \sqrt{N}\left(\frac{N^{-1}\tilde{S} - E(\tilde{z}_{it})}{\sqrt{var(\tilde{z}_{it})}}\right) \quad (105)$$

where $E(\tilde{z}_{it}) = k$ is the mean and variance is described by $var(\tilde{z}_{it}) = \frac{2k(T-k-1)}{T} + 1$. Results of the slope heterogeneity tests can be found in Table 3.

Cross-sectional dependence was tested by the first four tests. The null hypothesis of all cross-sectional dependence tests was rejected, thus the described SUR method is more appropriate than the pooled OLS estimation. Rejecting the null hypothesis implies that shocks in one country are somehow transmitted into other countries. Another important point for using SUR instead of OLS is casual linkages between population growth and the GDP growth per capita, which we have to take into account. Therefore, causality results given by SUR are more solid than from the OLS regression.

Tests for the slope homogeneity in Table 3 are the last two tests. The null hypothesis is rejected, thus countries have unique heterogeneity. Therefore, the direction of linkages between the population growth and the GDP growth per capita could differ in African countries.

Table 3: Test statistics result

Test	Statistic	Probability
Breusch – Pagan LM test	2098.947	0.0000
Pesaran scaled LM	7.755885	0.0000
Bias – corrected scaled LM	6.755885	0.0000
Pesaran CD	-3.148716	0.0016
Pesaran, Yamaguta slope test	2.662	0.0004
Adjusted Pesaran, Yamaguta slope test	2.732	0.0003

Note: Test statistics is estimated on the full sample of 30 African countries in time period 1960 – 2018. Own construction.

3.3. Results of BPGC

Table 4: BPGC results of population growth does not cause GDP growth for 30 countries

N = 30

T = 58

lag = 1

5 000 replications

Population growth does not cause GDP growth

Country	Wald statistics	Bootstrap critical value		
		1%	5%	10%
Algeria	0.181	40.574	21.212	15.162
Benin	0.489	28.032	17.784	12.859
Botswana	1.396	40.205	21.758	14.367
Burkina Faso	1.387	33.946	21.450	15.018
Burundi	0.108	38.862	21.682	15.575
Cameroon	1.123	39.473	20.734	15.760
Central African Republic	1.529	37.585	21.877	15.9277
Congo, Dem.	0.846	40.833	23.774	16.630
Congo, Rep.	20.675*	40.520	23.133	15.857
Core d Ivore	0.269	30.396	19.478	13.173
Egypt	2.087	34.778	23.401	16.571
Gabon	5.993	40.098	23.536	16.445
Ghana	5.045	37.526	22.811	15.235
Chad	16.157	36.038	24.247	16.819
Kenya	8.525	48.336	27.054	17.624
Lesotho	4.175	40.252	21.414	15.369
Madagascar	3.045	57.273	26.749	17.373
Malawi	2.285	67.064	23.731	17.300

Mauritania	3.870	40.528	17.232	10.451
Niger	8.973	46.793	26.573	18.321
Nigeria	4.447	47.098	24.202	15.740
Rwanda	1.053	61.463	29.955	20.837
Senegal	10.520	35.942	20.617	14.299
Seychelles	0.488	37.514	22.951	17.320
Sierra Leone	0.097	33.124	21.633	14.312
South Africa	1.868	35.528	22.091	16.434
Sudan	2.198	43.176	27.607	19.503
Togo	13.046	53.718	25.254	17.330
Zambia	0.813	31.164	20.635	13.976
Zimbabwe	0.574	40.692	19.497	14.899

Note: *** Wald statistics > Bootstrap critical value at 1% level, ** Wald statistics > Bootstrap critical value at 5% level, * Wald statistics > Bootstrap critical value at 10% level. Own construction.

Table 5: BPGC results of GDP growth does not caused population growth for 30 countries

N = 30

T = 58

lag = 1

5 000 replications

GDP growth does not cause population growth

Country	Wald statistics	Bootstrap critical value		
		1%	5%	10%
Algeria	0.167	35.327	20.175	15.499
Benin	22.001*	37.042	22.873	15.389
Botswana	0.242	31.679	19.471	13.843
Burkina Faso	0.327	31.304	19.525	13.942
Burundi	0.932	33.067	20.007	13.959
Cameroon	30.754**	38.412	24.291	17.677

Central African Republic	1.490	46.374	24.785	15.961
Congo, Dem.	8.575	46.902	22.415	16.820
Congo, Rep.	0.280	38.550	23.166	16.010
Core d Ivore	3.739	41.705	27.204	20.421
Egypt	0.274	34.928	20.987	14.938
Gabon	0.064	37.083	22.168	14.637
Ghana	1.051	39.563	23.867	16.115
Chad	3.619	36.755	18.367	14.039
Kenya	0.382	38.883	21.615	15.119
Lesotho	0.379	32.534	18.140	12.790
Madagascar	0.761	35.690	20.422	14.804
Malawi	0.316	56.815	20.531	13.011
Mauritania	1.680	36.508	22.959	16.318
Niger	0.305	35.089	19.285	13.953
Nigeria	1.858	30.388	17.919	13.218
Rwanda	1.654	38.872	24.892	17.463
Senegal	0.820	36.703	17.697	12.532
Seychelles	0.018	32.457	19.610	14.556
Sierra Leone	4.575	41.110	20.436	14.249
South Africa	0.095	57.776	28.539	18.881
Sudan	0.228	247.154	168.986	141.267
Togo	13.863	60.335	24.746	14.877
Zambia	5.121	69.445	28.965	17.562
Zimbabwe	11.096	36.020	20.403	14.861

Note: *** Wald statistics > Bootstrap critical value at 1% level, ** Wald statistics > Bootstrap critical value at 5% level, * Wald statistics > Bootstrap critical value at 10% level. Own construction.

Only one significant result at the 10% level for the Congo Republic indicates the possible influence of the population growth for the GDP growth. Two significant results (Cameroon and Benin) are estimated when the GDP growth caused the population growth. For all other countries, there is no significant impact of the population growth on the GDP growth per capita or vice versa.

After examining the results in Table 4 and Table 5, the Wald statistics for countries with a low population has a higher value than countries with a large population. To check this possible feature, I decide to test BPGC for small countries and large countries. To have a similar structure described above I decide to divide countries into subsections according to their population. Small countries consist of 9 countries and the biggest country has a population of 7,698,475 in the year 2017. The difference between the last country in small countries and the first country in large countries is around 3,000,000. I assume that this difference should eliminate the threshold bias. Again, I checked SBC and AIC to get the optimal lag length. Table 6 and Table 7 report a result for the optimal lag length for both sub-samples.

Table 6: Optimal lag length for small countries

LAG GDP	SBC	AIC	LAG POP	SBC	AIC
1	**32.80**	**31.83**	1	-34.70	-35.74
2	37.00	33.75	2	-46.31	-49.75
3	44.07	37.17	3	**-47.46**	**-54.68**
4	53.54	41.60	4	-39.62	-52.02
5	65.89	47.49	5	-26.62	-45.54
6	81.99	55.65	6	-10.63	-37.41
7	101.08	65.28	7	8.03	-28.09
8	123.19	76.38	8	27.83	-18.97

Note: Bold number is the lowest number in the column. The optimal lag length is tested on the sample of 9 African countries in time period 1960 – 2018. Own construction.

Table 7: Optimal lag length for large countries

LAG GDP	SBC	AIC	LAG POP	SBC	AIC
1	**61.73**	**59.47**	1	-138.65	-141.06
2	70.70	63.10	2	-160.87	-168.90
3	86.40	70.30	3	**-162.79**	**-179.65**

4	108.27	80.42	4	-147.70	-176.61
5	137.31	94.37	5	-115.62	-159.79
6	172.82	111.35	6	-76.90	-139.54
7	215.07	131.55	7	-33.31	-117.63
8	267.32	158.10	8	13.97	-95.24

Note: Bold number is the lowest number in the column. The optimal lag length is tested on the sample of 21 African countries in time period 1960 – 2018. Own construction.

From the results, there is the optimal lag length one for the GDP growth per capita, and three for the population growth for both sub-samples of countries according to SBC and AIC. In contrast with the last results, SBC and AIC have the minimum value in the same lag length. I checked the test statistic described above too. The results are reported in Table 8 for small countries and Table 9 for large countries:

Table 8: Test statistics for small countries

Test	Statistic	Probability
Breusch – Pagan LM test	1819.464	0.0024
Pesaran scaled LM	2.895140	0.0038
Bias – corrected scaled LM	-0.729860	0.4655
Pesaran CD	-1.476667	0.1398
Pesaran, Yamaguta slope test	0.541	0.294
Adjusted Pesaran, Yamaguta slope test	0.555	0.289

Note: Test statistics is estimated on the sample of 9 African countries in time period 1960 – 2018. Own construction.

Table 9: Test statistics for large countries

Test	Statistic	Probability
Breusch – Pagan LM test	2222.912	0.0000
Pesaran scaled LM	9.911881	0.0000
Bias – corrected scaled LM	8.461881	0.0000
Pesaran CD	-2.692937	0.0071
Pesaran, Yamaguta slope test	2.782	0.003

Adjusted Pesaran, Yamaguta slope test	2.856	0.002

Note: Test statistics is estimated on the sample of 21 African countries in time period 1960 – 2018. Own construction.

Table 9 shows a cross-section dependency with slope heterogeneity. Unfortunately, Pesaran CD and Bias – corrected scale LM test approve null hypothesis for small countries, thus the cross-section dependency is not present and there is slope homogeneity. This could be due to a low number of countries in this group, which has a negative effect on these tests. Therefore, even if small countries show some effect, this conclusion should be taken with caution, and BPGC is not the best model to use.

Results for the sub-sample are in tables below:

Table 10: BPGC results of population growth does not cause GDP growth for 21 countries

N = 21

T = 58

lag = 1

10 000 replications

Population growth does not cause GDP growth

Country	Wald statistics	Bootstrap critical value		
		1%	5%	10%
Algeria	3.901	16.096	8.941	6.155
Benin	1.623	16.721	9.432	6.475
Burkina Faso	0.322	17.785	9.781	6.803
Burundi	0.052	15.042	8.466	5.932
Cameroon	0.022	17.528	9.527	6.534
Congo, Dem.	0.009	18.148	9.462	6.722
Core d Ivore	0.988	17.497	9.629	6.729
Egypt	0.729	18.491	9.945	6.894
Ghana	0.791	17.690	9.871	6.776

Chad	0.005	20.592	10.550	7.042
Kenya	3.883	19.220	9.347	6.216
Madagascar	0.370	22.614	8.694	5.462
Malawi	0.205	30.753	10.947	6.743
Niger	1.126	23.926	10.578	6.951
Nigeria	0.243	20.667	10.210	6.773
Rwanda	0.241	15.819	8.399	5.578
Senegal	1.508	16.002	9.002	6.134
South Africa	0.657	17.595	9.745	6.932
Sudan	0.000	16.358	8.966	6.287
Zambia	1.786	16.989	9.137	6.229
Zimbabwe	0.614	17.135	9.091	6.218

Note: *** Wald statistics > Bootstrap critical value at 1% level, ** Wald statistics > Bootstrap critical value at 5% level, * Wald statistics > Bootstrap critical value at 10% level. Own construction.

Table 11: BPGC results of GDP growth does not cause population growth for 21 countries

N = 21

T = 58

lag = 3

10 000 replications

GDP growth does not cause population growth

Country	Wald statistics	Bootstrap critical value		
		1%	5%	10%
Algeria	0.120	11.405	6.342	4.220
Benin	0.095	10.825	5.929	4.104
Burkina Faso	4.066	12.559	6.768	4.555
Burundi	0.021	12.049	6.503	4.487
Cameroon	1.415	12.824	6.794	4.589
Congo, Dem.	9.003**	10.806	5.908	4.040

Core d Ivore	3.601	10.026	5.744	3.860
Egypt	0.000	11.851	6.636	4.623
Ghana	4.582*	12.425	6.338	4.361
Chad	0.271	15.160	8.316	5.833
Kenya	0.040	12.478	7.320	5.148
Madagascar	4.633*	9.126	5.006	3.485
Malawi	0.047	12.137	6.322	4.145
Niger	0.072	11.529	6.093	4.187
Nigeria	3.436	11.580	5.923	4.056
Rwanda	6.030**	11.326	5.918	4.119
Senegal	1.535	12.204	6.943	4.619
South Africa	0.846	15.542	8.532	5.786
Sudan	0.077	13.453	6.877	4.686
Zambia	2.238	14.238	7.190	4.825
Zimbabwe	15.055***	11.995	6.639	4.672

Note: *** Wald statistics > Bootstrap critical value at 1% level, ** Wald statistics > Bootstrap critical value at 5% level, * Wald statistics > Bootstrap critical value at 10% level. Own construction.

Table 12: BPGC results of population growth does not cause GDP growth for 9 countries

N = 9

T = 58

lag = 1

10 000 replications

population growth does not cause GDP growth

Country	Wald statistics	Bootstrap critical value		
		1%	5%	10%
Botswana	2.203	6.535	3.719	2.558
Central Africa Republic	14.670***	7.721	4.456	3.261
Congo, Rep.	0.764	8.266	4,456	3.020
Gabon	3.842*	7.331	4.027	2.746
Lesotho	9.108***	6.887	3.664	2.446
Mauritania	0.052	7.729	4.089	2.708
Seychelles	0.796	6.288	3.376	2.254
Sierra Leone	0.027	8.782	4.755	3.400
Togo	2.496	7.647	4.267	2.990

Note: *** Wald statistics > Bootstrap critical value at 1% level, ** Wald statistics > Bootstrap critical value at 5% level, * Wald statistics > Bootstrap critical value at 10% level. Own construction.

Table 13: BPGC results of GDP growth does not cause population growth for 9 countries

N = 9

T = 58

lag = 3

10 000 replications

GDP growth does not cause population growth

Country	Wald statistics	Bootstrap critical value		
		1%	5%	10%
Botswana	0.298	7.746	4.361	3.077
Central Africa Republic	5.235*	9.441	5.944	4.418
Congo, Rep.	0.176	6.857	3.849	2.660
Gabon	0.977	8.428	5.258	3.765
Lesotho	9.192***	5.908	3.467	2.408
Mauritania	0.010	6.893	4.033	2.819
Seychelles	0.126	6.816	3.980	2.850
Sierra Leone	0.000	7.894	4.349	2.998
Togo	0.017	7.086	3.784	2.601

Note: *** Wald statistics > Bootstrap critical value at 1% level, ** Wald statistics > Bootstrap critical value at 5% level, * Wald statistics > Bootstrap critical value at 10% level. Own construction

Lesotho, as the only country, shows a bidirectional causality between the population growth and the GDP growth per capita at a 1% level. The population growth influenced GDP growth in the Central Africa Republic at a 1% level of significance and the same causality for Gabon at the 10% level. The Central Africa Republic has bidirectional causality at the 10% level. In other countries, there is no significant effect.

The result for big countries shows no significant effect of population growth on the GDP growth per capita for all countries in this sub-sample. Causal relationship when the GDP growth per capita causes the population growth is significant in Zimbabwe at a 1% level, in

Rwanda at 5% level and Congo Republic, and in Madagascar and Ghana at 10% level. There is no effect in other countries.

In the conclusion of the BPGC model, most countries show no significant relationship between the population growth and the GDP growth per capita or vice versa. When I divided countries into two sub-samples, more countries are showing significant values. For the big countries, there is mainly the influence of the GDP growth per capita on the population growth. For the small countries, there is a bidirectional effect for 2 countries and one country has a significant value when the population growth cause the GDP growth per capita. But the results for small countries could not be valid as discussed above.

3.4. The Dynamic Panel Data model

As described in Nickel (1981), the standard pooled OLS model with random and fixed effects in my thesis cannot be used. As Nickel estimated, for small time values (T) and large cross-section values (N) there is the possibility of a correlation between regressor and error term due to the demeaning process which takes individual regressor to mean value of the dependent variable and explanatory variable. This correlation creates a bias in the estimate of the coefficient of lagged dependent variables which is not countered by an increase in N. Reggresor cannot be independently distributed on the error term. Adding more regressors is not helpful, because the correlation between lagged dependent and added regressors can create biased coefficients to some degree as well. Nickel (1981) points out that even if we have an uncorrelated error term process, the bias will be present. The same problem occurs with the random-effects model too. The error term enters every dependent value by assumption, thus the lagged dependent value cannot be independent of the composite of error process.

There are two possibilities, how to deal with the problems described above. First is the General Method of Moments (GMM), which uses the first difference approach to remove constant term and the individual effect. But there still could be autocorrelation between the lagged dependent and disturbance process, which is defined as a MA (1) process.

The second option is a dynamic panel data model (DPD). The foundation of this model was set by Holtz-Eakin, Newey, and Rosen (1988) and popularized by Arellano and Bond (1991).

They discuss the weakness of the Anderson-Hsiao estimator, which is used in the GMM. The authors claim that GMM is not able to implement all potential orthogonality conditions into estimations. This estimator is replaced by the Arellano-Bond strategy, which implies the assumption of a necessary “internal” instrument. External instruments can be included too. For dynamic panel model consider equations:

$$\begin{gathered} y_{it} = X_{it}\beta_1 + W_{it}\beta_2 + v_{it} \\ v_{it} = u_{it} + \epsilon_{it} \end{gathered} \tag{106}$$

where X_{it} includes strictly exogenous regressors, W_{it} represents predetermined regressors (mainly lags of y) and endogenous regressors, which can be correlated with u_{it}, the unobserved individual effect. With data, which I include in my empirical research, all variables are considered as exogenous regressors. None of them has a strong correlation with both dependent variables. Only endogenous variables are lag values of the dependent variable. The first differentiation technique removes u_{it}, which can cause omitted variable bias. The Arellano-Bond estimator uses the GMM with a specific system of equations per one period. Instruments used in each equation differ.

To apply DPD as a valid model, we need to use two tests. First is the Arellano – Bond serial correlation test. This test was proposed by Arellano and Bond (1991). The test has two separate statistics, one for the process of AR(1) and second for the process AR(2). If variables are independent and identically distributed, we expect that process AR(1) will be significant with a negative coefficient. Process AR(2) should be insignificant. The statistic is calculated as:

$$\begin{gathered} m_j = \frac{\rho_j}{\sqrt{VAR(\rho_j)}} \\ \rho_j = \frac{1}{T-3-j}\sum_{t=4+j}^{T}\rho_{tj} \\ \rho_{tj} = E(\Delta\epsilon_{i,t}, \Delta\epsilon_{i,t-j}) \end{gathered} \tag{107}$$

where ρ_j is the average of j-th order autocovariance. Because DPD uses instrument variables, I include the Sargan – Hansen test created by Sargan (1958) and extended by Hansen (1982).

The aim of this test is to over-identify restrictions imposed on the model. In other words, there are more instruments than endogenous variables. This test uses the chi-quadrat test where inputs are the value of J-statistic and instrument rank in the DPD.

3.5. Results of the Dynamic panel data model

3.5.1. Relationship with health control variables

I used the above methodology to estimate a relationship between GDP growth per capita and population growth. I combine DPD with standard Random and Fixed effect models for comparison, although the results are probably biased as described above. Lagged values of the dependent variable are taken as an instrument variable and tested by the Sargan – Hansen test in the case of DPD. For comparison, the same variables are used in the Random-effect model and the Fixed-effect model. The length of lagged values is set on the last value which is significant, and when the covariance matrix does not have NA value. I choose to use a group of variables that have an impact on the health condition of citizens for the first estimation.

Table 14: DPD results of GDP growth as dependent variable with health control variables

Dependent variable: growth of GDP per capita			
Variables	Fixed-effect model	Random-effect Model	DPD
C	4.077276	1.284531**	-
	(2.729819)	(0.609358)	
GDP_lag1	0.189392***	0.311232***	0.131123***
	(0.042992)	(0.040868)	(0.010259)
GDP_lag2	-0.097510**	0.004858	-0.099468***
	(0.041577)	(0.040179)	(0.021837)
GDP_lag3	-0.039089	0.042203	-0.090193***
	(0.041536)	(0.040179)	(0.002356)

GDP_lag4	-0.104237**	-0.011313	-0.133582***
	(0.040650)	(0.038409)	(0.009107)
Population_growth	2.618986***	0.021201	-0.785669
	(0.723027)	(0.241550)	(0.593667)
Prevalence of undernourishment	-0.021630	0.016062	0.138967
	(0.045510)	(0.13162)	(0.104332)
Prevalence of HIV	-1.155360***	-0.003444	-1.255593***
	(0.345138)	(0.025711)	(0.213445)
Malaria deaths	-0.017592	-0.002473	0.087496***
	(0.013141)	(0.003281)	(0.021435)
Time	14	14	13
Countries	41	41	41
Adj. R-square	0.174909	0.099989	-
Hausman test	-	39.271054***	-
J-statistic	-	-	37.01917
Arrelo – Bond AR(1)	-	-	-2.684011**
Arrelo – Bond AR(2)	-	-	0.082968
Sargan – Hansen test	-	-	0.648170

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) Prevalence of HIV data from World Bank: World Development Indicators [online], accessed at 19-09-2019, available at: https://datacatalog.worldbank.org/dataset/world-development-indicators
f) Malaria deaths data from: UNICEF, Malaria [online], accessed at 19-09-2019, available at: https://data.unicef.org/topic/child-health/malaria/
g) Prevalence of undernourishment data from: World Bank Prevalence of undernourishment (% of population) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/sn.itk.defc.zs
h) Result contains data for 41 countries from 2000 to 2017.
i) Own construction

When the GDP growth per capita is the dependent variable, the influence of the population growth does not seems to be significant in DPD and significant in the Fixed-effect model, which is a better alternative than the Random-effect model due to rejecting the null hypothesis of the Hausman test. The prevalence of HIV is a significant variable in both cases with a large negative coefficient, malaria deaths is a variable important only in the DPD model, and the prevalence of undernourishment is not significant in any of the models. In A-B statistics, we reject the null hypothesis with the negative coefficient in AR(1) process and approve the null hypothesis in AR(2) process. The p-value of the Sargan-Hansen test is more than 0.05, thus used instruments are valid.

Table 15: DPD result of population growth as dependent variable with health control variables

Dependent variable: population growth			
Variables	Fixed-effect model	Random-effect Model	DPD
C	0.113704***	0.012854***	-
	(0.022826)	(0.00273)	
pop_lag1	2.066935***	2.284706***	1.883628***
	(0.041694)	(0.237635)	(0.002032)
pop_lag2	-1.314842***	-1.579363**	-1.062907***
	(0.090194)	(0.620633)	(0.0016838)
pop_lag3	-0.149021	-0.103017	-0.234649***
	(0.099665)	(0.646320)	(0.000580)
pop_lag4	0.490403***	0.576013*	0.472835***
	(0.076495)	(0.330931)	(0.000654)
pop_lag5	-0.151937***	-0.185666**	-0.14275***
	(0.028046)	(0.076325)	(0.000303)
GDP_growth_per_capita	-0.000307	-0.000506*	-0.000169***
	(0.000317)	(0.000336)	(0.00001)
Prevalence of undernourishment	-0.001686***	0.000186*	0.005229***
	(0.000348)	(0.00008)	(0.000214)

Prevalence of HIV	-0.003356	0.00008	0.019484***
	(0.002759)	(0.000176)	(0.000648)
Malaria deaths	0.000165	0.00002	-0.000336***
	(0.000103)	(0.00003)	(0.00003)
Time	13	13	12
Countries	41	41	41
Adj. R-square	0.998997	0.998848	-
Hausman test	-	40.410835***	-
J-statistic	-	-	38.36842
Arrelo – Bond AR(1)	-	-	-0.999417
Arrelo – Bond AR(2)	-	-	0.151195
Sargan – Hansen test	-	-	0.672185

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) Result contains data for 41 countries from 2000 to 2017.
f) Own construction.

When the population growth is a dependent variable, it is highly correlated with the last lagged values, and other variables do not have so much influence. The GDP growth per capita negatively influences the population growth according to the DPD model, but the value is very small. We can even say that it has no influence. The same can be said for other control variables, except for the prevalence of HIV, which has a correlation coefficient of 0.02. The Hausman test found out that the Fixed-effect model is better than the Random-effect model in this case, thus we reject the null hypothesis. The Fixed-effect model does not find the GDP growth per capita as a significant variable. The only significant value seems to be the prevalence of undernourishment. Sagan – Hansen test rejects the alternative hypothesis, thus instrument variables are valid instruments. The Arellano – Bond serial correlation shows a serial correlation with a negative effect in process AR(1). As discussed in Habimana (2016) this is not a crucial problem, the importance lays in approving AR(2) process, thus there is no serial correlation in disturbances. Even so, the results for the population growth should be taken with more caution.

3.5.2. Relationship with safety variables

Table 16: DPD result of GDP growth as dependent variable with safety control variables

Dependent variable: growth of GDP per capita			
Variables	Fixed-effect model	Random-effect Model	DPD
C	0.213996	1.185583**	-
	(2.729819)	(0.638823)	
GDP_lag1	0.126479***	0.222214***	0.129049***
	(0.076062)	(0.065385)	(0.030140)
GDP_lag2	-0.014593	0.095628**	0.031738*
	(0.040523)	(0.034254)	(0.017031)
Population_growth	0.779372**	0.040621	0.820707**
	(0.627034)	(0.250491)	(0.273668)
War_intern	-0.809884	-0.150413	-1.702171*
	(0.535210)	(0.416155)	(0.889276)
War_outer	-0.071874	-0.145162	-2.509953***
	(0.803897)	(0.744884)	(0.386403)
More than 1000 deaths	-0.070309	0.075779	-0.201144
	(0.273427)	(0.278570)	(0.659255)
Tourism arrivals per capita	-0.000721	0.000502	-0.00006
	(0.001542)	(0.000341)	(0.004592)
Time	21	21	20
Countries	37	37	37
Adj. R-square	0.114657	0.068793	-

Hausman test	-	1.852493	-
J-statistic	-	-	31.95774
Arrelo – Bond AR(1)	-	-	-3.317116***
Arrelo – Bond AR(2)	-	-	-0.063805
Sargan – Hansen test	-	-	0.704573

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) War intern. War outer and more than 1000 deaths data from: Clio Infra, Armed conflicts (Internal) [online], accessed at 19-09-2019, available at: https://clio-infra.eu/Indicators/ArmedconflictsInternal.html and from: Uppsala Conflict Data Program, UCDP dataset download center [online], Accessed at 19-09-2019, available at: https://ucdp.uu.se/downloads/#d3
f) Tourism arrivals per capita data from: World Bank, International tourism, number of arrivals [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/ST.INT.ARVL
g) Result contains data for 37 countries from 1995 to 2017.
h) Own construction.

From the results above the Fixed-effect model and the Random-effect model did not find any significant values except for the lagged dependent variable. The estimation of these models seems to be biased. Wars should have a significant impact on the GDP as discussed in Chapter 2. DPD finds all variables significant at least at 1% level value except for the variable more than 1000 deaths and tourism arrivals per capita. Wars in the country negatively affect GDP growth per capita. Again, the test statistics approve the DPD as a valid method for modeling.

Table 17: DPD result of population growth as dependent variable with safety control variables

Dependent variable: population growth

Variables	Fixed-effect model	Random-effect model	DPD
C	1.273555***	0.211609***	-
	(0.350719)	(0.050025)	
pop_lag1	0.248490***	0.487408***	0.172258***
	(0.196470)	(0.038082)	(0.001948)
pop_lag2	0.371171***	0.492039***	0.336254***
	(0.075422)	(0.042297)	(0.001441)
pop_lag3	0.046147	0.044573	0.050312***
	(0.053957)	(0.046231)	(0.000683)
pop_lag4	0.043292	0.076442*	0.038012***
	(0.057454)	(0.046231)	(0.002831)
pop_lag5	-0.270022***	-0.186691***	-0.283309***
	(0.045795	(0.037142)	(0.005704)
Growth of GDP per capita	0.005375*	0.001316	0.008037***
	(0.002766)	(0.003124)	(0.000126)
War_intern	-0.007643	0.059729*	-0.092290***
	(0.041168)	(0.034206)	(0.0.012470)
War_outer	-0.068146	-0.065655	-0.054736**
	(0.058059)	(0.049269)	(0.027729)
More than 1000 deaths	-0.006888	0.023897	-0.058661**
	(0.2014957)	(0.029832)	(0.0.025786)
Tourism arrivals per capita	-0.000116	-0.000108***	-0.000157***
	(0.00140)	(0.00004)	(0.00002)

Time	18	18	17
Countries	37	37	37
Adj. R-square	0.896124	0.870163	-
Hausman test	-	13.471774**	-
J-statistic	-	-	28.56075
Arrelo – Bond AR(1)	-	-	-1.1377868
Arrelo – Bond AR(2)	-	-	1.903588*
Sargan – Hansen test	-	-	0.838519

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) Result contains data for 37 countries from 1995 to 2017.
f) Own construction.

Table 17 provides another interesting result, where DPD shows all variables as significant, but the AB test in AR(2) process at 1% level of significance shows a serial correlation in disturbances. For these reasons, results for population growth should be taken with more caution. Again, the Fixed-effect model seems to be a better estimator for panel data analysis instead of the Random-effect model. The GDP growth per capita is significant at 1% level and it has a very small positive value. All other variables are insignificant. The same applies to some lagged values of the population growth.

3.5.3. Results with economy variables

Table 18: DPD result of Growth of GDP per capita as dependent variable with economy control variables

Dependent variable: growth of GDP per capita

Variables	Fixed-effect model	Random-effect model	DPD
C	-1.562565	0.603630	-
	(0.961540)	(0.554436)	
GDP_lag1	0.083948**	0.178650***	0.109114***
	(0.076062)	(0.036360)	(0.012612)
Population growth	0.834092**	0.163640	0.682673**
	(0.350850)	(0.206829)	(0.206567)
Gross domestic saving rate	0.034943	-0.040256**	1.080477**
	(0.025622)	(0.012294)	(0.028905)
Total natural resources	0.055520	-0.030984*	0.156268***
	(0.036908)	(0.016856)	(0.016782)
Development aid per capita	0.001287	0.003849	-0.004564**
	(0.003892)	(0.003247)	(0.001886)
Tourism arrivals per capita	-0.000749	0.000209	0.004564
	(0.001212)	(0.000525)	(0.001886)
Time	22	22	21
Countries	32	32	32
Adj. R-square	0.086949	0.055991	-
Hausman test	-	54.216611***	-
J-statistic	-	-	28.27658
Arrelo – Bond AR(1)	-	-	-3.043293***

Arrelo – Bond AR(2)	-	-	1.281036
Sargan – Hansen test	-	-	0.655594

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) Gross domestic saving rate data from: World Bank Gross domestic savings (% of GDP) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/NY.GDS.TOTL.ZS?locations=LR
f) Total natural resources data from: World Bank Total natural resources rents (% of GDP) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS
g) Development aid per capita data from: World Bank Net official development assistance and official aid received (current US$) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/DT.ODA.ALLD.CD
h) Result contains data for 32 countries from 1995 to 2017.
i) Own construction.

When we include economy group variables, the only variable which is not significant is tourism arrivals per capita. After including these four economy variables, only the first lag is significant in the DPD model. Population growth has a positive impact and development aid has a negative effect. The gross domestic saving rate has a large coefficient. The Fixed-effect model shows only lagged value and population growth as significant variables. Both of these variables have a positive effect on the GDP growth per capita.

Table 19: DPD result of population growth as dependent variable with economy control variables

Dependent variable: population growth

Variables	Fixed-effect model	Random-effect Model	DPD
C	0.921404***	0.225239***	-
	(0.100321)	(0.055071)	
pop_lag1	0.359889***	0.528517***	0.306600***
	(0.040588)	(0.039924)	(0.001971)
pop_lag2	0.403702***	0.488931***	0.375839***
	(0.040920)	(0.043336)	(0.002120)

pop_lag3	-0.112573***	-0.083609*	-0.130023***
	(0.41018)	(0.044089)	(0.000871)
pop_lag4	-0.106352***	-0.055214	-0.126389***
	(0.037076)	(0.038878)	(0.004468)
Growth of GDP per capita	0.003861	0.000953	0.008221***
	(0.003405)	(0.003460)	(0.000317)
Gross domestic saving rate	0.001128	-0.001747**	0.002082**
	(0.002327)	(0.001167)	(0.000557)
Total natural resources	-0.003703	-0.002595	-0.012108***
	(0.003205)	(0.001608)	(0.000641)
Development aid per capita	0.002512***	0.001945***	-0.003269***
	(0.000324)	(0.000299)	(0.000007)
Tourism arrivals per capita	0.000245**	-0.000244***	0.000253***
	(0.000107)	(0.00005)	(0.00003)
Time	19	19	18
Countries	32	32	32
Adj. R-square	0.881200	0.857197	-
Hausman test	-	54.216611***	-
J-statistic	-	-	28.61082
Arrelo – Bond AR(1)	-	-	-1.070658
Arrelo – Bond AR(2)	-	-	-0.966451
Sargan – Hansen test	-	-	0.685546

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: Ordinary
e) Result contains data for 32 countries from 1995 to 2017.
f) Own construction.

For the population growth as a dependent variable, I include four lags. All variables are significant for the population growth according to the DPD model, but again AR(1) shows no serial correlation. The GDP growth per capita, gross domestic saving rate and tourism arrivals per capita have a positive effect on the population growth. Development aid per capita and total natural resources have a negative effect on population growth.

3.5.4. Results with index variables

Table 20: DPD result of Growth of GDP per capita as dependent variable with index control variables

Dependent variable: growth of GDP per capita

Variables	Fixed-effect model	Random-effect Model	DPD
C	-12.74833	4.240304**	-
	(12.36529)	(1.881273)	
GDP_lag1	0.322577***	0.382393***	0.279039***
	(0.053845)	(0.042255)	(0.014302)
GDP_lag2	-0.101297**	-0.056972	-0.088035***
	(0.063519)	(0.041694)	(0.013769)
GDP_lag3	0.194495***	0.199523***	0.191714***
	(0.132686)	(0.036900)	(0.006354)
GDP_lag4	-0.004485	0.005119	0.025437**
	(0.057346)	(0.035462)	(0.010654)
Population growth	1.294169*	-0.559705**	3.664461***
	(1.004571)	(0.240752)	(0.380618)
Free Market Index	-0.047416	0.007355	-0.033746
	(0.128000)	(0.023908)	(0.034951)
HDI Index	26.14576	-4.324707**	-34.17665***
	(27.03518)	(1.876551)	(5.509815)

Time	13	13	12
Countries	38	38	38
Adj. R-square	0.240209	0.217095	-
Hausman test	-	21.009654***	-
J-statistic	-	-	32.37920
Arrelo – Bond AR(1)	-	-	-3.464062***
Arrelo – Bond AR(2)	-	-	0.208551
Sargan – Hansen test	-	-	0.726430

Note:

a) Standard errors in parentheses
b) *** p < 0.01, ** p < 0.05, *p < 0.1
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) Free Market Index data from: Heritage.org index of economic freedom [online], accessed at 19-09-2019, available at: https://www.heritage.org/index/explore?view=by-region-country-year&u=636968127055364350
f) HDI Index data from: United Nations, HDI (1990 – 2018) [online], accessed at 19-09-2019, available at: http://hdr.undp.org/en/data#
g) Result contains data for 38 countries from 2000 to 2017.
h) Own construction.

When we include two indexes as variables, the DPD model shows a large negative impact of the HDI index on the GDP growth per capita. HDI contains a level of GDP per capita. This can be a reason for large coefficient in the DPD model. Another explanation is a convergence of the countries. In comparison with other control variables, the population growth has a much greater positive impact on the dependent variable. Lag values of the dependent variable are comparable with previous models with different control variables. Test statistic shows DPD as a valid model.

Table 21: DPD result of population growth as dependent variable with index control variables

Dependent variable: population growth

Variables	Fixed-effect model	Random-effect Model	DPD
C	0.092965	0.048344***	-
	(0.073920)	(0.014055)	
pop_lag1	2.027449***	2.200936***	1.939604***
	(0.043574)	(0.037984)	(0.001578)
pop_lag2	-1.224615***	-1.421144***	-1.133216***
	(0.091539)	(0.089181)	(0.002195)
pop_lag3	-0.184364**	-0.170206*	-0.182176***
	(0.099216)	(0.100929)	(0.001004)
pop_lag4	0.465738***	0.541231***	0.428944***
	(0.077262)	(0.076110)	(0.001065)
pop_lag5	-0.132949***	-0.160729***	-0.124406***
	(0.028849)	(0.026994)	(0.000787)
Growth of GDP per capita	0.000278	-0.000139	0.000729***
	(0.000339)	(0.000309)	(0.00004)
Free market index	0.000247	-0.00006	0.000770***
	(0.000485)	(0.000172)	(0.00006)
HDI Index	-0.020830	-0.040649**	-0.142016***
	(0.139443)	(0.014164)	(0.017542)
Time	12	12	11
Countries	38	38	38
Adj. R-square	0.999102	0.999052	-
Hausman test	-	6.585021*	-

J-statistic	-	-	33.18358
Arrelo – Bond AR(1)	-	-	-1.057316
Arrelo – Bond AR(2)	-	-	-0.174899
Sargan – Hansen test	-	-	0.685546

Note:

a) Standard errors in parentheses
b) *** $p < 0.01$, ** $p < 0.05$, *$p < 0.1$
c) Random-effect model and Fixed-effect model coef. covariance method: Ordinary
d) DPD coef. covariance method: White period
e) Result contains data for 38 countries from 2000 to 2017.
f) Own construction.

If the population growth is a dependent variable, the HDI index has a larger impact than other control variables. Lag values are again comparable with previous models. As in the same estimation with other control variables, process AR (1) shows no serial correlation.

Conclusion

The first part discusses the literature review of the theory of endogenous population growth. If we increase a shadow price of quality relative to the shadow price of quantity of children and shadow price of consumption, parents will have less children. On the other hand, decrease in the number of children leads to a higher quality of children. Another prediction is that change in income has a higher influence on the number of children than on the quality of children.

The difference in the Solow–Swan model with the exogenous population growth and the endogenous population growth is in the number and stability of equilibrium points. The exogenous form has only one stable equilibrium. The endogenous form depends highly on the form of the dynamics of the capital per capita. There is a possibility of multiple equilibria, and some of them are unstable.

The Nerlove – Raut model estimates the importance of the minimum wage ratio on the capital-labor ratio or per capita output. This model shows an influence of the third production factor, which can have a negative coefficient, on the population growth and output growth per capita. The Nerlove – Raut equilibrium depends on minimum wage in relation to output or capital-labor ratio. If this ratio has a large value, equilibrium might be unstable. The Barro-Becker model conclusion depends on the form of recursion. If we have nonrecursive formulation, the number of children depends only on costs for raising a child. When we have recursive formulation, the number of children depends on the interest rate, initial capital, wage, costs of raising a child, and the degree of altruism of parents.

The next parts consist of the data description and empirical analysis. The empirical analysis examines two models and their results. The Bootstrapped Panel Granger Causality model finds only one country where the population growth influences the GDP growth per capita, and two countries where the GDP growth per capita influences the population growth. Other 27 countries show no relationship. Next, I divide countries into two sub-samples and use the same methodology to estimate the relationship. The sub-sample large countries shows no effect of the population growth on the growth of the GDP per capita. The GDP growth per capita causes the population growth in five countries. The rest (16 countries) shows no relationship. The sub-sample small countries displays two bidirectional causalities and one relationship when the GDP growth per capita causes the population growth. Other six countries show no relationship. Nevertheless, the Bootstrapped Panel Grange Causality model is not the best model to use for the small countries sub-sample due to the slope coefficient homogeneity.

The second model, Dynamic Panel Data, examines the relationship of the GDP growth per capita and the population growth with control variables. The model estimates that the GDP growth per capita is in all results a significant variable when the population growth is a dependent variable. Coefficient is negative when I include health control variables, but positive in all others. In all cases the value of coefficient is very small in comparison with other variables. The population growth is significant in all cases, when the GDP per capita is dependent variable, expect when the health control variables are included. In all significant cases the population growth has positive coefficient and the value of coefficient was not as small as in the case of the GDP growth per capita.

The aim of the thesis is fulfilled. The relationship between the population growth and the GDP growth per capita depends mainly on the used methodology and the time period. In the first model, most of the countries show no relationship. In the other model, both variables are significant, but the population growth has a much bigger coefficient value. This can be caused by two reasons. Because the population growth is a variable which can have a large dependency on lagged values, we cannot find a relationship with data which we have at our disposal. Most of the data are gathered from 1990 with different quality. To be able to estimate a relationship we need larger datasets with more control variables. The second reason might be that quality of data for Africa is not sufficient. Even though the data are from reliable sources, they can be corrupted or manipulated by African governments.

As further extension, other models could be used, such as regression trees or neural networks. Next researchers could look for the data with long time span, which can be used as control variables, and the data should cover as many countries as possible.

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References

Acemoglu, Daron. “Innovation, Reallocation, and Growth.” *American Economic Review*, vol. 108, no. 11, 2018, pp. 3450-3491. https://economics.mit.edu/files/16903. Accessed 1 Apr. 2020.

Acemoglu, Daron. “Technical Change, Inequality, and the Labor Market.” *Journal of Economic Literature*, vol. 40, no. 1, 2002, pp. 7–72. *JSTOR*, www.jstor.org/stable/2698593. Accessed 1 Apr. 2020.

Afzal, Mohammad. “Population Growth and Economic Development in Pakistan“. *The Open Demography Journal*, No.2 , pp. 1-7, 2009. https://benthamopen.com/ABSTRACT/TODEMOJ-2-1, Accessed 29 Apr. 2020.

Aghion, Philippe, and Peter Howitt. “A Model of Growth Through Creative Destruction.” *Econometrica*, vol. 60, no. 2, 1992, pp. 323–351. *JSTOR*, www.jstor.org/stable/2951599. Accessed 1 Apr. 2020.

Arellano, Manuel, and Stephen Bond. “Some Tests of Specification for Panel Data: Monte Carlo Evidence and an Application to Employment Equations.” *The Review of Economic Studies*, vol. 58, no. 2, 1991, pp. 277–297. *JSTOR*, www.jstor.org/stable/2297968. Accessed 30 Apr. 2020.

Arrow, Kenneth J. “The Economic Implications of Learning by Doing.” *The Review of Economic Studies*, vol. 29, no. 3, 1962, pp. 155–173. *JSTOR*, www.jstor.org/stable/2295952. Accessed 1 Apr. 2020.

Azariadis, Costas and Allan, Drazen. ”Endogenous Fertility in Models of Growth Endogenous Fertility in Models of Growth.” *Revista de Análisis Económico (RAE)*, vol. 8, 1993, *Researchgate*, www.researchgate.net/publication/49594307_Endogenous_Fertility_in_Models_of_Growth_Endogenous_Fertility_in_Models_of_Growth. Accessed 1 Apr. 2020.

Baltagi, Badi; Feng, Qu; and Kao, Chihwa. "A Lagrange Multiplier Test for Cross-Sectional Dependence in a Fixed Effects Panel Data Model". Center for Policy Research. 193, 2012. https://surface.syr.edu/cpr/193, Accessed 29 Apr. 2020.

BARRO, Robert J. and Xavier SALA-I-MARTIN. *Economic growth*. 2nd ed. Cambridge, Mass.: MIT Press, c2004. ISBN 9780262025539.

Barro, Robert J., and Gary S. Becker. "Fertility Choice in a Model of Economic Growth." *Econometrica*, vol. 57, no. 2, 1989, pp. 481–501. *JSTOR*, www.jstor.org/stable/1912563. Accessed 2 Apr. 2020.

Becker, Gary S. " An Economic Analysis of Fertility" *Demographic and Economic Change in Developed Countries*, 1960, pp. 209 - 240. *NBER Chapters*, https://www.nber.org/chapters/c2387.pdf, Accessed 1 Apr. 2020.

Becker, Gary S., and H. Gregg Lewis. "On the Interaction between the Quantity and Quality of Children." *Journal of Political Economy*, vol. 81, no. 2, 1973, pp. S279–S288. *JSTOR*, www.jstor.org/stable/1840425. Accessed 1 Apr. 2020.

Becker, Gary S., et al. "Human Capital, Fertility, and Economic Growth." *Journal of Political Economy*, vol. 98, no. 5, 1990, pp. S12–S37. *JSTOR*, www.jstor.org/stable/2937630. Accessed 12 May 2020.

Bloom, Nicholas. "Are Ideas harded to find? " *The American Economic Review*, vol. 110, no. 4, 2017, pp. 1104 - 1144., https://www.aeaweb.org/articles?id=10.1257/aer.20180338&&from=f. Accessed 1 Apr. 2020.

Bosi, Stefano, Raouf, Boucekkine and Thomas Seegmuller. "The dynamics of wealth inequality under endogenous fertility: A remark on the Barro-Becker model with heterogenous endowments". 2010. *HAL* , https://halshs.archives-ouvertes.fr/halshs-00503195/document, Accessed 1 Apr. 2020.

Breusch, T. S., and A. R. Pagan. "The Lagrange Multiplier Test and Its Applications to Model Specification in Econometrics." *The Review of Economic Studies*, vol. 47, no. 1, 1980, pp. 239–253. *JSTOR*, www.jstor.org/stable/2297111. Accessed 29 Apr. 2020.

Cass, David. "Optimum Growth in an Aggregative Model of Capital Accumulation." *The Review of Economic Studies*, vol. 32, no. 3, 1965, pp. 233–240. *JSTOR*, www.jstor.org/stable/2295827. Accessed 1 Apr. 2020.

Cebula, Richard J. and Clark, J. R. “The impact of economic freedom on per capita real GDP: A study of OECD nations“. *Journal of Regional analysis and Policy*, vol.43, no.2, 2013, pp. 34-41. *ResearchGate*, https://www.researchgate.net/publication/255970025_The_impact_of_economic_freedom_on_per_capita_real_GDP_A_study_of_OECD_nations, , Accessed 30 Apr. 2020.

Chang, Tsangyao, Hsiao-Ping, Chu, Dale, Frederick, Gupta, Randa, Miller Stephen. “The Relationship between Population Growth and Economic Growth Over 1870-2013: Evidence from a Bootstrapped Panel-Granger Causality Test“. Working Papers from University of Pretoria, Department of Economics, no. 201431, 2014. https://econpapers.repec.org/paper/prewpaper/201431.htm, Accessed 29 Apr. 2020.

Clio Infra, Armed conflicts (Internal) [online], accessed at 19-09-2019, available at: https://clio-infra.eu/Indicators/ArmedconflictsInternal.html

Collier, Paul, 2008. "The Bottom Billion: Why the Poorest Countries are Failing and What Can Be Done About It," OUP Catalogue, Oxford University Press, number 9780195374636.

Corchón, Diaz Luis Carlos. “A Malthus-Swan-Solow Model of Economic Growth “ 2016. SSRN, https://ssrn.com/abstract=2013962, Accessed 17 May 2020

Dao , Minh Quang. “Population and Economic Growth in Africa - Popolazione e crescita economica in Africa, “ Economia Internazionale / International Economics, Camera di Commercio Industria Artigianato Agricoltura di Genova, vol. 66(2), pages 215-230. https://ideas.repec.org/a/ris/ecoint/0680.html, Accessed 29 Apr. 2020.

Darrat, A. F., and Y. K. Al-Yousif. “On the Long-Run Relationship between Population and Economic Growth: Some Time Series Evidence for Developing Countries.” *Eastern Economic Journal*, vol. 25, no. 3, 1999, pp. 301–313. *JSTOR*, www.jstor.org/stable/40325933. Accessed 29 Apr. 2020.

De Tray, Dennis N. “Child Quality and the Demand for Children.” *Journal of Political Economy*, vol. 81, no. 2, 1973, pp. S70–S95. *JSTOR*, www.jstor.org/stable/1840413. Accessed 1 Apr. 2020.

Decker, Ryan A., et al. “Declining Business Dynamism: What We Know and the Way Forward.” *The American Economic Review*, vol. 106, no. 5, 2016, pp. 203–207., www.jstor.org/stable/43861015. Accessed 1 Apr. 2020.

Decker, Ryan A., et al. “Declining Business Dynamism: What We Know and the Way Forward.” *The American Economic Review*, vol. 106, no. 5, 2016, pp. 203–207., www.jstor.org/stable/43861015. Accessed 5 May 2020.

Domar, Evsey D. “Capital Expansion, Rate of Growth, and Employment.” *Econometrica*, vol. 14, no. 2, 1946, pp. 137–147. *JSTOR*, www.jstor.org/stable/1905364. Accessed 1 Apr. 2020.

Duesenberry, James. “ Comment: An Economic Analysis of Fertility” *Demographic and Economic Change in Developed Countries*, 1960, pp. 209 - 240. *NBER Chapters*, https://www.nber.org/chapters/c2387.pdf, Accessed 1 Apr. 2020.

Fumitaka Furuoka and Qaiser Munir, (2011) "Can population growth contribute to economic development? New evidence from Singapore", *Economics Bulletin*, Vol. 31 No. 4, 2011, pp. 3226-3239, https://scholar.google.com/scholar?hl=cs&as_sdt=0%2C5&q=Can+population+growth+contribute+to+economic+development%3F+New+evidence+from+Singapore&btnG=, Accesed 29 Apr. 2020.

Furman, Jeffrey L., and Megan J. MacGarvie. “ Academic science and the birth of industrial research laboratories in the U.S. pharmaceutical industry” *Journal of Economic Behavior & Organization*, vol. 63, no. 4, 2007, pp. 756–776. *ScienceDirect*, https://www.sciencedirect.com/science/article/abs/pii/S0167268107000157. Accessed 1 Apr. 2020.

FURUOKA, Fumitaka. “ Population and economic development in Indonesia: A revisit with new data and methods.” *Acta Oeconomica*, vol. 63, no. 4, 2013, pp. 451–467., www.jstor.org/stable/24857647. Accessed 29 Apr. 2020.

Galenson, Walter, and Harvey Leibenstein. “Investment Criteria, Productivity, and Economic Development.” *The Quarterly Journal of Economics*, vol. 69, no. 3, 1955, pp. 343–370. *JSTOR*, www.jstor.org/stable/1885846. Accessed 1 Apr. 2020.

Galor, Oded, and David N. Weil. “Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond.” *The American Economic Review*, vol. 90, no. 4, 2000, pp. 806–828. *JSTOR*, www.jstor.org/stable/117309. Accessed 5 May 2020.

Garza-Rodriguez, Jorge and Andrade-Velasco, Cecillia and Martinez-Silva, Karen and Renteria-Rodriguez, Francisco and Vallejo-Castillo, Pedro, “The Relationship Between Population Growth and Economic Growth in Mexico“, *Economics Bulletin*, Vol. 36 No. 1 pp. 97-107, 2016. https://ssrn.com/abstract=2728681, Accessed 29 Apr. 2020.

Goldin, Claudia; Katz, Lawrence. “The Race Between Education and Technology“. Bibliovault OAI Repository, the University of Chicago Press, 2009. *ResearchGate*, https://www.researchgate.net/publication/37713153_The_Race_Between_Education_and_Technology. Accessed 1 Apr. 2020.

Grossman, Gene M.; Helpman, Elhanan. “ Trade, knowledge spillovers, and growth.” *European Economic Review*, vol. 35, *No. 2–3, 1991*, pp. 517-526. *ScienceDirect*, https://www.sciencedirect.com/science/article/pii/001429219190153A, Accessed 1 Apr. 2020.

Habimana, O. “Do flexible exchange rates facilitate external adjustment? A dynamic approach with time-varying and asymmetric volatility“. *Int Econ Econ Policy*, vol. 14, 2017, pp. 625–642, https://doi.org/10.1007/s10368-016-0341-7, Accessed 30 Apr. 2020.

Hansen, Lars Peter. “Large Sample Properties of Generalized Method of Moments Estimators.” *Econometrica*, vol. 50, no. 4, 1982, pp. 1029–1054. *JSTOR*, www.jstor.org/stable/1912775. Accessed 30 Apr. 2020.

Harrod, R. F. “An Essay in Dynamic Theory.” *The Economic Journal*, vol. 49, no. 193, 1939, pp. 14–33. *JSTOR*, www.jstor.org/stable/2225181, Accessed 1 Apr. 2020.

Heritage.org index of economic freedom [online], accessed at 19-09-2019, available at: https://www.heritage.org/index/explore?view=by-region-country-year&u=636968127055364350

Holtz-Eakin, Douglas, et al. "Estimating Vector Autoregressions with Panel Data." *Econometrica*, vol. 56, no. 6, 1988, pp. 1371–1395. *JSTOR*, www.jstor.org/stable/1913103. Accessed 30 Apr. 2020.

Hudakova, Jarka. "Relationship between gross domestic product and human development index." 2018, *ResearchGate,* https://www.researchgate.net/publication/324803815_RELATIONSHIP_BETWEEN_GROSS_DOMESTIC_PRODUCT_AND_HUMAN_DEVELOPMENT_INDEX, Accessed 30 Apr. 2020.

in India: A co-integration analysis." *Scientific Journal of Pure and Applied Sciences,* vol.1, No.3, pp.90-96., 2012. http://sjournals.com/index.php/SJPAS/article/view/450, Accessed 29 Apr. 2020.

Kodoru, Bhanu Phani Krishna, Tatavarthi, Archana. "Effect of Population Growth on Economic Development in India". 2016, https://www.researchgate.net/publication/309158950_Effect_of_Population_Growth_on_Economic_Development_in_India, Accessed 29 Apr. 2020.

Kónya, László. "Exports and growth: Granger causality analysis on OECD countries with a panel data approach". *Economic modelling,* Vol.23, No.6, pp. 978-992, 2006. https://www.sciencedirect.com/science/article/abs/pii/S0264999306000496, Accessed 29 Apr. 2020.

Koopmans, Tjalling C. " On the Concept of Optimal Economic Growth." *Cowles Foundation Discussion Papers,* vol. 163, 1963. *IDEAS,* https://ideas.repec.org/p/cwl/cwldpp/163.html, Accessed 1 Apr. 2020.

Lee, R.D. , "Malthus and Boserup: a dynamic synthesis", in: D. Coleman and R. Schofield, eds., The state of population theory: forward from Malthus (Basil Blackwell, Oxford) pp. 96-130, 1986.

Lentz, Rasmus, and Dale T. Mortensen. "An Empirical Model of Growth through Product Innovation." *Econometrica,* vol. 76, no. 6, 2008, pp. 1317–1373. *JSTOR*, www.jstor.org/stable/40056508. Accessed 1 Apr. 2020.

Munir, Qaiser, Kasim Mansur, and Fumitaka Furuoka. "Inflation and Economic Growth in Malaysia: A Threshold Regression Approach." *ASEAN Economic Bulletin*, vol. 26, no. 2, 2009, pp. 180–193. *JSTOR*, www.jstor.org/stable/41317057. Accessed 29 Apr. 2020.

Nerlove, Marc and Lakshmi Raut. " Growth models with endogenous population: A general framework." *Handbook of Population and Family Economics*, vol. 1, part B, 1997, pp. 1117-1174. *ScienceDirect*, https://www.sciencedirect.com/science/article/pii/S1574003X97800128, Accessed 2 Apr. 2020

Nerlove, Marc, Assaf Razin and Efraim Sadka. *Household and Economy. Welfare Economics of Endogenous Fertility*. Florida, Academic Press, 2014. ISBN 1483274683

Nerlove, Marc. "Procreation, Fishing, and Hunting: Renewable Resources and Dynamic Planar Systems." *American Journal of Agricultural Economics*, vol. 75, no. 1, 1993, pp. 59–71. *JSTOR*, www.jstor.org/stable/1242954. Accessed 2 Apr. 2020.

Nickell, Stephen. "Biases in Dynamic Models with Fixed Effects." *Econometrica*, vol. 49, no. 6, 1981, pp. 1417–1426. *JSTOR*, www.jstor.org/stable/1911408. Accessed 30 Apr. 2020.

Niehans, Jürg. "Economic Growth with Two Endogenous Factors." *The Quarterly Journal of Economics*, vol. 77, no. 3, 1963, pp. 349–371. *JSTOR*, www.jstor.org/stable/1879567. Accessed 5 May 2020.

Nishimura, Kazuo, and Junsen Zhang. "Sustainable Plans of Social Security with Endogenous Fertility." *Oxford Economic Papers*, vol. 47, no. 1, 1995, pp. 182–194. *JSTOR*, www.jstor.org/stable/2663671. Accessed 1 Apr. 2020.

Pesaran, M. Hashem, et al. "A Bias-Adjusted LM Test of Error Cross-Section Independence." *The Econometrics Journal*, vol. 11, no. 1, 2008, pp. 105–127. *JSTOR*, www.jstor.org/stable/23116064. Accessed 29 Apr. 2020.

Pesaran, M. Hashem, Yamagata, Takashi. "Testing slope homogeneity in large panels". *Journal of Econometrics,* vol.124, no. 1, pp. 50-93, 2008. https://www.sciencedirect.com/science/article/abs/pii/S0304407607001224, Accessed 29 Apr. 2020.

Pesaran, M. Hashem. “General Diagnostic Tests for Cross Section Dependence in Panels“. *CESifo Working Paper Series,* No. 1229; IZA Discussion Paper, No. 1240, 2004. https://ssrn.com/abstract=572504, Accessed 29 Apr. 2020.

Pestieau Pierre and Grégory Ponthière. ”Optimal Lifecycle Fertility in a Barro Becker Economy”. 2012, *HAL*, https://halshs.archives-ouvertes.fr/halshs-00676500/document, Accessed 1 Apr. 2020.

Ramsey, Frank P. “ A Mathematical Theory of Saving.” *The Economic Journal*, vol. 38, no. 152, 1928, pp. 543-559. *JSTOR*, https://www.jstor.org/stable/2224098. Accessed 1 Apr. 2020.

Raut, Lakshimi.), ”Subgame perfect manipulation of children by overlapping generations of agents with two-sided altruism and endogenous fertility”, 1996, ResearchGate, https://www.researchgate.net/publication/23746913_Subgame_perfect_manipulation_of_children_by_overlapping_generations_of_agents_with_two-sided_altruism_and_endogenous_fertility, Accessed 1 Apr. 2020.

Ricardo, David: On the Principles of Political Economy and Taxation (1 ed.). London: John Murray, 1817

Romer, Paul M. “Increasing Returns and Long-Run Growth.” *Journal of Political Economy*, vol. 94, no. 5, 1986, pp. 1002–1037. *JSTOR*, www.jstor.org/stable/1833190. Accessed 1 Apr. 2020

Sargan, J. D. “The Estimation of Economic Relationships Using Instrumental Variables.” *Econometrica*, vol. 26, no. 3, 1958, pp. 393–415. *JSTOR*, www.jstor.org/stable/1907619. Accessed 30 Apr. 2020.

Schumpeter, Joseph. “ The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle.” 1934. SSRN, https://ssrn.com/abstract=1496199, Accessed 1 Apr. 2020.

Sheshinksi, Eytan: “Optimal Accumulation with Learning by Doing“. *Karl, S., Ed., Essays on the Theory of Optimal Economic Growth, MIT Press, Cambridge*, 1967, pp.31-52.

Singha, Komol, and M. S. Jaman. (2012). “Nexus between population and economic growth“

Smith, Adam: An Inquiry into the Nature and Causes of the Wealth of Nations. 1 (1 ed.). London: W. Strahan,1776.

Solow, Robert M. “A Contribution to the Theory of Economic Growth.” *The Quarterly Journal of Economics*, vol. 70, no. 1, 1956, pp. 65–94. *JSTOR*, www.jstor.org/stable/1884513. Accessed 1 Apr. 2020.

Stamova, Ivanka M. and Gani A. Stamov . “On the stability of the solutions of an impulsive Solow model with endogenous population. Economic Change and Restructuring.“ *Economic Change and Restructuring*, vol. 46, no. 2, 2012, pp. 203–217. *SpringerLink*, https://link.springer.com/article/10.1007/s10644-012-9124-5#auth-1, Accessed 17 May 2020

Swan, Trevor W. “ Economic growth and capital accumulation.” *Economic Record*, vol. 32, no. 2, 1956, pp. 334-361. *Wiley online library*, https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1475-4932.1956.tb00434.x, Accessed 1 Apr. 2020.

Thornton, John. “Population Growth and Economic Growth: Long-Run Evidence from Latin America.” *Southern Economic Journal*, vol. 68, no. 2, 2001, pp. 464–468. *JSTOR*, www.jstor.org/stable/1061606. Accessed 29 Apr. 2020.

Tir, Jaroslav and Diehl Paul F. “Demographic Pressure and Interstate Conflict: Linking Population Growth and Density to Militarized Disputes and Wars, 1930-89“. *Journal of Peace Research,* vol. 35, no. 3, 1998. *Journal of Peace research,* https://journals.sagepub.com/doi/abs/10.1177/0022343398035003004, Accessed 30 Apr. 2020.

Ufuk, Akcigit and Sina Ates., et al. “ Innovation and trade policy in a globalized world.” *The American NBER working paper series 24543*, 2018. *NBER*, https://www.nber.org/papers/w24543.pdf. Accessed 1 Apr. 2020.

Ufuk, Akcigit and William R. Kerr, "Growth through Heterogeneous Innovations," *Journal of Political Economy,* vol. 126, no. 4, 2018, pp. 1374-1443. *The University of Chicago Press Journals*, https://www.journals.uchicago.edu/doi/abs/10.1086/697901, Accessed 1 Apr. 2020.

UNICEF, Malaria [online], accessed at 19-09-2019, available at: https://data.unicef.org/topic/child-health/malaria/

Uppsala Conflict Data Program, UCDP dataset download center [online], Accessed at 19-09-2019, available at: https://ucdp.uu.se/downloads/#d3

Willis, Robert J. "A New Approach to the Economic Theory of Fertility Behavior." *Journal of Political Economy*, vol. 81, no. 2, 1973, pp. S14–S64. *JSTOR*, www.jstor.org/stable/1840411. Accessed 1 Apr. 2020.

World Bank Gross domestic savings (% of GDP) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/NY.GDS.TOTL.ZS?locations=LR

World Bank Net official development assistance and official aid received (current US$) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/DT.ODA.ALLD.CD

World Bank Prevalence of undernourishment (% of population) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/sn.itk.defc.zs

World Bank Total natural resources rents (% of GDP) [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS

World Bank, Growth of GDP per capita [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/NY.GDP.PCAP.KD.ZG

World Bank, International tourism, number of arrivals [online], accessed at 19-09-2019, available at: https://data.worldbank.org/indicator/ST.INT.ARVL

World Bank, Population growth [online], accessed at 19-09-2019, available at: https://databank.worldbank.org/reports.aspx?source=2&series=SP.POP.GROW&country=#

World Bank: World Development Indicators [online], accessed at 19-09-2019, available at: https://datacatalog.worldbank.org/dataset/world-development-indicators

United Nations, HDI (1990 – 2018) [online], accessed at 19-09-2019, available at: http://hdr.undp.org/en/data#

A Appendix

This appendix is a derivation of law of motion for capital for the population growth in two forms (exogenous and endogenous). We define K_{t+1}as:

$$K_{t+1} = (1-\delta)K_t + I_t = sF(K_t, L_t) + (1-\delta)K_t \tag{A.1}$$

Next, we define the population growth in the exogenous form:

$$\begin{gathered} L_{t+1} = (1+\bar{n})L_t \\ \frac{L_{t+1}}{L_t} = (1+\bar{n}) \\ L_t = \frac{L_{t+1}}{(1+\bar{n})} \\ L_t = (1+n)^t L_0 \end{gathered} \tag{A.2}$$

Equation (A.1) in aggregate terms is divided by L_t, note that $L_t = \frac{L_{t+1}}{(1+\bar{n})}$:

$$\begin{gathered} K_{t+1} = sf(k_t) + (1-\delta)K_t \\ \frac{K_{t+1}}{L_t} = s\frac{F(K_t, L_t)}{L_t} + (1-\delta)\frac{K_t}{L_t} \\ \frac{K_{t+1}}{\frac{L_{t+1}}{(1+n)}} = s\frac{F(K_t, L_t)}{L_t} + (1-\delta)\frac{K_t}{L_t} \\ (1+n)\frac{K_{t+1}}{L_{t+1}} = s\frac{F(K_t, L_t)}{L_t} + (1-\delta)\frac{K_t}{L_t} \\ (1+n)k_{t+1} = sf(k_t) + (1-\delta)k_t \\ k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+\bar{n}} \end{gathered} \tag{A.3}$$

When we have the population growth in the endogenous form which depends on exogenous saving rate in per capita term. L_t is:

$$\frac{L_{t+1}}{L_t} = 1 + n[(1-s)f(k_t)]$$
$$L_t = \frac{L_{t+1}}{1 + n[(1-s)f(k_t)]} \tag{A.4}$$

Again, Equation (A.1) is divided by L_t, note that $L_t = \frac{L_{t+1}}{1+n[(1-s)f(k_t)]}$:

$$K_{t+1} = sf(k_t) + (1-\delta)K_t$$
$$\frac{K_{t+1}}{L_t} = s\frac{F(K_t, L_t)}{L_t} + (1-\delta)\frac{K_t}{L_t}$$
$$\frac{K_{t+1}}{\frac{L_{t+1}}{1+n[(1-s)f(k_t)]}} = s\frac{F(K_t, L_t)}{L_t} + (1-\delta)\frac{K_t}{L_t}$$
$$1 + n[(1-s)f(k_t)]\frac{K_{t+1}}{L_{t+1}} = s\frac{F(K_t, L_t)}{L_t} + (1-\delta)\frac{K_t}{L_t} \tag{A.5}$$
$$1 + n[(1-s)f(k_t)]k_{t+1} = sf(k_t) + (1-\delta)k_t$$
$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1 + n[(1-s)f(k_t)]}$$

B Appendix

Appendix B provides a more detailed solution for chapter 1.5.1 The Nerlove – Raut model example. First, we define production function, law of motion for capital, and equation for the endogenous population growth.

	$y_{t=}k_t^{\sigma} z_t^{\mu}, 0 < \sigma; \mu, \sigma + \mu < 1,$ $1 + n(k_t, z_t) = \frac{L_{t+1}}{L_t} = \frac{w_t}{w_m}, w_m > 0,$	(B.1)

$$k_{t+1} = \frac{(1-\delta)k_t - s_t}{w_t/w_m},$$
$$s_t = (\sigma+\mu)y_t,$$
$$w_t = y_t - s_t = (1-\sigma-\mu)y_t$$

Note that all equations above are written in per capita terms. Then we define first function from the law of motion for capital in steady-state.

$$k^* = \frac{(1-\delta)k^* + (\sigma+\mu)(k^*)^\sigma(z^*)^\mu}{\frac{(1-\sigma-\mu)(k^*)^\sigma(z^*)^\mu}{w_m}}$$
$$k^*\frac{(1-\sigma-\mu)(k^*)^\sigma(z^*)^\mu}{w_m} = (1-\delta)k^* + (\sigma+\mu)(k^*)^\sigma(z^*)^\mu$$
$$k^*\left[\frac{(1-\sigma-\mu)(k^*)^\sigma(z^*)^\mu}{w_m}\right] - (\sigma+\mu)(k^*)^\sigma(z^*)^\mu = (1-\delta)k^*$$
$$(k^*)^\sigma(z^*)^\mu k^*\left[\frac{(1-\sigma-\mu)}{w_m}\right] - (\sigma+\mu) = (1-\delta)k^*$$
$$(k^*)^\sigma(z^*)^\mu = \frac{(1-\delta)k^*}{\left[\frac{(1-\sigma-\mu)}{w_m}\right]k^* - (\sigma+\mu)} \tag{B.2}$$
$$(z^*)^\mu = \frac{(1-\delta)k^*(k^*)^{-\sigma}}{\left[\frac{(1-\sigma-\mu)}{w_m}\right]k^* - (\sigma+\mu)}$$
$$(z^*)^\mu = \frac{(1-\delta)(k^*)^{1-\sigma}}{\left[\frac{1-(\sigma+\mu)}{w_m}\right]k^* - (\sigma+\mu)}$$
$$z^* = \left\{\frac{(1-\delta)(k^*)^{1-\sigma}}{\left[\frac{1-(\sigma+\mu)}{w_m}\right]k^* - (\sigma+\mu)}\right\}^{\frac{1}{\mu}}$$
$$M^{-1}(k^*) = z^*$$

Another function we need to set is for factor *z*. Thus a N(z*) is:

$$z_{t+1} = k_t^\alpha z_t^\beta, 0 < \beta < 1 \tag{B.3}$$

$$z^* = (k^*)^\alpha (z^*)^\beta$$

$$\frac{z^*}{(z^*)^\beta} = (k^*)^\alpha$$

$$(z^*)^{1-\beta} = (k^*)^\alpha$$

$$z^* = (k^*)^{\frac{\alpha}{1-\beta}}$$

$$N(z^*) = (k^*)^{\frac{\alpha}{(1-\beta)}}$$

Now we can show how population growth is determined:

$$1 + n(k_t, z_t) = \frac{L_{t+1}}{L_t} = \frac{w_t}{w_m}, w_m > 0$$

$$1 + n(k_t, z_t) = \frac{L_{t+1}}{L_t} = \frac{(1-\sigma-\mu)(k^*)^\sigma (z^*)^\mu}{w_m} \tag{B.4}$$

When we substitute term z* by N(z*), we receive equation for population growth:

$$1 + n(k_t, z_t) = \frac{L_{t+1}}{L_t} = \frac{(1-\sigma-\mu)(k^*)^\sigma (k^*)^{\frac{\alpha\mu}{(1-\beta)}}}{w_m} \tag{B.5}$$

C Appendix

Appendix C is a detail solution for the maximization problem in Chapter 1.6:

$$u^*(b_t) = \max_{(c,n,b_{t+1}} \{v(c_t, n_t) + \beta(n_t) n_t u^*(b_{t+1})\},$$

$$such\ that\ c_t + (b_{t+1} + a) n_t = I + b_t, \tag{C.1}$$

Under the assumption described in the same chapter, we need to solve the Lagrangian function:

$$u_0^* \equiv \sum_{i=0}^{\infty}\left\{c_i^\sigma \beta_0^i N_i^{1-\beta_1} - \lambda_i[c_i + (k_i + a)n_i - w_i - (1 + r_i)k_{i-1}]\right\} \tag{C.2}$$

Equation (C.2) equals to zero:

$$\begin{aligned}
\frac{\partial u_0^*}{\partial c_j} &= \frac{\sigma V_j}{c_j} - \lambda_j = 0, \\
\frac{\partial u_0^*}{\partial n_j} &= \sum_{i=j+1}^{\infty} \frac{(1-\beta_1)V_i}{N_i}\frac{\partial N_i}{\partial n_j} - \lambda_j(k_j + a), \\
&= \frac{1}{n_j}\sum_{i=j+1}^{\infty}(1-\beta_1)V_i - \lambda_j(k_j + a) = 0, \\
\frac{\partial u_0^*}{\partial k_j} &= -\lambda_j n_j + \lambda_{j+1}(1 + r_{j+1}) = 0, j = 1,2, \dots.,
\end{aligned} \tag{C.3}$$

Next, we take ratios which are described in the relevant chapter.